Ramsey-like Operators

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Starting from measurability upwards, larger large cardinals are usually characterized by the existence of certain elementary embeddings of the universe, or dually, the existence of certain ultrafilters. However, below measurability, we have a somewhat similar picture when we consider certain embeddings with set-sized domain, or ultrafilters for small collections of sets. I will present some new results, and also review some older ones, showing that not only large cardinals, but also several related concepts can be characterized in such a way, and I will also provide a sample application of these characterizations.
Large Cardinals
Ramsey cardinals

\( \kappa \) is a Ramsey cardinal if every function \( c : [\kappa]^{<\omega} \to 2 \)
has a homogeneous set \( H \) of size \( \kappa \).

**Theorem (Mitchell (70ies) / Gitman, Sharpe, Welch (2011))**

\( \kappa \) is a Ramsey cardinal if and only if for every \( y \subseteq \kappa \) there is a weak \( \kappa \)-model \( M \ni y \), and a \( \kappa \)-amenable, countably complete and \( M \)-normal \( M \)-ultrafilter \( U \) on \( \kappa \).

- A weak \( \kappa \)-model \( M \) is a transitive model of \( \mathrm{ZFC}^- \) with \( |M| = \kappa \) and \( \kappa + 1 \subseteq M \).

- An \( M \)-ultrafilter \( U \) on \( \kappa \) is a filter that measures all subset of \( \kappa \) in \( M \). \( U \) is \( M \)-normal if it is closed under diagonal intersections in \( M \).

- \( U \) is countably complete if any countable intersection (in \( V \)) of elements of \( U \) is nonempty (equivalently, unbounded in \( \kappa \)).

- \( U \) is \( \kappa \)-amenable if whenever \( X \) is a set of size \( \kappa \) in \( M \), then \( X \cap U \in M \).

We require all our filters to be uniform: they only have elements of size \( \kappa \).
Ineffable cardinals

- A $\kappa$-list is a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ s.t. $a_\alpha \subseteq \alpha$ for every $\alpha < \kappa$.
- $H$ is homogeneous for $\vec{a}$ if $a_\alpha = a_\beta \cap \alpha$ for $\alpha < \beta$ both in $H$.

$\kappa$ is ineffable if every $\kappa$-list has a stationary homogeneous set.

**Proposition (Abramson et al (70ies) / Holy, Lücke (2020))**

$\kappa$ is an ineffable cardinal if and only if for every $y \subseteq \kappa$ there is a weak $\kappa$-model $M \ni y$, and an $M$-ultrafilter $U$ on $\kappa$ such that any diagonal intersection of $U$ is stationary – we write: $\Delta U$ is stationary.
Large Cardinal Ideals
The Ramsey ideal

**Lemma (Baumgartner (70ies))**

\( \kappa \) is a Ramsey cardinal iff every regressive function \( f : [\kappa]^{<\omega} \rightarrow \kappa \) has a homogeneous set of size \( \kappa \).

He used this to define Ramseyness of a subset \( A \) of \( \kappa \):

\( A \subseteq \kappa \) is Ramsey if every regressive function \( f : [\kappa]^{<\omega} \rightarrow \kappa \) has a homogeneous set \( H \subseteq A \) of size \( \kappa \).

The *Ramsey ideal* on a cardinal \( \kappa \) is the collection of all subsets of \( \kappa \) that are not Ramsey.

**Theorem (Mitchell (70ies) / Sharpe, Welch (2011))**

\( A \subseteq \kappa \) is Ramsey if and only if for every \( y \subseteq \kappa \) there is a weak \( \kappa \)-model \( M \ni y \), and a \( \kappa \)-amenable, countably complete and \( M \)-normal \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \).
Baumgartner (70ies) also made the following definition:

\[ A \subseteq \kappa \text{ is ineffable if every } \kappa\text{-list has a stationary homogeneous set } H \subseteq A. \]

The *ineffable ideal* on a cardinal \( \kappa \) is the collection of all subsets of \( \kappa \) that are not ineffable.

**Proposition (Abramson et al (70ies) / Holy, Lücke (2020))**

\[ A \subseteq \kappa \text{ is ineffable if and only if for every } y \subseteq \kappa \text{ there is a weak } \kappa\text{-model } M \supseteq y, \text{ and an } M\text{-ultrafilter } U \text{ on } \kappa \text{ such that any diagonal intersection of } U \text{ is stationary, with } A \in U. \]
Why, for example, we should care about large cardinal ideals:

A result of Baumgartner
Subtlety and Pre-Ramseyness

Definition (Baumgartner)

\( A \subseteq \kappa \) is **subtle** if for every club \( C \subseteq \kappa \) and every \( \kappa \)-list \( \vec{a} \), there are \( \alpha < \beta \) in \( A \cap C \) such that \( a_{\alpha} = a_{\beta} \cap \alpha \).

