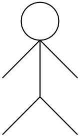
Rank-to-rank hypotheses and the failure of GCH

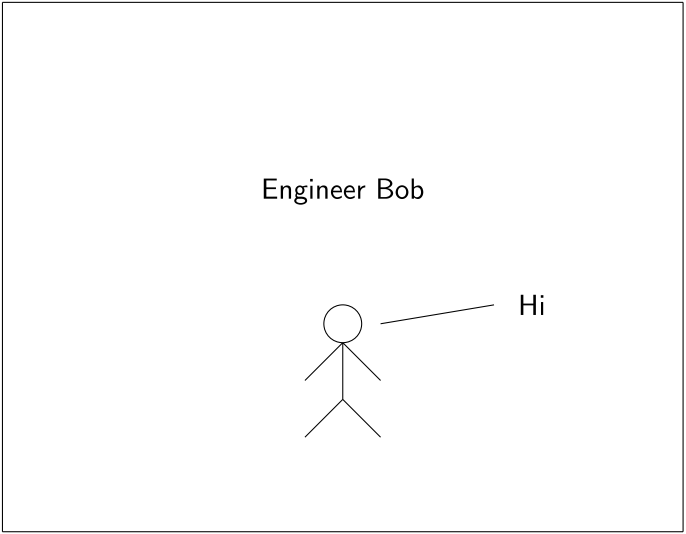
Joint work with Sy Friedman

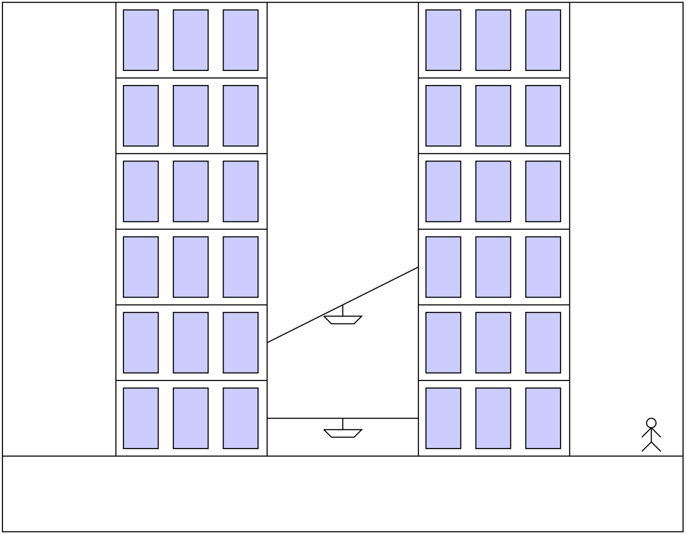
Vincenzo Dimonte

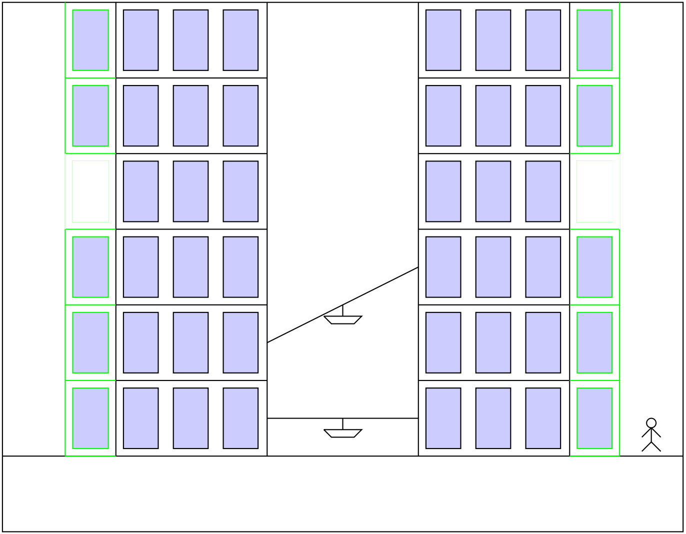
21 June 2012

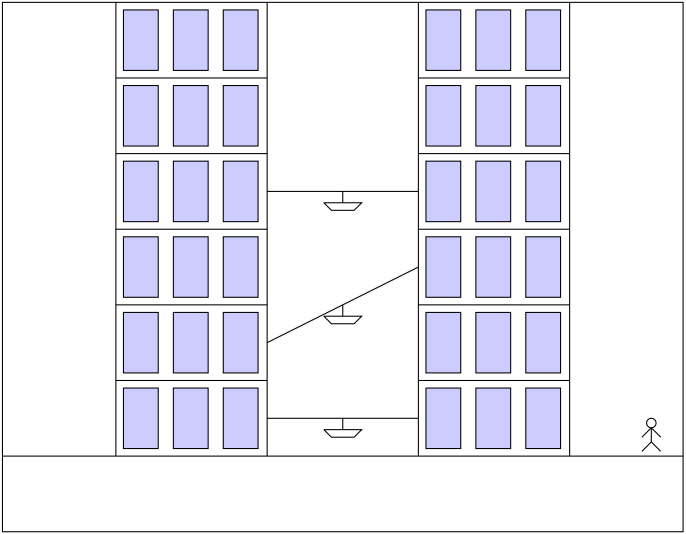
Engineer Bob

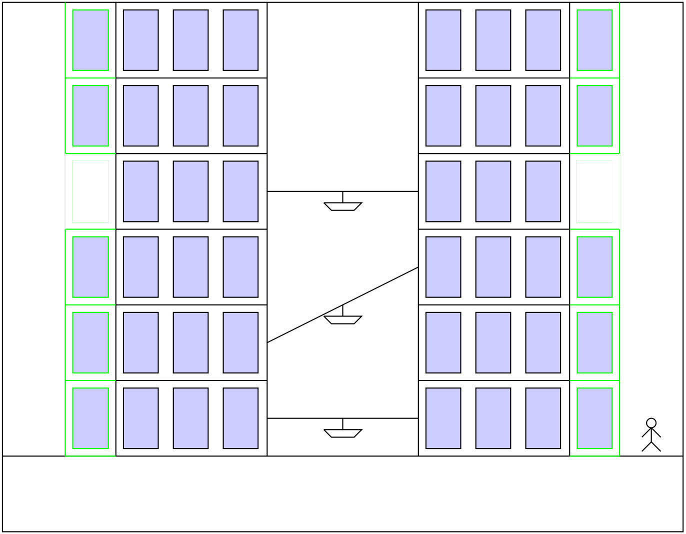


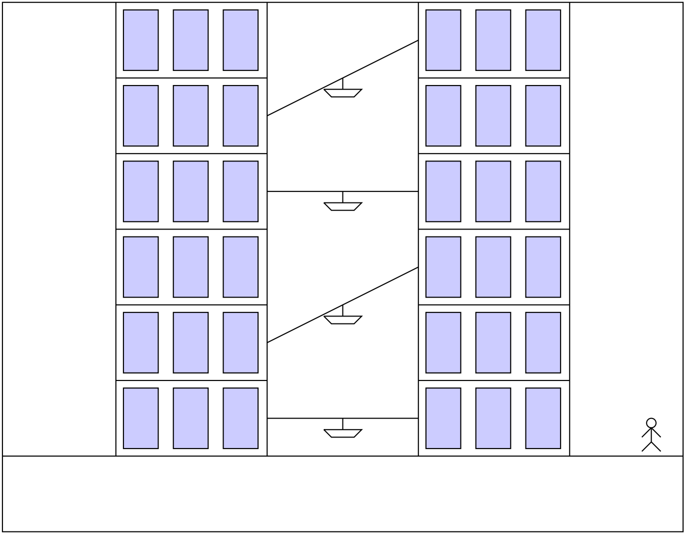

Engineer Bob
Hi

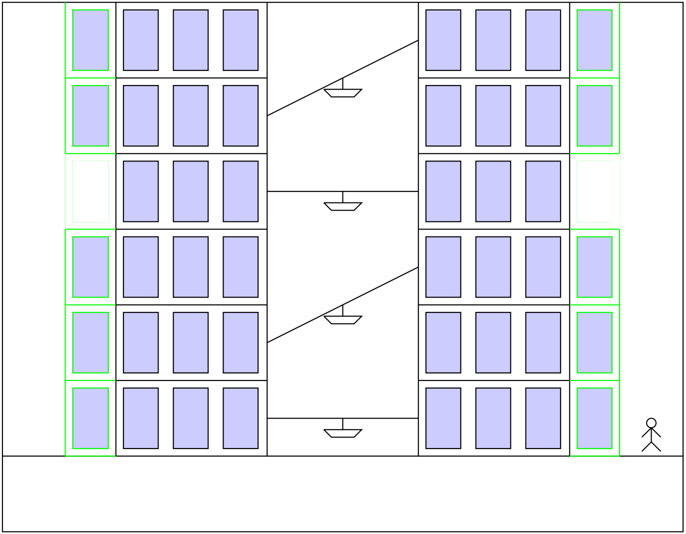


