The Fine Structure of Game Lambda Models

Pietro Di Gianantonio and Gianluca Franco

Dipartimento di Matematica e Informatica
Università di Udine
Via delle Scienze, 206 - 33100 Udine - Italy
Tel. +39 0432 558469
Fax +39 0432 558499
email:pietro,gfranco@dimi.uniud.it

Abstract. We study models of the untyped lambda calculus in the setting of game semantics. In particular, we show that, in the category of games $\mathcal{G}$, introduced by Abramsky, Jagadeesan and Malacaria, all λ-models can be partitioned in three disjoint classes, and each model in a class induces the same theory (*i.e.* the set of equations between terms), that are the theory $\mathcal{H}^*$, the theory which identifies two terms iff they have the same Böhm tree and the theory which identifies all the terms which have the same Lévy-Longo tree.

Key Words: Games Semantics, Lambda Calculus, Böhm-trees

Introduction

In this paper we explore the methodology for giving denotational semantics based on games, recently introduced by Abramsky, Jagadeesan, Malacaria and Hyland, Ong (see [AJM96,HO00]). We use game semantics to build models of the untyped λ-calculus, focusing on which λ-theories can be modeled. λ-theories are congruences over λ-terms, which extend pure β-conversion. Their interest lies in the fact that they correspond to the possible *operational* (*observational*) semantics of the λ-calculus. Although researchers have mainly focused on only three such operational semantics, namely those given by head reduction, head lazy reduction or call-by-value reduction, the class of λ-theories is, in effect, unfathomly rich, see *e.g.* [Bar84,HR92,HL95,Ber97] for interesting examples of this complexity. Brute force, purely syntactical techniques are usually extremely difficult to use in the study of λ-theories. Therefore, since the seminal work of Dana Scott on D_∞ in 1969 [Sco72], semantical tools have been extensively investigated.

Games semantics has been extremely successful in modeling sequential languages, and it has also been used to define a "fully abstract" model for the lazy λ-calculus [AM95].

In this paper we complete the work initiated in [GFH99] and give a complete characterization of the theories induced by general games models. In [GFH99] we considered just extensional models, *i.e.* models in which the η-rule holds. In order to obtain our new results new proof techniques have been introduced.

In particular we consider the class $\mathcal{D}$ of models of the untyped λ-calculus built in the Cartesian closed category $K_!(\mathcal{G})$ of games and history-free strategies; however we do not considered the category of extensionally collapsed games $\mathcal{E}$ [AJM96].

We show that the theory induced by each λ-model in $K_!(\mathcal{G})$ is either: the theory $\mathcal{H}^*$ (the maximal sensible theory), the theory $\mathfrak{B}$ which equates two terms if and only if that have the same Böhm tree or the theory $\mathcal{L}$ which equates two terms if and only if they have the same Lévy-Longo tree.

This result suggests that there exists a strong connection between a *strategy* which interprets a term in the game semantics setting and the tree form of the term. The current notion of game appears indeed to carry a very strong bias towards *head reduction*. A new notion of game seems to be necessary to model different λ-theories. This appears to be rather problematic, since we feel that "head reduction" is intrinsic to games for which we can observe only *interactions with the environment*.

This hypothesis seems to be confirmed also by the recent results presented in the related works [KNO00,KNO99], in the slight different game semantics paradigm of Hyland and Ong [HO00]. There, two particular games λ-models are built and it is proved, using techniques quite different from ours, that the two models induce respectively the theories $\mathcal{H}^*$ and $\mathfrak{B}$.

The present paper is organized as follows. In section 1, we introduce the categories of games that we shall utilize, namely $\mathcal{G}$, $K_!(\mathcal{G})$ and their extensions $\mathcal{G}^s$, $K_!(\mathcal{G}^s)$. In Section 2 we introduce the main tool for the study of the fine structure of the models built in the categories of Section 1, that is the *approximations strategies* whose intended meaning is to give a finite approximation of the interpretation of a λ-term. Section 3 is devoted to the study of the models previously introduced and to the proof of the main theorem of this work, *i.e.* the characterization of all the λ-theories induced by the models of the untyped λ-calculus in the category $K_!(\mathcal{G})$.

We assume the reader familiar with the basic notions and definitions of λ-calculus, see e.g. [Bar84]. This paper is self-contained as far as the theory of games, however the reader can refer to [AJM96,HO00] for more details on this topic.

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1 Categories of games.

This section is devoted to the introduction of the basic notions of games semantics that will be used in the rest of the paper. Essentially, we will make use of the category $\mathcal{G}$ of games and history-free strategies [AJ94,AJM96,AM95] and most definitions are standard, with the only exception of the definition of the category

$\mathcal{G}^s$ of games and history-sensitive strategies that is new. This new category is a straightforward extension (super-category) of $\mathcal{G}$ and it has been introduced for technical reasons.

Definition 1 (Games). *A* game *has two participants: the* Player *and the* Opponent. *A game A is a quadruple $(M_A, \lambda_A, P_A, \approx_A)$ where*

- *M_A is the set of* moves *of the game.*
- *$\lambda_A : M_A \to \{O, P\} \times \{Q, A\}$ is the* labeling *function: it tells us if a move is taken by the Opponent or by the Player, and if it is a Question or an Answer. We can decompose λ_A into $\lambda_A^{OP} : M_A \to \{O, P\}$ and $\lambda_A^{QA} : M_A \to \{Q, A\}$ and put $\lambda_A = \langle \lambda_A^{OP}, \lambda_A^{QA} \rangle$. We denote by $^{-}$ the function which exchanges Player and Opponent, i.e. $\overline{O} = P$ and $\overline{P} = O$. We also denote with $\overline{\lambda_A^{OP}}$ the function defined by $\overline{\lambda_A^{OP}}(a) = \overline{\lambda_A^{OP}(a)}$. Finally, we denote with $\overline{\lambda_A}$ the function $\langle \overline{\lambda_A^{OP}}, \lambda_A^{QA} \rangle$.*
- *P_A is a non-empty and prefix-closed subset of the set $M_A^{\circledast}$ (which will be written as $P_A \subseteq^{nepref} M_A^{\circledast}$), where $M_A^{\circledast}$ is the set of all sequences of moves which satisfy the following conditions:*
  - $s = at \Rightarrow \lambda_A(a) = OQ$
  - $(\forall i : 1 \leq i \leq |s|)[\lambda_A^{OP}(s_{i+1}) = \overline{\lambda_A^{OP}(s_i)}]$
  - $(\forall\ t \sqsubseteq s)[|t \restriction M_A^A| \leq |t \restriction M_A^Q|]$

  where M_A^A and M_A^Q denote the subsets of game moves labeled respectively as Answers and as Questions, $s \restriction M$ denotes the set of moves of M which appear in s and $\sqsubseteq$ is the substring relation. P_A is called the set of positions *of the game A.*
- *$\approx_A$ is an equivalence relation on P_A which satisfies the following properties:*
  - $s \approx_A s' \Rightarrow |s| = |s'|$
  - $sa \approx_A s'a' \Rightarrow s \approx_A s'$
  - $s \approx_A s' \ \wedge\ sa \in P_A \Rightarrow (\exists a')[sa \approx_A s'a']$

In the above s, s', t and t' range over sequences of moves, while a, a', b and b' range over moves. The empty sequence is written ϵ.