Lemma (Baumgartner (70ies))

\( A \subseteq \kappa \) is subtle iff for every club \( C \subseteq \kappa \) and every \( \kappa \)-list \( \vec{a} \), there is \( \alpha \in A \) and a stationary subset \( H \) of \( C \cap A \cap \alpha \) such that \( H \) is homogeneous for \( \vec{a} \).

Definition (Baumgartner)

\( A \subseteq \kappa \) is **pre-Ramsey** if for every club \( C \subseteq \kappa \) and every regressive function \( f : [\kappa]^{<\omega} \to \kappa \), there is \( \alpha \in A \) and an unbounded subset \( H \) of \( C \cap A \cap \alpha \) such that \( H \) is homogeneous for \( f \).

As before, we can define the subtle and the pre-Ramsey ideals. We will later see that those can also be characterized using small models and ultrafilters.
Indescribability

A $\Sigma^1_n$-formula is a formula that starts with $n$ alternating blocks of second order quantifiers $\exists$ and $\forall$, starting with $\exists$, followed by a formula with only first order quantifiers. $\Pi^1_n$-formulae are defined analogously, starting with $\forall$.

**Definition (Levy, 70ies)**

$A \subseteq \kappa$ is $\Pi^1_n$-indescribable if whenever $P \subseteq \kappa$ and $\varphi$ is a $\Pi^1_n$-formula such that $\langle V_\kappa, \in, P \rangle \models \varphi$, then there is $\alpha \in A$ such that $\langle V_\alpha, \in, P \cap V_\alpha \rangle \models \varphi$.

The $\Pi^1_n$-indescribable ideal $\Pi^1_n(\kappa)$ on $\kappa$ is the collection of all subsets of $\kappa$ that are not $\Pi^1_n$-indescribable. Note that $\Pi^1_0(\kappa) = NS_\kappa$, and that $\Pi^1_1$-indescribability $\equiv$ weak compactness.

There’s an extension of this hierarchy, that allows one to consider $\Pi^1_\xi$-indescribability for arbitrary ordinals $\xi < \kappa$, independently due to Sharpe and Welch (2011), and Joan Bagaria (2019).
Baumgartner’s results

We say that two ideals $I$ and $J$ on $\kappa$ generate (an ideal) $K = I \cup J$ on $\kappa$ in case $K$ consists of all unions $x \cup y$ with $x \in I$ and $y \in J$.

**Theorem (Baumgartner, 70ies)**

$\kappa$ is Ramsey if the pre-Ramsey and the $\Pi^1_1$-indescribable ideal on $\kappa$ generate a nontrivial ideal. This then is the Ramsey ideal on $\kappa$.

Ideals are necessary in this statement: the least cardinal that is pre-Ramsey and $\Pi^1_1$-indescribable is strictly below the least Ramsey cardinal.

**Theorem (Baumgartner, 70ies)**

$\kappa$ is ineffable if the subtle and the $\Pi^1_2$-indescribable ideal on $\kappa$ generate a nontrivial ideal. This then is the ineffable ideal on $\kappa$.

Ideals are necessary in this statement: the least cardinal that is subtle and $\Pi^1_2$-indescribable is strictly below the least ineffable cardinal.
Large Cardinal Operators
If $I$ is an ideal, we let $I^+$ denote the collection of subsets of $\kappa$ that are not in $I$, i.e. the complement of $I$. Sets in $I^+$ are usually also called $I$-positive.

We will often define certain ideals $I$ by actually defining the collection of $I$-positive sets in the following.
Large cardinal operators are maps between ideals on (large) cardinals $\kappa$. The Ramsey operator $\mathcal{R}$ was introduced by Qi Feng (1989).

Given an ideal $I$ on $\kappa$, let $\mathcal{R}(I)^+$ be the set of all $A \subseteq \kappa$ such that any regressive function $f : [\kappa]^{<\omega} \to \kappa$ has a homogeneous set $H \subseteq A$ in $I^+$.

We next introduce what we want to call the *model version* of the Ramsey operator.

