Small Models, Large Cardinals, and Large Cardinal Ideals
(joint work with Philipp Lücke)

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Ramsey cardinals

Victoria Gitman isolated the following from work of William Mitchell from the late 70’ies.

Theorem

\( \kappa \) is a Ramsey cardinal if for every \( A \subseteq \kappa \) there is a transitive weak \( \kappa \)-model \( M \) with \( A \in M \) and with a (uniform) \( \kappa \)-amenable, countably complete and \( M \)-normal ultrafilter \( U \) on \( \kappa \).

- A weak \( \kappa \)-model \( M \) is a model of \( \text{ZFC}^- \) such that \( |M| = \kappa \) and \( \kappa + 1 \subseteq M \).
- An \( M \)-ultrafilter \( U \) is \( M \)-normal if it closed under diagonal intersections in \( M \).
- \( U \) is countably complete if any countable intersection (in \( V \)) of filter elements is nonempty.
- \( U \) is \( \kappa \)-amenable if whenever \( X \) is a set of size \( \kappa \) in \( M \), then \( X \cap U \in M \).

Note: We will require all our filters to be uniform.
Varying the parameters

What happens if we vary the requirements on $M$ and on $U$? For example:

- Instead of the countable completeness of $U$, only require the ultrapower of $M$ by $U$ to be well-founded.
- Do not require well-foundedness of the ultrapower.

Or require $U$ to be ...

- *stationary-complete*: Every countable intersection from $U$ (in $\mathbf{V}$) is stationary in $\kappa$.
- *genuine*: Every diagonal intersection of $U$ is unbounded in $\kappa$.
- *normal*: Every diagonal intersection of $U$ is stationary in $\kappa$.

We may also require that $M \prec H(\theta)$ for sufficiently large regular $\theta$ instead of transitivity of $M$ in any of the above.
A table of results

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Completely ineffable cardinals

**Definition**

$S \subseteq \mathcal{P}(\kappa)$ is a stationary class if $S \neq \emptyset$ is a collection of stationary subsets of $\kappa$.

**Definition**

A cardinal $\kappa$ is completely ineffable if there is a stationary class $S \subseteq \mathcal{P}(\kappa)$ such that whenever $A \in S$ and $f : [A]^2 \to 2$, then there is $H \subseteq A$ in $S$ that is homogeneous for $f$.

**Theorem (Kleinberg, 1970ies)**

$\kappa$ is completely ineffable iff for every sufficiently large regular $\theta$ and every / some countable $M \prec H(\theta)$ with $\kappa \in M$, there is an $M$-normal, $\kappa$-amenable $M$-ultrafilter $U$ on $\kappa$. 
Another characterization of complete ineffability

Results from below papers essentially show the following theorem (using completely different proofs than the above result about countable models):

- Holy-Schlicht (2018): A hierarchy of Ramsey-like cardinals, characterized through the (non-)existence of winning strategies for certain infinite games, with \(\omega\)-Ramsey cardinals at the bottom.

**Theorem**

\(\kappa\) is completely ineffable iff for every sufficiently large regular \(\theta\) and every \(x \in H(\theta)\) there is a weak \(\kappa\)-model \(M \prec H(\theta)\) with \(x \in M\) and with a \(\kappa\)-amenable, \(M\)-normal ultrafilter \(U\) on \(\kappa\).
Uniform large cardinal ideals

These large cardinal characterizations also allow for highly uniform definitions of corresponding *large cardinal ideals*. Let \( \varphi \) denote a large cardinal property that is characterized (as are Ramseyness or complete ineffability above) through the existence of certain models \( M \) with \( M \)-ultrafilters \( U \) having a certain property \( \varphi^* \). We define \( I_\varphi \) and \( I_\prec \varphi \) as follows:

- \( A \in I_\varphi \) if there is \( x \subseteq \kappa \) such that for all transitive weak \( \kappa \)-models \( M \) with \( x \in M \) and every \( M \)-ultrafilter \( U \) with Property \( \varphi^* \), we have \( A \notin U \).

- \( A \in I_\prec \varphi \) if for all sufficiently large regular \( \theta \) there is \( x \in H(\theta) \) such that for all weak \( \kappa \)-models \( M \prec H(\theta) \) with \( x \in M \) and every \( M \)-ultrafilter \( U \) with Property \( \varphi^* \), we have \( A \notin U \).