Definition 2 (Tensor product).
Given games A and B the tensor product $A \otimes B$ is the game defined as follows:

- $M_{A \otimes B} = M_A + M_B$
- $\lambda_{A \otimes B} = [\lambda_A, \lambda_B]$
- *$P_{A \otimes B} \subseteq M_{A \otimes B}^{\circledast}$ is the set of positions, s, which satisfy the following conditions:*
  1. *the projections on each component (written as $s \restriction A$ or $s \restriction B$) are positions for the games A and B respectively;*
  2. *every answer in s must be in the same component game as the corresponding question.*
- $s \approx_{A \otimes B} s' \iff s \restriction A \approx_A s' \restriction A, s \restriction B \approx_B s' \restriction B, (\forall i)[s_i \in M_A \Leftrightarrow s'_i \in M_A]$

Here + denotes disjoint union of sets, that is $A+B = \{in_l(a) \mid a \in A\} \cup \{in_r(b) \mid b \in B\}$, and $[-,-]$ is the usual (unique) decomposition of a function defined on disjoint unions.

It is easy to see that in such a game only the Opponent can switch component.

Definition 3 (Unit). *The unit element for the tensor product is given by the empty game $I = (\varnothing, \varnothing, \{\epsilon\}, \{(\epsilon, \epsilon)\})$.*

Definition 4 (Linear implication). *Given games A and B the compound game $A \multimap B$ is defined as follows:*

- $M_{A \multimap B} = M_A + M_B$
- $\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$
- $P_{A \otimes B} \subseteq M^{\circledast}_{A \otimes B}$ *is the set of positions, s, which satisfy the following conditions:*
  1. *the projections on each component are positions for the games A and B respectively;*
  2. *every answer in s must be in the same component game as the corresponding question.*
- $s \approx_{A \multimap B} s' \iff s \restriction A \approx_A s' \restriction A, s \restriction B \approx_B s' \restriction B, (\forall i)[s_i \in M_A \Leftrightarrow s'_i \in M_A]$

It is easy to see that in such a game only the Player can switch component.

Definition 5 (Exponential). *Given a game A the game $!A$ is defined by:*

- $M_{!A} = \omega \times M_A = \sum_{i \in \omega} M_A$
- $\lambda_{!A}(\langle i, a \rangle) = \lambda_A(a)$
- $P_{!A} \subseteq M^{\circledast}_{!A}$ *is the set of positions, s, which satisfy the following conditions:*
  1. $(\forall i \in \omega)[s \restriction \langle i, A \rangle \in P_{\langle i, A \rangle}]$;
  2. *every answer in s is in the same index as the corresponding question.*
- $s \approx_{!A} s' \iff \exists$ *a permutation of indexes $\alpha \in S(\omega)$ such that:*
  - $\pi_1^*(s) = \alpha^*(\pi_1^*(s'))$
  - $(\forall i \in \omega)[\pi_2^*(s \restriction \alpha(i)) \approx \pi_2^*(s \restriction i)]$

  where π_1 and π_2 are the projections of $\omega \times M_A$, π_1^* and π_2^* are the (unique) extensions of π_1 and π_2 to sequences of moves and $s \restriction i$ is an abbreviation of $s \restriction \langle i, A \rangle$.

Definition 6 (Strategies). *A strategy for the Player in a game A is a non-empty set $\sigma \subseteq P_A^{even}$ of positions of even length such that $\overline{\sigma} = \sigma \cup dom(\sigma)$ is prefix-closed, where $dom(\sigma) = \{t \in P_A^{odd} \mid (\exists! a)[ta \in \sigma]\}$, and P_A^{odd} and P_A^{even} denote the sets of positions of odd and even length respectively.*

A strategy can be seen as a set of rules which tells (in some position) the Player which move to take after the last move by the Opponent.

The equivalence relation on positions $\approx_A$ can be extended to strategies in the following way.

Definition 7. *Let σ, τ be strategies, $\sigma \approx \tau$ if and only if*

1. $sab \in \sigma, s'a'b' \in \tau, sa \approx_A s'a' \Rightarrow sab \approx_A s'a'b'$
2. $s \in \sigma, s' \in \tau, sa \approx_A s'a' \Rightarrow (\exists b)[sab \in \sigma]$ *iff* $(\exists b')[s'a'b' \in \tau]$

Such an extension is not in general an equivalence relation since it might lack reflexivity. If σ is a strategy for a game A such that $\sigma \approx \sigma$, we write $\sigma : A$ and denote with $[\sigma]$ the equivalence class containing σ.

Definition 8 (History-free strategies). *A strategy σ for a game A is history-free if it satisfies the following properties:*

1. $sab, tac \in \sigma \Rightarrow b = c$
2. $sab, t \in \sigma, ta \in P_A \Rightarrow tab \in \sigma$

Together with the "standard category" $\mathcal{G}$ of games and history-free strategies, we need to introduce also the category $\mathcal{G}^s$ having as morphisms all the strategies. We call these morphisms history-sensitive strategies. Notice that almost all the definitions in the two categories coincide.

Definition 9 (The category of games $\mathcal{G}$ and $\mathcal{G}^s$). *The category $\mathcal{G}$ has as objects games and as morphisms, between games A and B, the equivalence classes, w.r.t. relation $\approx_{A \multimap B}$, of the history-free strategies $\sigma : A \multimap B$.*

The category $\mathcal{G}^s$ has as objects games and as morphisms, between games A and B, the equivalence classes, w.r.t. relation $\approx_{A \multimap B}$, of strategies $\sigma : A \multimap B$.

The identity, in both categories, for each game A, is given by the (equivalence class) of the copy-cat *strategy $id_A = \{s \in P_{A' \multimap A''} \mid s \upharpoonright A' = s \upharpoonright A''\}$ where the superscripts are introduced to distinguish between the two different occurrences of the game A.*

Composition is given by the extension on equivalence classes of the following composition of strategies. Given strategies $\sigma : A \multimap B$ and $\tau : B \multimap C$, $\tau \circ \sigma : A \multimap C$ is defined by

$$\tau \circ \sigma = \{s \upharpoonright (A, C) \mid s \in (M_A + M_B + M_C)^* \ \wedge \ s \upharpoonright (A, B) \in \overline{\sigma}, s \upharpoonright (B, C) \in \overline{\tau}\}^{even}$$

It is not difficult to check that the above definitions are well posed, that $\mathcal{G}$ is a faithful sub-category of $\mathcal{G}^s$ and that the constructions introduced in Definitions 2, 4 and 5 can be made functorial in both categories, with coincident definitions.

The introduction of $\mathcal{G}^s$ will be motivated in the next sections: it essentially allows a more flexible notion of "approximation" for strategies.