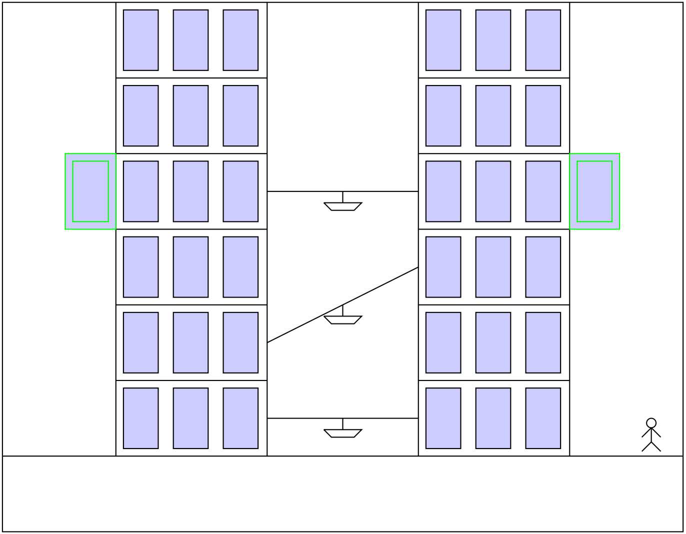




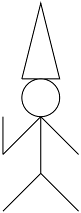


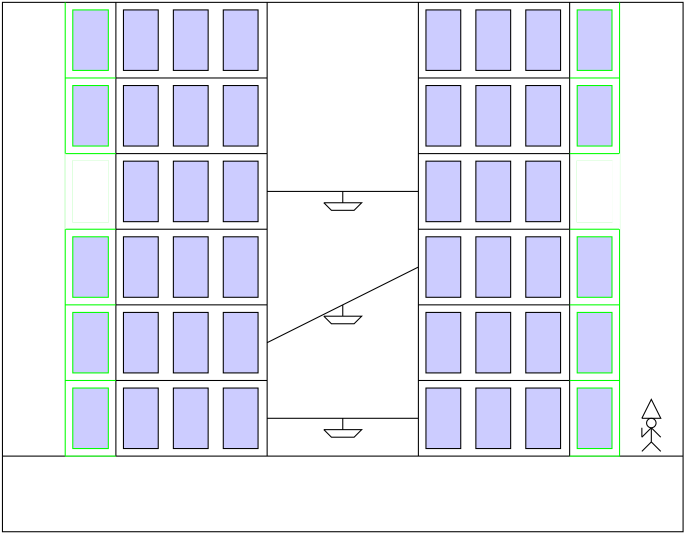






The Sorcerer Bob





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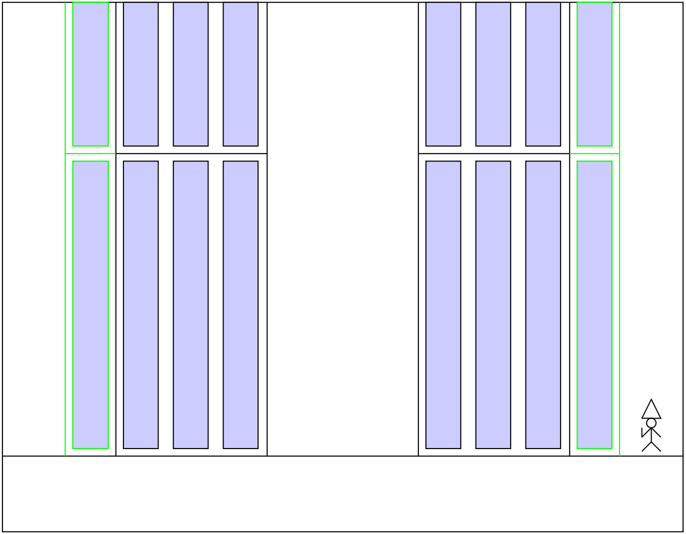

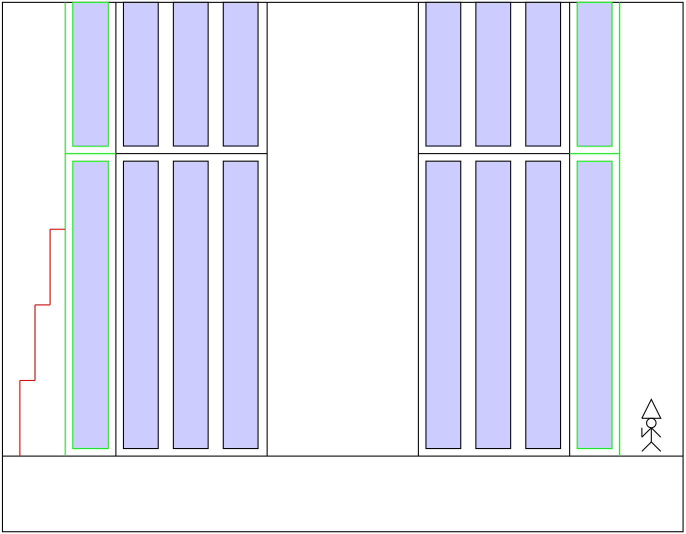

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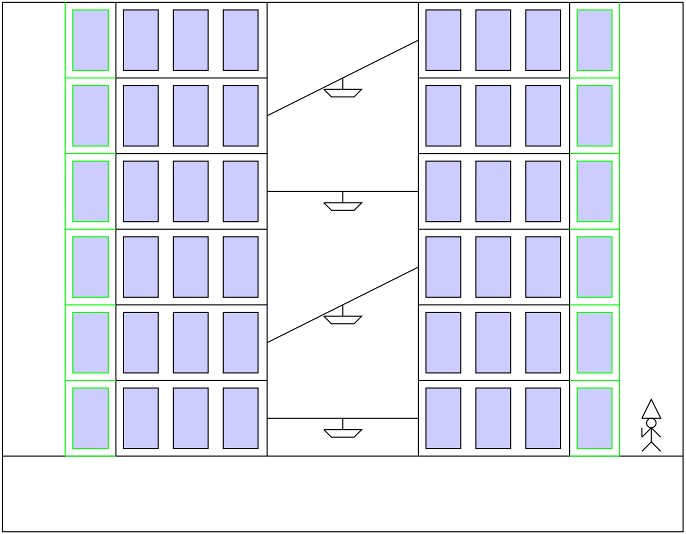

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Let E : Reg $\to$ Card a class function such that

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Then there exist definable, directed closed, reverse Easton iterations $\mathbb{P}$ of length the ordinals such that, if G is generic for $\mathbb{P}$, $V[G] \vDash$ GCH, or $V[G] \vDash \forall\kappa(\kappa \text{ regular } \to 2^\kappa = E(\kappa))$.

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We summarize the last sentence as "everything goes for the regulars".

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Theorem (Solovay, 1974)

Let κ be a strongly compact cardinal. Let λ be a singular strong limit cardinal greater than κ. Then $2^\lambda = \lambda^+$.

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This leaves room for a new breed of large cardinal hypotheses:

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- I0 is incompatible with $L(V_{\lambda+1}) \vDash$AC
- under I0 $L(V_{\lambda+1})$ is similar to $L(\mathbb{R})$ under AD.

