Higher Determinacy Axioms

Vincenzo Dimonte

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Chart of Cardinals

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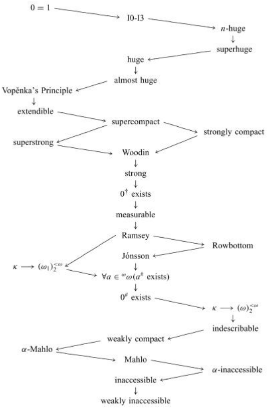

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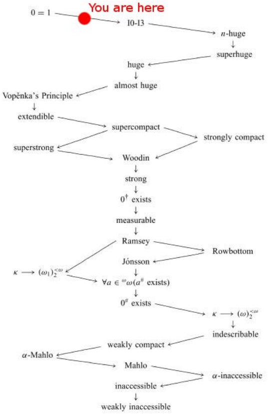

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(Reinhardt, 70) κ is a Reinhardt cardinal iff it's the critical point of an elementary embedding $j : V \prec V$.

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There is no $j : V_\eta \prec V_\eta$, with $\eta \geq \lambda + 2$.

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- I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Technical note: if $j, k : V_{\lambda+1} \prec V_{\lambda+1}$ and $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$.

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These results, however, after a while became obsolete, and nowadays there is no equivalence result.

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I0 in fact is more interesting than the other axioms, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

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I0 in fact is more interesting than the other axioms, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.
Since λ has cofinality ω, V_λ is similar to V_ω, so $V_{\lambda+1}$ is similar to $\mathbb{R}$.

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In fact these analogies hold for every model of $\mathrm{HOD}_{V_{\lambda+1}}$.

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ω_1 is measurable	λ^+ is measurable
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The most immediate corollary for the Coding Lemma is: For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$.

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- $L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

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This goes in the direction of "The club filter on ω_1 is an ultrafilter".

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- there is an elemntary embedding $j_c : L(X, Y, V[c]_{\lambda+1}) \to L(X, Y, V[c]_{\lambda+1})$ with $\operatorname{crt}(j) < \lambda$

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I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms? The first step is to find analogies with $L(A,\mathbb{R})\vDash \mathrm{AD}$, with $A\subseteq\mathbb{R}$.

$$j: L(X, V_{\lambda+1})\prec L(X, V_{\lambda+1}),\ \text{with } X\subset V_{\lambda+1};$$

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Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$.
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So the "important part" of j is under $L_\Theta(X, V_{\lambda+1})$.

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The sequence stops when $L(\Gamma_\alpha) \nvDash \mathrm{AD}$ or $\Gamma_\alpha = \Gamma_{\alpha+1}$

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The situation is more complicated, because even the first analogy can be false.

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Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

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If j is proper, the third analogy hold.

Informal definition of $X^\sharp$:
Suppose that there exists a class I of indiscernibles of $(L(X), \in, \{a : a \in X\}, X)$ such that every cardinal $> |X|$ is in I. Then $X^\sharp$ is the theory of the indiscernibles in the language $\{\in\} \cup \{a : a \in X\} \cup \{X\}$, i.e.

$$\begin{aligned} X^\sharp = \{\varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) : a_1, \dots, a_n \in X, \\ L(X) \vDash \varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) \text{ for some (any)} \\ \text{indiscernibles } i_1 < \dots < i_n \in I\} \end{aligned}$$

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$X^\sharp$ contains the "truth" of $L(X)$, so it cannot be in $L(X)$.

Definition

The sequence

$$\langle E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}}\rangle$$

is defined as:

- $E^0_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}$;
- for α limit, $E^0_\alpha(V_{\lambda+1}) = L(\bigcup_{\beta<\alpha} E^0_\beta(V_{\lambda+1})) \cap V_{\lambda+2}$;
- for α limit,
  - if $(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}) < \lambda)^{L(E^0_\alpha(V_{\lambda+1}))}$ then
  $$E^0_{\alpha+1}(V_{\lambda+1}) = L((E^0_\alpha(V_{\lambda+1}))^\lambda) \cap V_{\lambda+2};$$
  - if $(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}))^{L(E^0_\alpha(V_{\lambda+1}))} > \lambda$ then
  $$E^0_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(E^0_\alpha(V_{\lambda+1}))) \cap V_{\lambda+2};$$