Notice that there is a natural isomorphism in the category of sets between $(M_A + M_B) + M_C$ and $M_A + (M_B + M_C)$ which induces a natural transformation

$$\Lambda^l_{A,B,C} : \hom(A \otimes B, C) \to \hom(A, B \multimap C)$$

in $\mathcal{G}$ and $\mathcal{G}^s$, that is the categories $\mathcal{G}$ and $\mathcal{G}^s$ are monoidal close. If we define for each pair of games B and C of $\mathcal{G}$ the strategy

$$ev^l_{B,C} = \{s \in P_{A' \multimap B' \otimes A'' \multimap B''} \mid s \upharpoonright A' = s \upharpoonright A'' \ \&\ s \upharpoonright B' = s \upharpoonright B''\}$$

we have, for each strategy $\sigma : A \otimes B \multimap C$ the identity $[\sigma] = [ev^l_{B,C}] \circ (\Lambda^l_{A,B,C}([\sigma]) \otimes [id_B])$. However $\mathcal{G}$ and $\mathcal{G}^s$ are not Cartesian.

Definition 10 (The Cartesian closed categories of games $K_!(\mathcal{G})$ and $K_!(\mathcal{G}^s)$). *The categories $K_!(\mathcal{G})$ and $K_!(\mathcal{G}^s)$ are the categories obtained by taking the co-Kleisli category over $\mathcal{G}$ and $\mathcal{G}^s$ respectively, over the co-monad $(!, \mathtt{der}, \delta)$ [AJM96], where for each game A the (history-free) strategies $\mathtt{der}_A : !A \multimap A$ and $\delta_A : !A \multimap !!A$ are defined as follows:*

- $\mathtt{der}_A = \{s \in P_{!A \multimap A} \mid s \upharpoonright \langle 0, A\rangle = s \upharpoonright A\}$
- $\delta_A = \{s \in P_{!A \multimap\, !!A} \mid s \upharpoonright \langle p(i,j), A\rangle = s \upharpoonright \langle j, \langle i, A\rangle\rangle\}$ *where* $p : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *is a pairing function*

It is worthwhile to observe that $(!, [\mathtt{der}], [\delta])$ will not be a co-monad if we do not consider strategies up to equivalence. By the above definition the categories $K_!(\mathcal{G})$ and $K_!(\mathcal{G}^s)$ have as objects games and as morphisms between games A and B the equivalence class of the strategies for the game $!A \multimap B$. Moreover, $K_!(\mathcal{G})$ and $K_!(\mathcal{G}^s)$ are Cartesian.

Definition 11 (Cartesian product). *The Cartesian product $A \times B$ of two games A and B is defined by:*

- $M_{A \times B} = M_A + M_B$
- $\lambda_{A \times B} = [\lambda_A, \lambda_B]$
- $P_{A \times B} = P_A + P_B$
- $\approx_{A \times B} = \approx_A + \approx_B$

The projection morphism $\pi_A^{A,B} : A \times B \to A$ is defined by

$$\pi_A^{A,B} = [\{s \in P_{A' \times B \multimap A''} \mid s \upharpoonright A' = s \upharpoonright A''\} \circ \mathtt{der}_{A \times B}]$$

From the isomorphisms $!(A \times B) \cong !A \otimes !B$ and $!I \cong I$ it follows easily that $K_!(\mathcal{G})$ and $K_!(\mathcal{G}^s)$ are Cartesian closed [AJM96].

Definition 12 (Exponent). *The exponent game $A \Rightarrow B$ is the game $!A \multimap B$. The natural transformation $\Lambda_{A,B,C} : hom(A \times B, C) \to hom(A, B \Rightarrow C)$ is the strategy $\Lambda^l_{!A,!B,C}$, and $ev_{B,C} = ev^l_{!B,C} \circ (\mathtt{der}_{!B \multimap C} \times id_{!B})$.*

Definition 13. *Given a game A and strategies $\sigma : A$ and $\tau : A$ (hence such that $\sigma \approx \sigma$ and $\tau \approx \tau$) we define*

$$\sigma \sqsubseteq \tau \Longleftrightarrow \forall s \in \sigma\ \exists t \in \tau\ .\ s \approx t$$

and then $[\sigma] \sqsubseteq [\tau] \Longleftrightarrow \sigma \sqsubseteq \tau$.

The above definition does not coincide with the standard one, but can be easily proved equivalent. It induces a partial order on equivalence classes of strategies.

2 Approximation strategies.

Let us now introduce the general concept of *approximation strategy*, which can be seen as a finite approximation of a strategy. It will be used to prove that the interpretation of a term is the least upper bound of the interpretations of its "approximate normal forms".

Definition 14. *1. Let D be a game. We indicate with D^n the sub-game of D ($D^n \trianglelefteq D$) in which $P_{D^n} = \{s \in P_D \mid |s| \leq n\}$.*
2. Let B be a sub-game of A and let σ be a strategy for the game A. We write $\sigma|B$ for the strategy $\{s \in \sigma \mid s \in P_B\}$.
3. Let $\sigma : A \multimap B$ be a strategy. We indicate with σ^n the history-sensitive strategy $\sigma|A \multimap B^n$ and with $[\sigma]^n$ the equivalence class $[\sigma^n]$.

Observe that if $\sigma \approx \tau$ then $\sigma^n \approx \tau^n$, since equivalent positions have the same length. Thus we can write $[\sigma]^n$ with no ambiguity.

In general the strategy σ^n can be history-sensitive also if the strategy σ is history-free. This is because σ^n can reply to a move a of the Opponent in some position and does not reply in some others. In order to accommodate and freely use the strategies σ^n we introduced the category $\mathcal{G}^s$ of games and history-sensitive strategies.

The strategies σ^n can be seen as a finite approximation of the strategy σ, and they will be use to prove an approximation theorem along the same line of the works [HR92,Hyl76,Wad78]. In these works the approximation of a semantical point is obtained through a series of projection functions. We use a different approach because, in the context of games, it is simpler and more direct.

We need to state a series of properties enjoyed by the approximation strategies. The basic ones are the following:

Proposition 1. *For each pair of games A and B and strategy $\sigma : A \Rightarrow B$, the following properties hold:*

1. $\sigma^0 = \{\epsilon\}$
2. $\sigma^n \subseteq \sigma^{n+1}$
3. $\bigcup_{n\in\omega}\{\sigma^n\} = \sigma$
4. $(\sigma^n)^m = \sigma^{min\{m,n\}}$

Proof. *The property 1 follows from the fact that the first move in the game $A \Rightarrow B$ has to be in B. The other proofs are immediate.* □

Properties concerning function spaces and approximations are the following:

Lemma 1. *For each pair of games A and B we have:*

1. $(A \multimap B)^{n+1} \trianglelefteq A^n \multimap B^{n+1}$
2. $ev^l_{A,B}|(A^n \multimap B^m) \otimes A \multimap B = ev^l_{A,B}|(A \multimap B) \otimes A^n \multimap B^m$

Proof. 1. *Let $s \in P_{(A \multimap B)^{n+1}}$ be a position. We obviously have that $|s \upharpoonright B| \leq n+1$, and since the first move of s has to be made in B we have that $|s \upharpoonright A| \leq n$.*
2. $ev^l_{A,B}|(A^n \multimap B^m) \otimes A \multimap B =$
$\{s \in P_{(A_1^n \multimap B_1^m) \otimes A_2 \multimap B_2} \mid s \upharpoonright A_1^n = s \upharpoonright A_2 \ \& \ s \upharpoonright B_1^m = s \upharpoonright B_2\} =$
$\{s \in P_{(A_1 \multimap B_1) \otimes A_2^n \multimap B_2^m} \mid s \upharpoonright A_1 = s \upharpoonright A_2^n \ \& \ s \upharpoonright B_1 = s \upharpoonright B_2^m\} =$
$ev^l_{A,B}|(A \multimap B) \otimes A^n \multimap B^m$.