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But in this case it is not unique: let $j : V_\lambda \prec V_\lambda$, then $j(j) = \bigcup_{n\in\omega} j(j \cap V_{\kappa_n})$ is an elementary embedding.
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$\mathrm{crt}(j(j)) = \kappa_1$.
So, we can think of λ as a large cardinal. But it is singular.

Reminder: lifting lemma

Lifting Lemma

Assume $j : M \prec M$ is an elementary embedding between models of ZF, G is M-generic for a poset $\mathbb{P} \in M$, and $j``G \subseteq G$.

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- $[\kappa_0, \lambda]$ this is the sensitive part.

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New Theorem!

Con(I^*+ everything goes at regulars)

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Now $V_{\lambda+1}[G] = V[G]_{\lambda+1}$ and $L(V[G]_{\lambda+1}) \subseteq L(V_{\lambda+1})[G]$. $\square$

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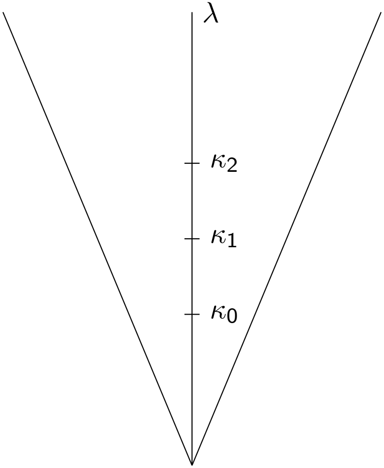
What about singular cardinals?
Solovay restricts us: κ_0 is strongly compact in V_λ, so the strong limit singulars satisfy GCH co-boundedly in κ_0 (and κ_1, $\kappa_2, \dots$).


λ
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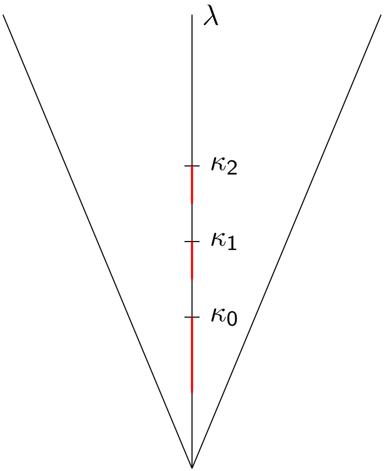
κ_1
κ_0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Therefore we change strategy, and we will use deep work by Woodin on I0.

Generic Absoluteness

Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ be a proper elementary embedding. Let (M_ω, j_ω) be the ω-th iterate of j. Then for all $\alpha < \lambda^+$ there exists an elementary embedding

$$\pi : L_\alpha(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$$

such that $\pi \upharpoonright \lambda$ is the identity.

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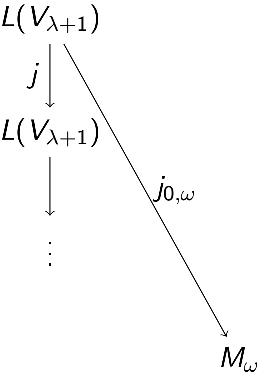
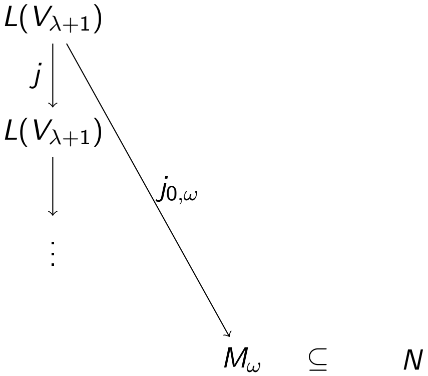
- $N \cap V_{\lambda+1} = (V_{\lambda+1})^N \prec V_{\lambda+1}$;
- exists $\pi : L_1(N \cap V_{\lambda+1}) = (L_1(V_{\lambda+1}))^N \prec L_1(V_{\lambda+1})$.

