Baire property and the Ellentuck-Prikry topology

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Inspiration (Woodin)

I0 is a large cardinal similar to AD.

Motivation

- Proving theorems that reinforce such statement
- Understanding the deep reasons behind such similarity

Definition (Woodin, 1980)

We say that I0(λ) holds iff there is an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ such that $j \upharpoonright V_{\lambda+1}$ is not the identity.

It is a large cardinal: if I0(λ) holds, then λ is a strong limit cardinal of cofinality ω, limit of cardinals that are n-huge for every $n \in \omega$.

These are some similiarities with AD:

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0(λ)
DC	DC$_\lambda$
Θ is regular	$\Theta^{L(V_{\lambda+1})}$ is regular
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds

Theorem (Laver)

Let $\langle \kappa_n : n \in \omega \rangle$ be a cofinal sequence in λ. For every $A \subseteq V_\lambda$:

- A is Σ^1_1-definable in $(V_\lambda, V_{\lambda+1})$ iff there is a tree $T \subseteq \prod_{n\in\omega} V_{\kappa_n} \times \prod_{n\in\omega} V_{\kappa_n}$ whose projection is A;
- A is Σ^1_2-definable in $(V_\lambda, V_{\lambda+1})$ iff there is a tree $T \subseteq \prod_{n\in\omega} V_{\kappa_n} \times \lambda^+$ whose projection is A.

Let us go a bit deeper.

We define a topology on $V_{\lambda+1}$: Since $V_{\lambda+1} = \mathcal{P}(V_\lambda)$, the basic open sets of the topology are, for any $\alpha < \lambda$ and $a \subseteq V_\alpha$,

$$O_{(a,\alpha)} = \{b \in V_{\lambda+1} : b \cap V_\alpha = a\}.$$

Theorem (Cramer, 2015)

Suppose I0(λ). Then for every $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$, either $|X| \leq \lambda$ or ${}^\omega\lambda$ can be continuously embedded inside X (${}^\omega\lambda$ with the bounded topology).

This is similar to AD: in fact, under AD every subset of the reals has the Perfect Set Property.

But the proof is completely different: Cramer uses heavily elementary embeddings (inverse limit reflection), while in the classical case involves games.

In recent work, with Motto Ros we clarified the similarity.

The classical case is:

- Large cardinals $\Rightarrow$ every set of reals in $L(\mathbb{R})$ is homogeneously Suslin
- Every homogeneously Suslin set is determined (so $L(\mathbb{R}) \models \mathrm{AD}$)
- Every determined set has the Perfect Set Property

But there is a shortcut for regularity properties:

- Infinite Woodin cardinals $\Rightarrow$ every set of reals in $L(\mathbb{R})$ is weakly homogeneously Suslin
- Every weakly homogeneously Suslin set has the Perfect Set Property

D.-Motto Ros

Let λ be a strong limit cardinal of cofinality ω, and let $\langle \kappa_n : n \in \omega \rangle$ be a increasing cofinal sequence in λ. Then the following spaces are isomorphic:

- ${}^{\lambda}2$, with the bounded topology;
- ${}^{\omega}\lambda$, with the bounded topology, and the discrete topology in every copy of λ;
- $\prod_{n\in\omega} \kappa_n$, with the bounded topology and the discrete topology in every κ_n;
- if $|V_\lambda| = \lambda$, $V_{\lambda+1}$, with the previously defined topology.

Moreover, they are λ-Polish, i.e., completely metrizable and with a dense subset of cardinality λ.

So, for example, we can rewrite Cramer's result as:

Theorem (Cramer, 2015)

Suppose I0(λ). Then $L({}^{\lambda}2) \vDash \forall X\, X \subseteq {}^{\lambda}2$ has the λ-PSP.

For any λ strong limit of cofinality ω, we defined *representable* subsets of ${}^{\omega}\lambda$, a generalization of weakly homogeneously Suslin sets.

D.-Motto Ros

Let λ strong limit of cofinality ω. Then every representable subset of ${}^{\omega}\lambda$ has the λ-PSP.