□

Finally we need to establish some properties concerning approximation and retracts. In a generic category an object B is a *retract* of an object A if there exists a pair of morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g = id_B$. We write $(B \lhd A, f, g)$ to indicate that B is a retract of A via f and g.

Lemma 2. *For each retract object $(B \lhd A, [\varphi], [\psi])$ in the category $\mathcal{G}$, $\varphi : A \multimap B$ does not contain any position sab with $a, b \in M_B$.*

Proof. By contradiction. Suppose there exists a position $sab \in \varphi$ with $a, b \in M_B$. Let $s \upharpoonright B = b_1 \ldots b_n$. The position $b_1 b_1' b_2' b_2 \ldots b_n' b_n \in P_{B' \multimap B}$ belongs to the strategy id_B, therefore there exists an equivalent position s' in the strategy $\varphi \circ \psi \approx id_B$. It follows that $s'ab$ belongs to $\varphi \circ \psi$, and this is in contradiction with the fact that $\varphi \circ \psi \approx id_B$, since the sequence $s'ab$ cannot be equivalent to a copy-cat sequence of moves. □

Lemma 3. *For each retract $(B \lhd A, [\varphi], [\psi])$ in $\mathcal{G}$, for each strategy $\sigma : C \multimap A$ and $n \in \omega$ we have:*

$$\varphi \circ \sigma^n \subseteq (\varphi \circ \sigma)^n$$

Proof. By the previous lemma, for any position $s \in \varphi : A \multimap B$, we have $|s \upharpoonright B| \leq |s \upharpoonright A|$, and using the definition of composition one readily proof the thesis. □

We remind here the definition of categorical λ-model. A categorical λ-model is a reflexive object in a Cartesian closed category, that is a retract $(D \Rightarrow D \lhd D, f, g)$, between an object and its exponent. We write $\langle D, (f, g)\rangle$ is indicate that D is a reflexive object via morphisms f and g. Given a reflexive object $\langle D, (f, g)\rangle$ and a generic object A, one can easily obtain a λ-algebra $\langle \hom(A, D), \cdot \rangle$ by defining the operation of application $\cdot$ as:

$$x \cdot y \stackrel{def}{=} ev_{D,D} \circ \langle (f \circ x), y \rangle$$

Finally we can state the fundamental properties satisfied by approximations in each game λ-model.

Proposition 2. *Let A be a game, $\langle D, ([\varphi], [\psi])\rangle$ be a reflexive object in the Cartesian closed category of games $K_!(\mathcal{G})$ and $\sigma, \tau : A \Rightarrow D$ be two strategies. Then we have:*

1. $\sigma^0 \cdot \tau = \epsilon_{A \Rightarrow D}$

2. $\sigma^{n+1} \cdot \tau \sqsubseteq (\sigma \cdot \tau^n)^{n+1}$.

Proof. *1. The following chain of relations holds:*
$\sigma^0 \cdot \tau = ev_{D,D} \circ \langle(\varphi \circ \sigma^0), \tau\rangle \subseteq ev_{D,D} \circ \langle(\varphi \circ \sigma)^0, \tau\rangle = ev_{D,D} \circ \langle\epsilon_{A \Rightarrow (D \Rightarrow D)}, \tau\rangle$
which, from the definition of $ev_{D,D}$*, is equal to* $\epsilon_{A \Rightarrow D}$.

2. The following chains of relations holds:
$\sigma^{n+1} \cdot \tau = ev_{D,D} \circ \langle(\varphi \circ \sigma^{n+1}), \tau\rangle \subseteq$
$ev_{D,D} \circ \langle(\varphi \circ \sigma)^{n+1}, \tau\rangle =$
$(ev_{D,D}|(D \Rightarrow D)^{n+1} \times\ D \Rightarrow D) \circ \langle(\varphi \circ \sigma), \tau\rangle \subseteq$
$(ev_{D,D}|(D^n \Rightarrow D^{n+1}) \times\ D \Rightarrow D) \circ \langle(\varphi \circ \sigma), \tau\rangle =$
$(ev_{D,D}|(D \Rightarrow D) \times\ D^n \Rightarrow D^{n+1}) \circ \langle(\varphi \circ \sigma), \tau\rangle =$
$(ev_{D,D} \circ \langle(\varphi \circ \sigma), \tau^n\rangle)^{n+1} = (\sigma \cdot \tau^n)^{n+1}$.

□

3 The fine structure of the game models

In this section the study of the λ-theory (*i.e.* the set of equations between λ-terms) supported by a λ-model, for models built in $K_!(\mathcal{G})$ is carried out. The theory induced by a model is also known as its *fine structure*. The equations on terms are described by means of the equality of some tree of the terms. The trees we consider are the Lévy-Longo trees [Lév75,Lon83] and the Böhm trees [Bar84,Hyl76]. We remind briefly the definitions.

Definition 15. *Let* $\Sigma^1 = \{\lambda x_1 \dots x_n.\bot \mid n \in \omega\} \cup \{T\} \cup \{\lambda x_1 \dots x_n.y \mid n \in \omega\}$, *let* $\Sigma^2 = \{\bot\} \cup \{\lambda x_1 \dots x_n.y \mid n \in \omega\}$, *let* $x_1, \dots x_n, y \in Var$ *and let* $M \in \Lambda$ *be a term.*

1. *The Lévy-Longo tree of* M, $LLT(M)$ *is a* Σ^1*-labelled infinitary tree defined informally as follows:*
   – $LLT(M) = T$ *if* M *is unsolvable of order* ∞, *that is for each natural number* n *there exists a lambda term* $\lambda x_1 \dots x_n.M' =_\beta M$
   – $LLT(M) = \lambda x_1 \dots x_n.\bot$ *if* M *is unsolvable of order* n
   – $LLT(M) = \lambda x_1 \dots x_n.y$
   $$\begin{array}{c} \lambda x_1 \dots x_n.y \\ / \quad \backslash \\ LLT(M_1) \dots LLT(M_m) \end{array}$$
   if M *is solvable and has principal head normal form* $\lambda x_1 \dots x_n.yM_1 \dots M_m$.
2. *The Böhm tree of* M, *BT(M) is a* Σ^2*-labelled tree defined informally as follows:*
   – $BT(M) = \bot$ *if* M *is unsolvable*
   – $BT(M) = \lambda x_1 \dots x_n.y$
   $$\begin{array}{c} \lambda x_1 \dots x_n.y \\ / \quad \backslash \\ BT(M_1) \dots BT(M_m) \end{array}$$
   if M *is solvable and has principal head normal form* $\lambda x_1 \dots x_n.yM_1 \dots M_m$.