Let $\mathcal{R}_M(I)^+$ be the set of all $A \subseteq \kappa$ such that for any $y \subseteq \kappa$, there is a weak $\kappa$-model $M \ni y$, and a $\kappa$-amenable $M$-normal $M$-ultrafilter $U$ on $\kappa$ with $A \in U$, such that any countable intersection of elements of $U$ is in $I^+$.

**Theorem (Sharpe, Welch (2011))**

*For any ideal $I$ on $\kappa$,*

$$\mathcal{R}_M(I) = \mathcal{R}(I).$$
The ineffability operator $\mathcal{I}$ was introduced by Baumgartner (70ies).

Given an ideal $I$ on $\kappa$, let $\mathcal{I}(I)^+$ be the set of all $A \subseteq \kappa$ such that any $\kappa$-list has a homogeneous set $H \subseteq A$ in $I^+$.

We also introduce a model version.

Let $\mathcal{I}_M(I)^+$ be the set of all $A \subseteq \kappa$ such that for any $y \subseteq \kappa$, there is a weak $\kappa$-model $M \ni y$, and an $M$-ultrafilter $U$ on $\kappa$ with $A \in U$ and $\Delta U \in I^+$. 

Proposition

For any ideal $I \supseteq NS_\kappa$ on $\kappa$,

$$\mathcal{I}_M(I) = \mathcal{I}(I).$$
Pre-operators
Local Instances for Ineffability

The ineffability operator

- \( A \in \mathcal{I}(I)^+ \) if any \( \kappa \)-list \( \vec{a} \) has a homogeneous set \( H \subseteq A \) in \( I^+ \).
- For any \( \kappa \)-list \( \vec{a} \), \( A \in \mathcal{I}^{\vec{a}}(I)^+ \) if \( \vec{a} \) has a homogeneous set \( H \subseteq A \) in \( I^+ \).

...and its model version

- \( A \in \mathcal{I}_M(I)^+ \) if for any \( y \subseteq \kappa \), there is a weak \( \kappa \)-model \( M \ni y \), and an \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \) and \( \Delta U \in I^+ \).
- For any \( y \subseteq \kappa \), \( A \in \mathcal{I}_M^y(I)^+ \) if there is a weak \( \kappa \)-model \( M \ni y \), and an \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \) and \( \Delta U \in I^+ \).

It seems that the local instances of the operators \( \mathcal{I} \) and \( \mathcal{I}_M \) do not agree. Similarly, this is also the case for the local instances for Ramseyness:
For any regressive $f : [\kappa]^\omega \rightarrow \kappa$, $A \in \mathcal{R}^f(I)^+$ if $f$ has a homogeneous set $H \subseteq A$ in $I^+$.

... and there’s a similar definition for $\mathcal{R}^y_M$...
Sequences of Ideals

We refer to a sequence $\vec{I} = \langle I_\alpha \mid \alpha \leq \kappa \rangle$ such that each $I_\alpha$ is an ideal on $\alpha$, and $\alpha$ ranges over inaccessible cardinals, as a sequence of ideals. Typical examples are when each $I_\alpha = [\alpha]^{<\alpha}$, each $I_\alpha = NS_\alpha$, or for some fixed $\beta$, each $I_\alpha = \Pi^1_\beta(\kappa)$ (for $\alpha > \beta$, and trivial otherwise).

If $\vec{I}$ is uniformly defined (say for example $I_\alpha = NS_\alpha$ for every $\alpha$), we sometimes identify $\vec{I}$ and $I_\kappa$. 

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Definition

Given an operator \( \mathcal{O} \) with local instances \( \mathcal{O}^p \), and a sequence \( \vec{I} \) of ideals, let \( \mathcal{O}_0(\vec{I})^+ \) be defined as

\[
\{ A \subseteq \kappa \mid \forall p \forall C \subseteq \kappa \text{ club } \exists \alpha \in A A \cap C \cap \alpha \in \mathcal{O}^p|\alpha(l_\alpha)^+\}. 
\]

\( \mathcal{I}_0(\text{NS}_{\kappa}) \) is the subtle ideal on \( \kappa \),
\( \mathcal{R}_0([\kappa]^{<\kappa}) \) is the pre-Ramsey ideal on \( \kappa \).