Cramer's analysis of I0 finalizes the similarity with AD:

Theorem (Cramer, to appear)

Suppose I0(λ). Then every $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$ is representable.

- Infinite Woodin cardinals $\Rightarrow$ every set of reals in $L(\mathbb{R})$ is weakly homogeneously Suslin
- Every weakly homogeneously Suslin set has the Perfect Set Property

- I0 $\Rightarrow$ every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ is representable
- Every representable set has the λ-Perfect Set Property

This approach is more scalable, as it works even in λ-Polish spaces such that λ does not satisfy I0:

D.-Motto Ros

Suppose I0(λ). Then it is consistent that there is κ strong limit of cofinality ω such that all the subsets of ${}^{\omega}\kappa$ in $L(V_{\kappa+1})$ have the κ-PSP, and $\neg$I0(κ).

The next step would be to analyze the Baire Property.

The most natural thing is to define nowhere dense sets as usual, λ-meager sets as λ-union of nowhere dense sets and λ-comeager sets as complement of λ-meager sets.

The most natural thing is to define nowhere dense sets as usual, λ-meagre sets as λ-union of nowhere dense sets and λ-comeagre sets as complement of λ-meagre sets.

Let $f : \omega \to \omega$.

Then $D_f = \prod_{n\in\omega} \kappa_{f(n)}$ is nowhere dense in ${}^{\omega}\lambda$.

But ${}^{\omega}\lambda = \bigcup_{f\in{}^{\omega}\omega} D_f$, therefore the whole space is λ-meagre (in fact, it is $\mathfrak{c}$-meagre), and the Baire property in this setting is just nonsense.

Or is it?

From now on, we work with λ strong limit of cofinality ω, $\langle \kappa_n : n \in \omega \rangle$ a strictly increasing cofinal sequence of *measurable* cardinals in λ.

The space we work in is $\prod_{n\in\omega} \kappa_n$.

Idea

- Baire category is closely connected to Cohen forcing
- The space ${}^{\kappa}2$, with κ regular, is κ-Baire (i.e., every nonempty open set is not κ-meagre) because Cohen forcing on κ is $< \kappa$-distributive
- "Cohen" forcing on λ singular is not $< \lambda$-distributive, and this is why ${}^{\lambda}2$ is not λ-Baire
- But there are other forcings on λ that are $< \lambda$-distributive, like Prikry forcing
- We can try to define Baire category via Prikry forcing instead of Cohen forcing.

Definition

Let λ be strong limit of cofinality ω, $\langle \kappa_n : n \in \omega \rangle$ a strictly increasing cofinal sequence of *measurable* cardinals in λ, and fix μ_n a measure for each κ_n. The *Prikry forcing* $\mathbb{P}_{\vec{\mu}}$ on λ respect to $\vec{\mu}$ has conditions of the form $\langle \alpha_1, \dots, \alpha_n, A_{n+1}, A_{n+2} \dots \rangle$, where $\alpha_i \in \kappa_i$ and $A_i \in \mu_i$.

$\langle \alpha_1, \dots, \alpha_n \rangle$ is the *stem* of the condition.

$\langle \beta_1, \dots, \beta_m, B_{m+1}, B_{m+2} \dots \rangle \leq \langle \alpha_1, \dots, \alpha_n, A_{n+1}, A_{n+2} \dots \rangle$ iff $m \geq n$ and

- for $i \leq n$ $\beta_i = \alpha_i$
- for $n < i \leq m$ $\beta_i \in A_i$
- for $i > m$ $B_i \subseteq A_i$.

$p \leq^* q$ if $p \leq q$ and they have the same stem.