On Lévy-Longo trees (Böhm trees) there is a natural order relation defined by $LLT(M) \subseteq LLT(N)$ iff $LLT(N)$ is obtained by $LLT(M)$ by replacing $\bot$ in some leaves of $LLT(M)$ by Lévy-Longo-trees of λ-terms or by replacing some $\lambda x_1 \ldots x_n.\perp$ by T ($BT(M) \subseteq BT(N)$ iff $BT(N)$ is obtained by $BT(M)$ by replacing $\perp$ in some leaves of $BT(M)$ by Böhm-trees of λ-terms).

Definition 16. *Let $\mathcal{D}$ be the class of all reflexive objects $\langle D, ([\varphi], [\psi])\rangle$ in the category $K_!(\mathcal{G})$. We define the following subclasses:*

1. $\mathcal{D}^{\mathcal{E}} = \{\langle D, ([\varphi], [\psi])\rangle \in \mathcal{D} \mid \psi \circ \varphi \approx id_D\}$
2. $\mathcal{D}^{\mathcal{B}} = \{\langle D, ([\varphi], [\psi])\rangle \in \mathcal{D} \mid \psi \circ \epsilon_{I \Rightarrow (D \Rightarrow D)} = \epsilon_{I \Rightarrow D}$ *and* $\psi \circ \varphi \not\approx id_D\}$
3. $\mathcal{D}^{\mathcal{L}} = \{\langle D, ([\varphi], [\psi])\rangle \in \mathcal{D} \mid \psi \circ \epsilon_{I \Rightarrow (D \Rightarrow D)} \neq \epsilon_{I \Rightarrow D}\}$

The main result of this paper states that, given a lambda model D in the category $K_!(\mathcal{G})$, the theory it induces is either

1. $\mathcal{H}^*$, the theory induced by the canonical D_∞ model of Scott [Sco72,Bar84] and [Wad78], if $\mathsf{D} \in \mathcal{D}^{\mathcal{E}}$;
2. $\mathfrak{B}$, the theory which identifies two terms iff they have the same Böhm tree, if $\mathsf{D} \in \mathcal{D}^{\mathcal{B}}$;
3. $\mathcal{L}$ the theory which identifies two terms iff they have the same Lévy-Longo tree if $\mathsf{D} \in \mathcal{D}^{\mathcal{L}}$.

The proof proceeds along the same lines of [Bar84,Wad78,Hyl76]. First we show that if two terms are equated in one of the above theories then they are equal on the corresponding model. In order to prove this we state an important property satisfied by all the models: the *approximation theorem*, which says that the interpretation of a term is the least upper bound of the interpretations of its approximants. The following definitions and lemmata are necessary to state this result.

Definition 17. *1. The set of $\lambda\Omega$-terms, $\Lambda(\Omega)(\ni M)$ is defined from a set of variables $Var(\ni x)$ as $M ::= x \mid MM \mid \lambda x.M \mid \Omega$.*
2. The set of (possibly) indexed terms *$\Lambda(\Omega)^{\mathbb{N}}(\ni M)$ is the superset of $\Lambda(\Omega)$ defined as $M ::= x \mid MM \mid \lambda x.M \mid \Omega \mid M^n$.*
3. A term is truly indexed *if it is of the shape M^n. A term is* completely indexed *if all its subterms of the shape variable, abstraction, and application are immediate subterms of truly indexed terms.*

Notice that in a truly indexed term the constant Ω could not be indexed. The reduction rules are extended to indexed terms as follows.

Definition 18. *1. The following reduction rules are definable on $\Lambda(\Omega)$:*
(Ω_1) $\lambda x.\Omega \to \Omega$ $\quad$ (Ω_2) $\Omega M \to \Omega$
2. The following reduction rules are definable on indexed terms of $\Lambda(\Omega)^{\mathbb{N}}$:
(Ω^n) $\Omega^n \to \Omega$ $\quad$ (Ω^0) $M^0 \to \Omega$
(β_I) $((\lambda x.P^n)^{m+1}Q^p)^h \to (P[Q^a/x])^b$ $\quad$ $(\beta_{i,j})$ $(M^i)^j \to M^{min\{i,j\}}$
where $b = min\{n, m+1, h\}, a = min\{m, p\}$

Lemma 4. *A completely indexed term Q is $\Omega^n\Omega^0\beta_I\beta_{i,j}$-normalizing.*

Proof. *The proof uses the same arguments as in Theorem 14.1.12 of [Bar84].* □

Denotational semantics is readily defined. The denotation of a pure λ-term $M \in \Lambda$ is defined along the usual categorical definition. To accommodate indexed terms we need to introduce two new rules and use the larger categories of games and history-sensitive strategies.

Definition 19. *Let $\mathsf{D} \in \mathcal{D}$ be a λ-model. The interpretation of a term $M \in \Lambda(\Omega)^{\mathbb{N}}$ (whose free variables are among the list $\Delta = \{x_1, \ldots, x_k\}$) in D, in the Cartesian closed category $K_!(\mathcal{G}^s)$, $[\![M]\!]^{\mathsf{D}}_{\Delta} : \mathsf{D}^{|\Delta|} \Rightarrow \mathsf{D}$ is the strategy inductively defined as follows:*

$[\![x]\!]^{\mathsf{D}}_{\Delta} = \pi^{\Delta}_{x}$;
$[\![MN]\!]^{\mathsf{D}}_{\Delta} = [\![M]\!]^{\mathsf{D}}_{\Delta} \cdot [\![N]\!]^{\mathsf{D}}_{\Delta}$;
$[\![\lambda x.M]\!]^{\mathsf{D}}_{\Delta} = \psi \circ \Lambda([\![M]\!]^{\mathsf{D}}_{\Delta,x})$;
$[\![M^n]\!]^{\mathsf{D}}_{\Delta} = ([\![M]\!]^{\mathsf{D}}_{\Delta})^n$
$[\![\Omega]\!]^{\mathsf{D}}_{\Delta} = \epsilon_{\mathsf{D}^{|\Delta|} \Rightarrow \mathsf{D}}$;

It is immediate to observe that for each term $M \in \Lambda$, the strategy $[\![M]\!]^{\mathsf{D}}_{\Delta}$ is history-free.

Theorem 1 (Validity of indexed reduction). *Rules (Ω_2), (Ω^n), (Ω^0), (β_I) and $(\beta_{i,j})$ are valid in each game model $\mathsf{D} \in \mathcal{D}$; the rule Ω_1 is valid in each model $\mathsf{D} \in \mathcal{D}^{\mathcal{E}} \cup \mathcal{D}^{\mathcal{B}}$. The Validity of a rule α is intended in the following sense: for each $P, Q \in \Lambda(\Omega)^{\mathbb{N}}$ if $(P \rightarrow_{\alpha} Q)$ then $[\![P]\!]^{\mathsf{D}}_{\Delta} \sqsubseteq [\![Q]\!]^{\mathsf{D}}_{\Delta}$.*

Proof. (Ω_2). $[\![\Omega M]\!]^{\mathsf{D}}_{\Delta} = ev_{\mathsf{D},\mathsf{D}} \circ \langle \varphi \circ [\![\Omega]\!]^{\mathsf{D}}_{\Delta}, [\![M]\!]^{\mathsf{D}}_{\Delta} \rangle$ *by definition 19*
$= ev_{\mathsf{D},\mathsf{D}} \circ \langle \varphi \circ \epsilon_{D^{|\Delta|} \Rightarrow D}, [\![M]\!]^{\mathsf{D}}_{\Delta} \rangle$ *by definition 19*
$= ev_{\mathsf{D},\mathsf{D}} \circ \langle \epsilon_{D^{|\Delta|} \Rightarrow (D \Rightarrow D)}, [\![M]\!]^{\mathsf{D}}_{\Delta} \rangle$ *by the definition of retract*
$= \epsilon_{D^{|\Delta|} \Rightarrow D}$ *for the structure of* $ev_{\mathsf{D},\mathsf{D}}$.