Given that \( \varphi(\kappa) \) holds, these ideals are easily seen to be proper ideals on \( \kappa \). If \( \varphi^* \) implies the \( M \)-normality of \( U \), then these ideals are normal ideals on \( \kappa \).
Example: The completely ineffable ideal

In all cases of large cardinals for which corresponding large cardinal ideals have already been defined, these coincide with our definitions: weakly compact, Ramsey, ineffably Ramsey. In some other cases, our ideals correspond to well-known set theoretic objects, and sometimes they are new.

Let $\kappa$ be completely ineffable, and let $I$ denote the completely ineffable ideal on $\kappa$. An adaption of the proof of the previous theorem yields the following.

**Theorem**

$I$ is the complement of the maximal (w.r.t. $\supseteq$) stationary class witnessing the complete ineffability of $\kappa$. 
We can show in most cases that these ideals are strictly $\subseteq$-increasing, in a way which also implies that the related large cardinal notions are strictly increasing in terms of consistency strength. For example: Weakly compact ideal $\subsetneq$ Ineffable Ideal $\subsetneq$ Completely Ineffable ideal $\subsetneq$ weakly Ramsey ideal $\subsetneq$ Ramsey ideal $\subsetneq$ $\prec$-Ramsey ideal $\subsetneq$ measurable ideal.
The measurable ideal

The measurable ideal $I^\kappa_{ms}$ on a measurable cardinal $\kappa$ is the complement of the union of all normal ultrafilters on $\kappa$, and also fits into our framework of large cardinal ideals. This ideal is not very interesting in small inner models (for example in $L[U]$). Moreover:

**Theorem**

*If any set of pairwise incomparable conditions in the Mitchell ordering at $\kappa$ has size at most $\kappa$, then the partial order $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomic.*

However, it is consistently non-trivial – adapting classical arguments from Kunen and Paris yields the following:

**Theorem**

*Every model with a measurable cardinal $\kappa$ has a forcing extension in which $\mathcal{P}(\kappa)/I^\kappa_{ms}$ is atomless.*
Normally Ramsey cardinals

Definition

An uncountable cardinal $\kappa$ is S-Ramsey / $\infty$-Ramsey / $\Delta$-Ramsey if for every regular $\theta > \kappa$, every $x \in H(\theta)$ is contained in a weak $\kappa$-model $M \prec H(\theta)$ with a $\kappa$-amenable, $M$-normal ultrafilter $U$ on $\kappa$ that is stationary-complete / genuine / normal.

Generalizing results from Holy and Schlicht shows the following.

Theorem

$\kappa$ is S-Ramsey / $\infty$-Ramsey / $\Delta$-Ramsey if for all regular $\theta > \kappa$, Player I does not have a winning strategy in the game of length $\omega$ in which Player I plays a $\subseteq$-increasing sequence of $\kappa$-models $M_i \prec H(\theta)$ with union $M$, and Player II responds with a $\subseteq$-increasing sequence of $M_i$-ultrafilters $U_i$ with union $U$. Player I also has to ensure that $M_i$ and $U_i$ are both elements of $M_{i+1}$ for every $i \in \omega$. Player II wins if $U$ is an $M$-normal filter that is stationary-complete / genuine / normal.
... are equivalent to some seemingly weaker Ramsey-like cardinals

Lemma

\( S\text{-Ramsey} \equiv \infty\text{-Ramsey} \equiv \Delta\text{-Ramsey}. \)

Proof: Assume that \( \kappa \) is \( S\text{-Ramsey} \), that \( \theta > \kappa \) is regular, and let \( x \in H(\theta) \). Let \( M_0 < H(\theta) \) with \( x \in M_0 \) be a weak \( \kappa \)-model. Consider a run of the game for \( S\text{-Ramseyness} \), in which Player I starts by playing \( M_0 \), and which Player II wins – with resulting model \( M = \bigcup_{i<\omega} M_i \) and \( M \)-ultrafilter \( U = \bigcup_{i<\omega} U_i \). This means that \( M < H(\theta) \) is a weak \( \kappa \)-model with \( x \in M \), and \( U \) is \( \kappa \)-amenable, \( M \)-normal and stationary-complete. But \( \Delta U \supseteq \bigcap_{i<\omega} \Delta U_i \) (modulo a non-stationary set). Since each \( \Delta U_i \in U \), it follows that \( \Delta U \) is stationary, for it is stationary-complete. But this means that \( U \) is normal, and hence \( \kappa \) is \( \Delta\text{-Ramsey}. \) \( \square \)