Definition

The *Ellentuck-Prikry $\vec{\mu}$-topology* (in short EP-topology) on $\prod_{n\in\omega}\kappa_n$ is the topology generated by the family $\{O_p : p \in \mathbb{P}_{\vec{\mu}}\}$, where if $p = \langle \alpha_1, \ldots, \alpha_n, A_{n+1}, A_{n+2} \ldots \rangle$, then

$$O_p = \{x \in \prod_{n\in\omega}\kappa_n : \forall i \leq n\ x(i) = \alpha_i,\ \forall i > n\ x(i) \in A_i\}.$$

The EP-topology is a refinement of the bounded topology: if a set is open in the bounded topology, it is open also in the EP-topology, but not viceversa (in fact, many open sets in the EP-topology are nowhere dense in the bounded topology).

There is a connection between the concepts of "open" and "dense" relative to the forcing and relative to the topology:

$\mathbb{P}_{\vec{\mu}}$ (forcing)		$\prod_{n\in\omega}\kappa_n$ (EP-topology)
O open	$\rightarrow$	${}^{I}O = \{x \in \prod_{n\in\omega}\kappa_n : \exists p \in \mathbb{P}_{\vec{\mu}}\ x \in O_p\}$ open
${}_{I}U = \{p \in \mathbb{P}_{\vec{\mu}} : O_p \subseteq U\}$ open	$\leftarrow$	U open
O open dense	$\rightarrow$	${}^{I}O$ open dense
${}_{I}U$ open dense	$\leftarrow$	U open dense

${}^{I}({}_{I}U) = U$, but not viceversa.

Definition

Let X be a topological space.

- a set $A \subseteq X$ is λ-meagre iff it is the λ-union of nowhere dense sets
- a set $A \subseteq X$ is λ-comeagre iff it is the complement of a λ-meagre set
- a set $A \subseteq X$ has the λ-Baire property iff there is an open set U such that $A \triangle U$ is λ-meagre
- X is a λ-Baire space iff every nonempty open set in X is not λ-meagre, i.e., the intersection of λ-many open dense sets is dense.

The key to prove that the space $\prod_{n\in\omega}\kappa_n$ is λ-Baire resides in this combinatorial property of Prikry forcing:

Strong Prikry condition

Let $D\subseteq\mathbb{P}_{\vec{\mu}}$ be open dense. Then for every $p\in\mathbb{P}_{\vec{\mu}}$ there are $p'\leq^* p$ and $n\in\omega$ such that for every $q\leq p'$ with stem of length at least n, $q\in D$.

Topologically: Let $D\subseteq\prod_{n\in\omega}\kappa_n$ be open dense. Then for every $p\in\mathbb{P}_{\vec{\mu}}$ there is a $p'\leq^* p$ such that $O_{p'}\subseteq D$.

Coupled with the fact that if $p\in\mathbb{P}_{\vec{\mu}}$ has stem of length n, then the intersection of $<\kappa_n$-many $\leq^*$-extensions of p is still in $\mathbb{P}_{\vec{\mu}}$, we have:

Proposition (D.-Shi)

The space $\prod_{n\in\omega}\kappa_n$ with the EP-topology is λ-Baire.

(Generalized) Mycielski Theorem (D.-Shi)

In $\prod_{n\in\omega}\kappa_n$ with the EP-topology every λ-comeagre set contains a λ-perfect set.

Conjecture

All the results in classical descriptive set theory that depend only on Baire category can be generalized to this setting.

Test case:

Kuratowski-Ulam Theorem

Let X, Y be second-countable spaces, and $A \subseteq X \times Y$ with the Baire property. Then A is meagre iff $\{x \in X : \{y \in Y : (x, y) \in A\}$ is meagre in $Y\}$ is λ-comeagre in X.

The key lemma to prove the Kuratowski-Ulam Theorem is the following:

Lemma

Let X, Y be second-countable spaces. Then if $A \subseteq X \times Y$ is open dense, $\{x \in X : \{y \in Y : (x,y) \in A\}$ is open dense in $Y\}$ is comeagre in X.

Sketch of proof.