(Ω^n). $[\![\Omega^n]\!]^{\mathsf{D}}_{\Delta} = ([\![\Omega]\!]^{\mathsf{D}}_{\Delta})^n = \epsilon^n_{\mathsf{D}^{|\Delta|} \Rightarrow \mathsf{D}} = \epsilon_{\mathsf{D}^{|\Delta|} \Rightarrow \mathsf{D}} = [\![\Omega]\!]^{\mathsf{D}}_{\Delta}$.

(Ω^0). *By proposition 1.*

(β_I). $[\![((\lambda x.P^n)^{m+1} Q^p)^h]\!]^{\mathsf{D}}_{\Delta} = ([\![(\lambda x.P^n)^{m+1}]\!]^{\mathsf{D}}_{\Delta} \cdot [\![Q^p]\!]^{\mathsf{D}}_{\Delta})^h$ *by definition 19*
$\sqsubseteq ([\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta} \cdot ([\![Q^p]\!]^{\mathsf{D}}_{\Delta})^m)^{min\{h,m+1\}}$ *by proposition 2*
$= ([\![\lambda x.P^n]\!]^{\mathsf{D}}_{\Delta} \cdot [\![Q^{min\{m,p\}}]\!]^{\mathsf{D}}_{\Delta})^{min\{h,m+1\}}$ *by proposition 1*
$= ([\![(\lambda x.P^n) Q^{min\{m,p\}}]\!]^{\mathsf{D}}_{\Delta})^{min\{h,m+1\}}$ *by definition 19*
$= [\![((\lambda x.P^n) Q^{min\{m,p\}})^{min\{h,m+1\}}]\!]^{\mathsf{D}}_{\Delta}$ *by definition 19*
$\approx [\![(P^n[Q^{min\{m,p\}}/x])^{min\{h,m+1\}}]\!]$ *by β-conversion*
$= [\![(P[Q^{min\{m,p\}}/x])^{min\{h,m+1,n\}}]\!]^{\mathsf{D}}_{\Delta}$ *by definition 19 and proposition 1.*

$(\beta_{i,j})$. $[\![(M^i)^j]\!]^{\mathsf{D}}_{\Delta} = ([\![M^i]\!]^{\mathsf{D}}_{\Delta})^j$ *by definition 19*
$= (([\![M]\!]^{\mathsf{D}}_{\Delta})^i)^j = [\![M^{min\{i,j\}}]\!]^{\mathsf{D}}_{\Delta}$ *by proposition 1.*

(Ω_1). $[\![\lambda x.\Omega]\!]^{\mathsf{D}}_{\Delta} = \psi \circ \Lambda([\![\Omega]\!]^{\mathsf{D}}_{\Delta})$ *by definition 19*
$= \psi \circ \Lambda(\epsilon_{D^{|\Delta|} \Rightarrow D})$ *by definition 19*
$= \psi \circ \epsilon_{D^{|\Delta|} \Rightarrow (D \Rightarrow D)}$ *by definition 12*

$= \epsilon_{D^{|\Delta|} \Rightarrow D}$ *by definition 16*
$= [\![\Omega]\!]^{\mathsf{D}}_{\Delta}$ *by definition 19.* $\square$

Each term $M \in \Lambda$ can be approximated by a "partially evaluated" term $A \in \Lambda(\Omega)$ which is called an *approximant*. Different notions of approximants arise for the different classes of models.

Definition 20. *For each term $M \in \Lambda$ the sets of its approximants are defined by:*

1. $\mathcal{A}^{\mathcal{E}}(M) = \{A \in \Lambda(\Omega) \mid BT(A[\Delta\Delta/\Omega]) \subseteq BT(M)$ *and A is in $\beta\eta\Omega_1\Omega_2$-nf* $\}$
2. $\mathcal{A}^{\mathcal{B}}(M) = \{A \in \Lambda(\Omega) \mid BT(A[\Delta\Delta/\Omega]) \subseteq BT(M)$ *and A is in $\beta\Omega_1\Omega_2$-nf* $\}$
3. $\mathcal{A}^{\mathcal{L}}(M) = \{A \in \Lambda(\Omega) \mid LLT(A[\Delta\Delta/\Omega]) \subseteq LLT(M)$ *and A is in $\beta\Omega_2$-nf* $\}$

Lemma 5. *For each game model $\mathsf{D} \in \mathcal{D}$, λ-term M and approximant $A \in \mathcal{A}^{\star}(M)$ with $\star \in \{\mathcal{E}, \mathcal{B}, \mathcal{L}\}$ we have:*

$$[\![A]\!]^{\mathsf{D}} \sqsubseteq [\![M]\!]^{\mathsf{D}}$$

Proof. *The lemma can be straightforwardly proved by structural induction on the approximant A, exploiting the fact that, in game categories, composition is monotone.* $\square$

Definition 21. *The erasing function $\mathcal{R} : \Lambda(\Omega)^{\mathbb{N}} \to \Lambda(\Omega)$ is inductively defined as follows:*

1. $\mathcal{R}(x) = x;\ \mathcal{R}(\Omega) = \Omega$ 2. $\mathcal{R}(PQ) = \mathcal{R}(P)\mathcal{R}(Q)$
3. $\mathcal{R}(\lambda x.P) = \lambda x.\mathcal{R}(P)$ 4. $\mathcal{R}(M^n) = \mathcal{R}(M)$

Lemma 6. *For each game model $\mathsf{D} \in \mathcal{D}^{\star}$, and for each completely indexed term $M \in \Lambda(\Omega)^{\mathbb{N}}$ there exists a term $N \in \Lambda(\Omega)^{\mathbb{N}}$ such that:*

$$[\![M]\!]^{\mathsf{D}} \sqsubseteq [\![N]\!]^{\mathsf{D}} \text{ and } \mathcal{R}(N) \in \mathcal{A}^{\star}(\mathcal{R}(M))$$

with $\star \in \{\mathcal{E}, \mathcal{B}, \mathcal{L}\}$.