Definition

- for $\alpha = \beta + 2$, if there exists $X \subseteq V_{\lambda+1}$ such that $E^0_{\beta+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ and $E^0_\beta(V_{\lambda+1}) < X$, then
$$E^0_{\beta+2}(V_{\lambda+1}) = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$$
otherwise we stop the sequence.
- $\forall\alpha < \Upsilon_{V_{\lambda+1}}$ $\exists X \subseteq V_{\lambda+1}$ such that $E^0_\alpha(V_{\lambda+1}) < X$ and $\exists j \colon L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ proper;
- $\forall\alpha$ limit $\alpha + 1 < \Upsilon_{V_{\lambda+1}}$ iff
$$(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}))^{L(E^0_\alpha(V_{\lambda+1}))} > \lambda \to$$
$$\exists Z \in E^0_\alpha(V_{\lambda+1})\ L(E^0_\alpha(V_{\lambda+1})) = (\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(E^0_\alpha(V_{\lambda+1}))}.$$

Definition

Let $N = L(\bigcup\{E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}}\}) \cap V_{\lambda+2}$. Suppose that

- $\mathrm{cof}(\Theta^N) > \lambda$;
- for all $Z \in N$ $L(N) \neq (\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(N)}$;
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- there is an elementary embedding $j : L(N) \prec L(N)$ with $\mathrm{crt}(j) < \lambda$.

Then $E^0_\infty(V_{\lambda+1})$ exists and $E^0_\infty(V_{\lambda+1}) = N$.

Three important facts:

Theorem

- If $\alpha < \beta < \Upsilon$, then $\Theta^{E^0_\alpha} < \Theta^{E^0_\beta}$.

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- The E^0_α sequence is absolute, i.e. for every M such that $L(M) \cap V_{\lambda+2}, V_{\lambda+1} \subseteq M$ for every $\alpha < \Upsilon^M$, $(\langle E^0_\beta : \beta < \alpha \rangle)^M = \langle E^0_\beta : \beta < \alpha \rangle$.

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- If $\alpha < \Upsilon$, then there exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$.

The E^0_α sequence can be considered as the “standard” example of Higher Determinacy Axioms.

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Theorem

Let $X \subset V_{\lambda+1}$ such that there exists $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$. Let $Y \in L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ such that $\Theta^{L(Y, V_{\lambda+1})} < \Theta^{L(X, V_{\lambda+1})}$. Then $(Y, V_{\lambda+1})^\sharp$ exists and $(Y, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$.

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So, two different concepts of “largeness” in this case coincide.

Theorem

Suppose $X \subset V_{\lambda+1}$ and there is an elementary embedding $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$. Then, either $\Upsilon^X = \Upsilon$ and $\sup_{\eta<\Upsilon} \Theta^{E^0_\eta} \leq \Theta^{L(X,V_{\lambda+1})}$, or there exists $\eta < \Upsilon$ such that $\Upsilon^X = \eta + 1$ and $\Theta^{E^0_\eta} = \Theta^{L(X,V_{\lambda+1})}$. Moreover, if $\alpha < \Upsilon^X$ then $E^X_\alpha = E^0_\alpha$.

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So, if there exists $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ and $\Theta^{L(X,V_{\lambda+1})} < \Theta^{E^0_\infty}$, then there exists $j : L(E^0_\eta) \prec L(E^0_\eta)$ and $\Theta^{E^0_\eta} = \Theta^{L(X,V_{\lambda+1})}$.

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Suppose $\alpha < \Upsilon$. If

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then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.

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Suppose $\alpha < \Upsilon$. If

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- α is a successor ordinal, or
- α is a limit ordinal with cofinality $> \omega$

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We will prove that both exist. The key Lemma is the following:

Lemma

Suppose $\alpha < \Upsilon$ and $\Theta^{E^0_\alpha}$ is regular in $L(E^0_\alpha)$. If $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper then the set of fixed points of j is cofinal in $\Theta^{E^0_\alpha}$.

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$$(E^0_\alpha)^\sharp_{\beta,n} = (E^0_\alpha)^\sharp \cap (\{\in\} \cup \{a : a \in E^0_\beta\} \cup \{X\} \cup \{i_1, \dots, i_n\})$$

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For every $\beta < \alpha$, $n \in \omega$ $(E^0_\alpha)^\sharp_{\beta,n} \in E^0_\alpha$, but $L(E^0_\alpha)$ doesn't know that they are sharp fragments.