For any $x \in X$, let $A_x = \{y \in Y : (x,y) \in A\}$, and let $\langle V_n : n \in \omega \rangle$ be a countable base. Then A_x is open, and A_x is dense iff $\forall n \in \omega\ A_x \cap V_n \neq \emptyset$. But then $\{x \in X : A_x$ is open dense$\} = \bigcap_{n \in \omega} \{x \in X : A_x \cap V_n \neq \emptyset\}$, a countable intersection of open dense sets, so comeagre. □

We can see why this proof cannot be generalized:

$\prod_{n\in\omega}\kappa_n$, with the EP-topology, has a base of cardinality 2^λ, so the set we want to be λ-comeagre is actually the intersection of 2^λ-many open dense sets, not λ-many.

The key is still the Strong Prikry condition, in this more general definition:

Strong Prikry condition

Let $A\subseteq\mathbb{P}_{\vec{\mu}}$ be an *open* set. Then for any $p\in\mathbb{P}_{\vec{\mu}}$, there is a $p^A\leq^* p$ such that if there is a $q\leq p^A$ with $q\in A$ with stem of length n, then for every $q\leq p^A$ with stem at least n, $q\in A$.

Topologically: Let $U\subseteq\prod_{n\in\omega}\kappa_n$ be an open set. Then for any $p\in\mathbb{P}_{\vec{\mu}}$, there is a $p^A\leq^* p$ such that either $O_{p^A}\subseteq A$, or $O_{p^A}\cap A=\emptyset$.

For any $s \in \bigcup_{m\in\omega}\prod_{n\leq m}\kappa_n$, let $1_s = s^\frown\langle\kappa_{m+1},\kappa_{m+2},\dots\rangle$.

Let $A \subseteq \prod_{n\in\omega}\kappa_n$ be open.

For any $s \in \bigcup_{m\in\omega}\prod_{n\leq m}\kappa_n$, fix 1_s^A as in the Strong Prikry condition.

Then A is dense iff for all $s \in \bigcup_{m\in\omega}\prod_{n\leq m}\kappa_n$ there is a $q \leq 1_s^A$ such that $q \in A$.

So to test that A open is dense, we do not need to test it for all the basic open sets, just for a subfamily of them of size λ!

(Generalized) Kuratowski-Ulam Theorem (D.)

Let $A \subseteq \prod_{n\in\omega}\kappa_n \times \prod_{n\in\omega}\kappa_n$ be with the λ-Baire property. Then A is λ-meagre iff $\{x \in X : A_x \text{ is } \lambda\text{-meagre}\}$ is λ-comeagre.

ERRATA CORRIGE

The last theorem is vacuously true. In fact:

Proposition

The $\prod_{n\in\omega}\kappa_n \times \prod_{n\in\omega}\kappa_n$, with the product topology of the EP-topology, is not $\mathfrak{c}$-Baire.

Proof.

For any $c\in {}^\omega 2$, consider $D_c=\{(x,y)\in\prod_{n\in\omega}\kappa_n\times\prod_{n\in\omega}\kappa_n : \exists n\in\omega(c(n)=0\land x(n)=y(n))\lor(c(n)=1\land x(n)\neq y(n)\}$. Then D_c is open dense and $\bigcap_{c\in{}^\omega 2}D_c=\emptyset$. □

The "right" product is the following:

$\prod_{n\in\omega}\kappa_n \ltimes \prod_{n\in\omega}\kappa_n=\{(x,y)\in\prod_{n\in\omega}\kappa_n\times\prod_{n\in\omega}\kappa_n : \exists n\in\omega\forall m>n\ x(m)<y(m)\}$

Then $\prod_{n\in\omega} \kappa_n \ltimes \prod_{n\in\omega} \kappa_n$, with the topology that is the restricted product of the EP-topologies, is λ-Baire and

(Generalized) Kuratowski-Ulam Theorem (D.)

Let $A \subseteq \prod_{n\in\omega} \kappa_n \ltimes \prod_{n\in\omega} \kappa_n$ be with the λ-Baire property. Then A is λ-meagre iff $\{x \in X : A_x \text{ is } \lambda\text{-meagre}\}$ is λ-comeagre.

Question

Are measurable cardinals necessary?

Question

What other forcings there are on λ that can generate interesting concepts?

Question

What about λ-universally Baire sets?

Thanks for watching