Proof. *Take for N the $\Omega^n\Omega^0\beta_I\beta_{i,j}\eta$-normal form of M if $\star = \mathcal{E}$, or the $\Omega^n\Omega^0\beta_I$–$\beta_{i,j}$-normal form of M if $\star \in \{\mathcal{B}, \mathcal{L}\}$.* $\square$

Lemma 7. *For each game model $\mathsf{D} \in \mathcal{D}$, λ-term M and natural number n there exists a completely indexed term M^* such that:*

$$[\![M^n]\!]^{\mathsf{D}} = [\![M^*]\!]^{\mathsf{D}}$$

Proof. *Structural induction on M by observing that: $[\![(\lambda x.P)^n]\!]^{\mathsf{D}} = [\![(\lambda x.P^n)^n]\!]^{\mathsf{D}}$ and $[\![(PQ)^n]\!]^{\mathsf{D}} = [\![(P^nQ^{n-1})^n]\!]^{\mathsf{D}}$ for Proposition 2.* $\square$

Theorem 2 (Approximation theorem). *For each game model $\mathsf{D} \in \mathcal{D}^{\star}$, λ-term M the following equality holds:*

$$[\![M]\!]^{\mathsf{D}} = \bigsqcup \{[\![A]\!]^{\mathsf{D}} \mid A \in \mathcal{A}^{\star}(M)\}$$

with $\star \in \{\mathcal{E}, \mathcal{B}, \mathcal{L}\}$.

Proof. $[\![M]\!]^{\mathsf{D}} = \bigsqcup_{n\in\omega}\{[\![M^n]\!]^{\mathsf{D}}\}$ *for Proposition 1*
$= \bigsqcup\{[\![M^*]\!]^{\mathsf{D}} \mid M^*$ *a completely indexing of* $M\}$ *for Lemma 7*
$\sqsubseteq \bigsqcup\{[\![N]\!]^{\mathsf{D}} \mid N \in \Lambda(\Omega)^{\mathbb{N}}$ & $\mathcal{R}(N) \in \mathcal{A}^{\star}(M)\}$ *for lemma 6*
$\sqsubseteq \bigsqcup\{[\![A]\!] \mid A \in \mathcal{A}^{\star}(M)\}$ *since* $[\![N]\!]^{\mathsf{D}} \sqsubseteq [\![\mathcal{R}(N)]\!]^{\mathsf{D}}$
$\sqsubseteq [\![M]\!]$ *for lemma 5.* □

From Theorem 2 we can readily conclude that if two terms have the same tree they also have the same interpretation in the different game models, that is:

Proposition 3. *For each model* $\mathsf{D} \in \mathcal{D}$, λ*-terms* M, N *we have:*

1. *if* $\mathsf{D} \in \mathcal{D}^{\mathcal{L}}$ *and* $LLT(M) = LLT(N)$ *then* $[\![M]\!]^{\mathsf{D}} = [\![N]\!]^{\mathsf{D}}$*;*
2. *if* $\mathsf{D} \in \mathcal{D}^{\mathcal{B}}$ *and* $BT(M) = BT(N)$ *then* $[\![M]\!]^{\mathsf{D}} = [\![N]\!]^{\mathsf{D}}$*;*
3. *if* $\mathsf{D} \in \mathcal{D}^{\mathcal{E}}$ *then* $M =_{\mathcal{H}^*} N \Longleftrightarrow [\![M]\!]^{\mathsf{D}} = [\![N]\!]^{\mathsf{D}}$.

Proof. The first point can be proved observing that if $LLT(M) = LLT(N)$ *then* $\mathcal{A}^{\mathcal{L}}(M) = \mathcal{A}^{\mathcal{L}}(N)$ *and hence* $[\![M]\!]^{\mathsf{D}} = [\![N]\!]^{\mathsf{D}}$ *by Theorem 2. A similar argument can be applied to the second point. Point three follows from the fact that* $\mathcal{H}^*$ *is a maximal theory and from the validity of the* η*-rule (and also the* $\eta\infty$*-rule [GFH99]) in the models* $\mathsf{D} \in \mathcal{D}^{\mathcal{E}}$*. We recall that two terms are equal in the theory* $\mathcal{H}^*$ *iff they have the same Böhm tree up to infinitary* η*-expansion.* □

In the following part of the section we shall prove that if two terms have different Lévy-Longo trees or different Böhm trees they also have different interpretation in corresponding game models of the λ-calculus. This will characterize completely the theories induced by game models and will substantiate the intuitive impression that the strategy which interprets a term is strongly connected with the tree of the term.

Definition 22. *Given two terms* M, $N \in \Lambda$*, we say that* M *and* N *are* similar *and we write* $M \sim N$ *if both* M *and* N *are unsolvable or they are solvable with principal head normal forms respectively* $\lambda x_1 \dots x_n.yM_1 \dots M_m$ *and* $\lambda x_1 \dots x_{n'}.y'N_1 \dots N_{m'}$ *in which* $y \equiv y'$ *and* $m - n = m' - n'$.

Lemma 8. *For each compositional not trivial model of the* λ*-calculus* D*, for each pair of lambda-terms* M, N *if* $M \not\sim N$ *then* $[\![M]\!]^{\mathsf{D}} \neq [\![N]\!]^{\mathsf{D}}$.

Proof. The proof follows immediately from the fact that terms which are not similar can be separated by a suitable context. A detailed proof can be found in [Bar84], Cap. 10. □

Lemma 9. *For each non-extensional game model* $\mathsf{D} \in \mathcal{D}^{\mathcal{B}} \cup \mathcal{D}^{\mathcal{L}}$*, for each variable* x *and* λ*-term* $M \equiv \lambda y.M'$ *we have that* $[\![x]\!]^{\mathsf{D}} \not\approx [\![\lambda y.M']\!]^{\mathsf{D}}$.

Proof. By contradiction: $[\![x]\!]^{\mathsf{D}} = id_{\mathsf{D}}$ *and* $[\![\lambda y.M']\!]^{\mathsf{D}} = \psi \circ \tau$ *for some suitable strategy* $\tau : \mathsf{D} \Rightarrow (\mathsf{D} \Rightarrow \mathsf{D})$*. If* $\psi \circ \tau \approx id_{\mathsf{D}}$*, this would mean that the strategy* ψ *has a left (by definition of retract) and a right inverse, which, by categorical arguments, need to coincide, contradicting the hypothesis that* $\psi \circ \varphi \not\approx id_{\mathsf{D}}$. □

Lemma 10. *Let M, N solvable λ-terms such that $M = \lambda x_1 \dots x_n.yM_1 \dots M_m$, $N = \lambda x_1 \dots x_{n'}.yN_1 \dots N_{m'}$ and $FV(M) \cup FV(N) \subseteq \Delta$, let D be a non extensional λ-model, $\mathsf{D} \in \mathcal{D}^{\mathcal{B}} \cup \mathcal{D}^{\mathcal{L}}$. If $n \neq n'$ then $[\![M]\!]^{\mathsf{D}} \neq [\![N]\!]^{\mathsf{D}}$.*

Proof. Suppose $n < n'$, it is not difficult to find a context $C[\ ]$ s.t. $C[M] = x$ and $C[N] = \lambda y.N'$. From the previous lemma we have the thesis. □

Lemma 11. *For each lazy game model $\mathsf{D} \in \mathcal{D}^{\mathcal{L}}$, and for each pair of λ-terms M, N both unsolvable but of different order, we have that $[\![M]\!]^{\mathsf{D}} \not\approx [\![N]\!]^{\mathsf{D}}$.*