$L(V_{\lambda+1})$

$$\begin{array}{c} L(V_{\lambda+1}) \\ j \downarrow \\ L(V_{\lambda+1}) \end{array}$$

$$\begin{array}{c} L(V_{\lambda+1}) \\ j \downarrow \\ L(V_{\lambda+1}) \\ \downarrow \\ \vdots \end{array}$$


$L(V_{\lambda+1})$
j
$L(V_{\lambda+1})$
$\vdots$
$j_{0,\omega}$
M_ω


$L(V_{\lambda+1})$
j
$L(V_{\lambda+1})$
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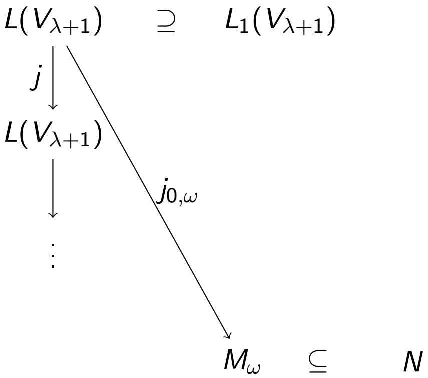
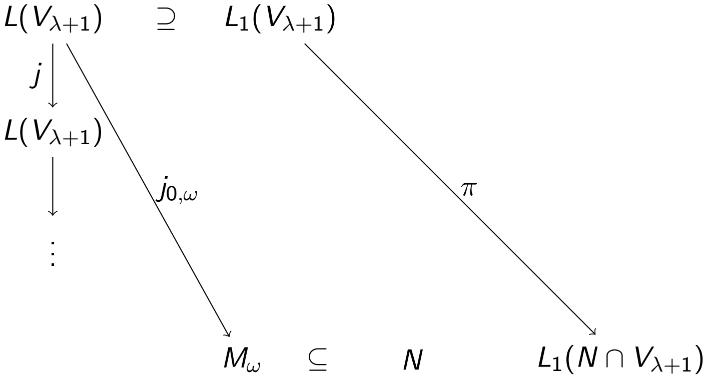
$j_{0,\omega}$
$M_\omega \subseteq N$


$L(V_{\lambda+1}) \supseteq L_1(V_{\lambda+1})$
j
$L(V_{\lambda+1})$
$\vdots$
$j_{0,\omega}$
$M_\omega \subseteq N$


$L(V_{\lambda+1})$
$\supseteq$
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π
M_ω
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N
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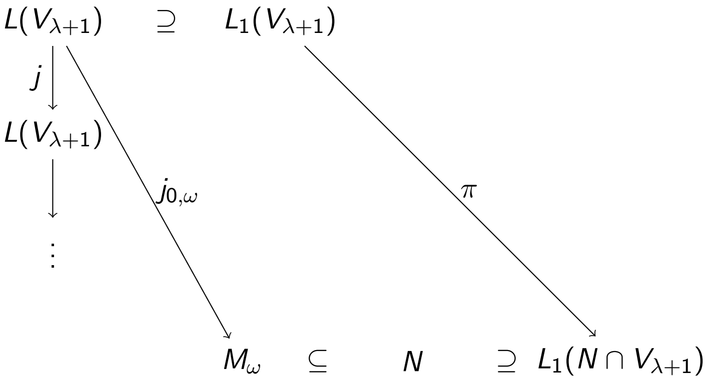
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Therefore we have $2^\lambda = \lambda^{++}$ in N. □

We can use te full power of generic absouluteness and Easton construction to prove:

Complete Theorem

Suppose I0. Then for any $\alpha < \lambda^+$ it is consistent ZFC $+ \exists j : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$ + everything goes below λ (at regulars) and at λ.

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Example

For any $\delta < \lambda$, we can have $2^\kappa = \kappa^{+\delta+1}$.

Open Questions

Is it possible to avoid I0?

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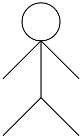
Conjecture

If $\exists j : L((V_{\lambda+1})^\sharp) \prec L((V_{\lambda+1})^\sharp)$ then is I0 + everything goes at λ consistent?

Thanks for your patience



Thanks for your patience