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So if $k : E^0_\beta \prec E^0_\alpha$, k(sharp fragment) can be anything.

Definition

We say that $k : E^0_\beta \prec E^0_\alpha$ is *sharp-friendly* if it maps sharp fragments to sharp fragments.

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Lemma

$k : E^0_\beta \prec E^0_\alpha$ is sharp-friendly, iff it's possible to extend it to $\hat{k} : L(E^0_\beta) \prec L(E^0_\alpha)$.

Define in $I = \{\beta < \Upsilon : \forall \gamma < \beta \ L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}$.

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Let $\beta \in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ, i.e. $(E^0_\gamma)^\sharp = (E^0_\beta)^\sharp_\gamma$.

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Let $\beta \in I$ such that $\mathrm{ot}(I_\beta) = \lambda$.

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Let $\beta\in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ, i.e. $(E^0_\gamma)^\sharp=(E^0_\beta)^\sharp_\gamma$.

Theorem

Let $\beta\in I$ such that $\mathrm{ot}(I_\beta)=\lambda$. Then β is totally non-proper.

By the Lemma above, we only have to find an α such that we know that there exists a proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$, By the Lemma above, we only have to find an α such that we know that there exists a proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$, and a sharp-friendly elementary embedding $k : E^0_\alpha \prec E^0_\alpha$ whose extension is not proper.

Define the game G_α in $L((E^0_\alpha)^\sharp)$:

$$\begin{array}{llllll} I & \langle k_0, \beta_0\rangle & & \langle k_1, \beta_1\rangle & & \langle k_2, \beta_2\rangle \\ & & & & & & \dots \\ II & & \eta_0 & & \eta_1 & \end{array}$$

with the following rules:

- $k_0 = \emptyset$;
- $k_{i+1} : E^0_{\beta_i} \prec E^0_{\beta_{i+1}}$ sharp-friendly;
- for every $\gamma < \beta_i$, $k_{i+1}(((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k_{i+1}(\gamma),n}$;
- $\beta_i, \eta_i < \alpha$;
- $\beta_{i+1} > \eta_i$;
- $k_i \subseteq k_{i+1}$ and $k_{i+1}(\beta_i) = \beta_{i+1}$;
- II wins if and only if I at a certain point can't play anymore.

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I.

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- $\alpha = \Theta^{E^0_\alpha} = \Theta^{(E^0_\alpha)^\sharp}$;
- $L(E^0_\alpha), L((E^0_\alpha)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$;
- α is regular in $L((E^0_\alpha)^\sharp)$.

Note that, since there exists $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$, $j \upharpoonright L((E^0_\alpha)^\sharp)$ is an elementary embedding, so in $L((E^0_\alpha)^\sharp)$ the first and second degree analogies hold.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

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Proof.

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Proof.

We fix $j : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$.
Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E^0_\alpha)^\sharp)$ from $V_{\lambda+1}$ to the set of all the k_n such that $\langle k_n, \beta_n\rangle$ is a legal move for I. $\square$

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So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$.

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Since C is definable and α is regular, C has ordertype α.
But then if I plays the κ_n-th element of C as β_n and $j \upharpoonright E^0_{\beta_{n-1}}$ as k_n, I wins, and that's a contradiction.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n\rangle) < \eta$. By the first claim C is a club.
Since C is definable and α is regular, C has ordertype α.
But then if I plays the κ_n-th element of C as β_n and $j \upharpoonright E^0_{\beta_{n-1}}$ as k_n, I wins, and that's a contradiction. $\square$

Since $\mathrm{cof}(\alpha) = \omega$ and for every $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$ $j \upharpoonright L(E^0_\alpha)$ is proper, then α is a partially non-proper ordinal.

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This proves that I_α is a club in α. Since $I_\alpha \in L((E^0_\alpha)^\sharp)$ and α is regular in $L((E^0_\alpha)^\sharp)$, we're done. $\square$

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- For every $j, k : L(E^0_\beta) \prec L(E^0_\beta)$ weakly proper if $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$.

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Some of these implications cannot be reversed.

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