Proof. By the approximation theorem 2 we have that for each unsolvable term P of order 0, $[\![M]\!]^{\mathsf{D}} = \epsilon$, while one can readily calculate that for a lazy game model D, for each term Q, $[\![\lambda x.Q]\!]^{\mathsf{D}} \not\approx \epsilon$. Using suitable contexts one obtains the thesis. □

Lemma 12. *Let $\mathsf{D} \in \mathcal{D}$ be a game λ-model, and let $M \equiv xM_1 \dots M_m$ and $N \equiv xN_1 \dots N_m$ be two untyped λ-terms, if $[\![M]\!]^{\mathsf{D}} \approx [\![N]\!]^{\mathsf{D}}$ then for each $1 \leq i \leq m$ we have that: $[\![M_i]\!]^{\mathsf{D}} \approx [\![N_i]\!]^{\mathsf{D}}$.*

Proof. A premise is here necessary. A position in a strategy describes the interaction between Player and Opponent on the hypothesis that Player follows the strategy and the Opponent exhibits a particular behavior. In defining the set of positions forming a strategy, any possible behavior of the Opponent has to be considered, also incoherent behavior that no Player is allowed to follow. For example in the game $!A \multimap B$ the Opponent can have completely different behaviors in the different components of A.

Now, suppose by contradiction that there exists $1 \leq i \leq m$ such that $[\![M_i]\!]^{\mathsf{D}} \not\approx [\![N_i]\!]^{\mathsf{D}}$. Then, by Definition 13, there exists then a play $s \in [\![M_i]\!]^{\mathsf{D}}$ such that $s \not\approx t$ for each $t \in [\![N_i]\!]^{\mathsf{D}}$. Using the definition of the interpretation of λ-terms, it is possible to calculate that the strategy $[\![M]\!]^{\mathsf{D}} : !\mathsf{D}_{x_1} \otimes \dots !\mathsf{D}_x \dots \otimes !\mathsf{D}_{x_n} \multimap \mathsf{D}$ replies to the initial question of the Opponent repeating the question on a particular copy, let say the j-th one, of D_x. If the Opponent behaves on the j-th copy of D_x, following the strategy $[\![\lambda x_1 \dots x_i.x_i]\!]^{\mathsf{D}}$, and on the other components with the same behavior that led to the position s, we obtain a position s' contained in $[\![M]\!]^{\mathsf{D}}$. The strategy $[\![N]\!]^{\mathsf{D}}$ contains a positions t' equivalent to s', only if $[\![N_i]\!]^{\mathsf{D}}$ contains a position t equivalent to s, but this is negated by hypothesis. Therefore, by Definition 13, $[\![M]\!]^{\mathsf{D}} \not\approx [\![N]\!]^{\mathsf{D}}$. □

We can finally state the following theorem:

Theorem 3. *Let M and N be two untyped λ-terms. Let $\mathsf{D} \in \mathcal{D}$ be a game λ-model. If $[\![M]\!]^{\mathsf{D}} = [\![N]\!]^{\mathsf{D}}$ then we have:*

1. *$LLT(M) = LLT(N)$ if $\mathsf{D} \in \mathcal{D}^{\mathcal{L}}$;*
2. *$BT(M) = BT(N)$ if $\mathsf{D} \in \mathcal{D}^{\mathcal{B}}$.*

Proof. We prove the converse. If $LLT(M) \neq LLT(N)$ then there exist a natural number n such that $LLT(M)$ and $LLT(N)$ differ at the level n, that is the restrictions of $LLT(M)$ and $LLT(N)$ to nodes having depths less that n are *different. We will prove that the semantic interpretation of M and N differ by induction of the number n.*

If $n = 0$ then one of the following cases need to occur:

1. *The two terms are not similar. In this case from Lemma 8 one readily obtains the thesis.*
2. *The two terms are both unsolvable but of different order. In this case from Lemma 11 the thesis follows.*
3. *The two terms are similar but with a different number of λ-abstraction. In this case Lemma 10 applies.*

If $n = n' + 1$, then M and N are similar, solvable, with the same number of lambda abstractions. Suppose they have hnf respectively $\lambda x_1 \dots x_n.xM_1 \dots M_m$ and $\lambda x_1 \dots x_n.yN_1 \dots N_m$. There exists i s.t. $LLT(M_i) \neq LLT(N_i)$ at the level n'. By induction hypothesis $[\![M_i]\!]^{\mathsf{D}} \not\approx [\![N_i]\!]^{\mathsf{D}}$, and from Lemma 12, using suitable contexts, one obtains the thesis.

The case for the Böhm-trees can be readily proved following the same lines.

□

Finally the characterization of the theories induced by models built in the Cartesian closed category $K_!(\mathcal{G})$ can be given.

Theorem 4. *Let $\mathsf{D} \in \mathcal{D}$ be a λ-model in $K_!(\mathcal{G})$. Then*

1. *if $\mathsf{D} \in \mathcal{D}^{\mathcal{E}}$ then $M = N \in Th(\mathsf{D})$ iff $M = N \in \mathcal{H}^*$*
2. *if $\mathsf{D} \in \mathcal{D}^{\mathcal{B}}$ then $M = N \in Th(\mathsf{D})$ iff $BT(M) = BT(N)$*
3. *if $\mathsf{D} \in \mathcal{D}^{\mathcal{L}}$ then $M = N \in Th(\mathsf{D})$ iff $LLT(M) = LLT(N)$*

4 Conclusions

In the present paper we have studied the λ-theories induced by the games models without performing the extensional collapse. Through the extensional collapse it is possible to identify strategies that have the same observational behavior. In general the extensional collapse is fundamental in order to obtain fully abstract games models of programming languages.

Therefore it is still possible to use game models to capture λ-theories that are strictly coarser than the three considered in this paper. An example of such a theory can be found in [AM95] where, through the extensional collapse of a model D in $\mathcal{D}^{\mathcal{L}}$, a fully abstract model of the lazy λ-calculus is obtained.

However, in general, models obtained through the extensional collapse are more difficult to study, *e.g.* the equivalence between strategies is not decidable also in the finite case.

Our main theorem defines precisely those theories that can be obtained using simple (not collapsed) games models, and hence it implies also that the theories obtained through the extensional collapse lie only in between the theories $\mathcal{L}$ and $\mathcal{H}^*$.

A second consideration concerns the class of the game models we consider in this work. We have focused on games and history-free strategies mainly for historical reasons. We claim that the paper can be easily reformulated in order to prove the same results for the category of games and innocent strategies [HO00]. We can substantiate our claim by observing that the main tools used in the proofs — history-sensitive strategies, approximation strategies, Lemma 12 — are not peculiar to the history-free strategies and can be reformulated and applied in the context of innocent strategies.

A final point concerns the constructions of games models. In this paper we do not build any example of games model for the lambda calculus; however in [GFH99] a general method to obtain non-initial solutions of recursive equations is presented. It is then quite simple to find extensional game models: several examples are presented there. Non-extensional games models can be obtained through the standard tricks used in the setting of the cpo models. For example a non-extensional model whose theory is $\mathfrak{B}$ $(\mathcal{L})$ can be obtained by taking the initial solution of the recursive equation $D = (D \Rightarrow D) \times A$ $(D = (D \Rightarrow D)_\perp \times A)$, where A is an arbitrary game.

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