Examples

The subtle operator is the operator \( \mathcal{I}_0 \), where

\[
\mathcal{I}_0(\vec{I})^+ = \{ A \subseteq \kappa \mid \forall \vec{a} \forall C \subseteq \kappa \text{ club } \exists \alpha \in A A \cap C \cap \alpha \in \mathcal{I}^{\vec{a}|\alpha}(l_\alpha)^+\}. 
\]

The pre-Ramsey operator is the operator \( \mathcal{R}_0 \), where

\[
\mathcal{R}_0(\vec{I})^+ = \{ A \subseteq \kappa \mid \forall f \forall C \subseteq \kappa \text{ club } \exists \alpha \in A A \cap C \cap \alpha \in \mathcal{R}^f|\alpha(l_\alpha)^+\}. 
\]
As for the operators $\mathcal{I}$ and $\mathcal{R}$, the above also defines pre-operators $(\mathcal{I}_M)_0$ and $(\mathcal{R}_M)_0$ that correspond to the operators $\mathcal{I}_M$ and $\mathcal{R}_M$. As a strong indicator towards the usefulness of our model versions, we can show that they induce equivalent pre-operators.

**Theorem**

For any ideal $I$ on $\kappa$, $(\mathcal{R}_M)_0(I) = R_0(I)$, and if $I \supseteq \text{NS}_\kappa$, then also $(\mathcal{I}_M)_0(I) = \mathcal{I}_0(I)$.

In particular, this gives us a way to characterize the subtle and the pre-Ramsey ideal using small models and ultrafilters.
A new operator
A new operator

The following large cardinal notion came up in recent joint work with Philipp Lücke. It is *Ramseyness without countable completeness*.

**Definition**

A cardinal $\kappa$ is $T^K\omega$-Ramsey if for every $y \subseteq \kappa$ there is a weak $\kappa$-model $M \in y$, and an $M$-normal $M$-ultrafilter $U$ on $\kappa$ that is $\kappa$-amenable for $M$.

Note that since we require all our filters to be uniform, we implicitly require that $U \subseteq ([\kappa]^{<\kappa})^+$ in the above. This naturally induces a weak version of the Ramsey operator.

**Definition**

$A \in T(I)^+$ if for every $y \subseteq \kappa$, there is a weak $\kappa$-model $M \in y$, and an $M$-normal $M$-ultrafilter $U$ on $\kappa$ with $A \in U$ that is $\kappa$-amenable for $M$, and such that $U \subseteq I^+$.

So, the difference to the Ramsey operator is that we only ask that $U \subseteq I^+$, rather than that all countable intersections from $U$ be in $I^+$. 
It is actually new!

In above-mentioned joint work with Philipp Lücke, we didn’t consider large cardinal operators, however our results show that
\[ \mathcal{I}([\kappa]^{<\kappa}) \subseteq \mathcal{T}([\kappa]^{<\kappa}) \subseteq \mathcal{R}([\kappa]^{<\kappa}). \]

We can extend this to indescribability ideals (remember: \( \Pi^1_0(\kappa) = \text{NS}_\kappa \)).

**Theorem**

*For any \( \beta < \kappa \), \( \mathcal{I}(\Pi^1_\beta(\kappa)) \subset \mathcal{T}(\Pi^1_\beta(\kappa)) \subset \mathcal{R}(\Pi^1_\beta(\kappa)). \)*

We can’t hope to obtain properness as above with respect to any ideal \( I \).
For example, if \( \kappa \) is measurable and \( I \) is the complement of any normal ultrafilter on \( \kappa \), then \( I \subseteq \mathcal{I}(I) \subseteq \mathcal{T}(I) \subseteq \mathcal{R}(I) = I. \)
A test application for large cardinal operators: Baumgartner’s result
By a uniform argument, we obtain the following.

**Theorem (for $\mathcal{I}$ and $\mathcal{R}$, this is due to Brent Cody (2020))**

For many operators $\mathcal{O}$, in particular also for $\mathcal{O} \in \{\mathcal{I}, \mathcal{T}, \mathcal{R}\}$, and all $\beta < \kappa$, we have

$$\mathcal{O}(\Pi^1_\beta(\kappa)) = \mathcal{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa).$$

Ideals are necessary in this statement: the least cardinal $\kappa$ such that $\kappa \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$ and $\kappa \in \Pi^1_{\beta+2}(\kappa)^+$ is strictly below the least cardinal $\kappa$ such that $\kappa \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$.

In most, but not all cases, letting $\Pi^1_{-1}(\kappa) = [\kappa]<\kappa$, the above also holds for $\beta = -1$. In fact, many further results on the ineffability operator and the Ramsey operator can be shown to carry over to a larger class of large cardinal operators, that includes the operators $\mathcal{I}$, $\mathcal{T}$, and $\mathcal{R}$, and potentially many other operators defined by the existence of ultrafilters for weak $\kappa$-models, by uniform arguments.