The Exact Strength of the Class Forcing Theorem

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October 30, 2017

The Forcing Theorem

The *forcing theorem* is the following classical theorem about set forcing.

Theorem

Let $\mathbb{P}$ be a notion of set forcing. For every particular first order formula φ, $\mathbb{P}$ admits a forcing relation for φ, that is there are definable classes for φ and for each of its subformulas, and these classes recursively obey the properties of the forcing relation.

As a consequence, one obtains the *truth lemma*.

Corollary

If M is a model of ZFC, G is $\mathbb{P}$-generic over M, and $\varphi(\vec{\sigma}^G)$ holds in V$[G]$, then there is $p \in G$ such that $p \Vdash \varphi(\vec{\sigma})$.

Failures of the forcing theorem

The recursion that defines the atomic forcing relation for class forcing notions is a recursion of classes (because quantifiers over conditions are unbounded, so the definition in the successor step of the recursion depends on a proper class of information), and thus may not have a solution. In fact, we have shown the following.

Theorem (Holy,Krapf,Lücke,Njegomir,Schlicht - 2016)

For any model of ZFC, there is a class forcing notion for which the forcing theorem fails. In fact, there are models of ZFC with a class forcing notion for which even the truth lemma fails.

Generalizing the setup

Rather than working within models of *ZFC*, we want to work within two-sorted models of *GBC*, consisting of a collection of sets and a collection of classes, the latter containing all definable classes. This generalizes the *ZFC* context, which is the special case where all classes are definable. When we generalize the statement of the forcing theorem to this new context, rather than requiring the forcing relations to be definable classes, we now plainly require them to exist (as classes).

In analogy to the above result, there are many models of *GBC* in which there exists a notion of class forcing that fails to satisfy the forcing theorem. On the other hand, the stronger system *KM* shows that every notion of class forcing satisfies the forcing theorem. We consider the following question in the *reverse mathematics of second order set theory*:

Question

Which second order set theory corresponds to the class forcing theorem?

ETR_{Ord}

The principle of *elementary transfinite recursion* ETR_{Ord} for recursions of length *Ord* is the assertion that there exists a solution to any *Ord*-length recursive first order definition of a sequence of classes, that may involve class parameters.

Definition

ETR_{Ord} is the scheme asserting of any first order formula $\varphi(x, X, A)$ with a class parameter A, that there is a class $S \subseteq Ord \times V$ that is a solution to the following recursion

$$S_\alpha = \{x \mid \varphi(x, S \upharpoonright \alpha, A)\},$$

where $S_\alpha = \{x \mid \langle\alpha, x\rangle \in S\}$ denotes the α^{th} slice of S and $S \upharpoonright \alpha = S \cap (\alpha \times V)$ is the part of the solution prior to stage α.

While *GBC* proves that there is a solution to any *Ord*-length recursive first order definition of a sequence of sets, this is not the case for classes.

Consequences of ETR_{Ord}

ETR_{Ord} implies the following.

- There exists a truth predicate for first order formulas.
- There exists a truth predicate for $\mathcal{L}_{Ord,Ord}$-formulas.
- For every notion of class forcing $\mathbb{P}$, there exists a uniform forcing relation for first order formulas.
- The Class Forcing Theorem holds, that is, the forcing theorem holds for all notions of class forcing.

Proof: Immediate, all of the above can be defined by ω-length or *Ord*-length recursive definitions of sequences of classes in a straightforward way. The main point of this talk will be that reversely, the Class Forcing Theorem also implies ETR_{Ord}.

What we will actually show

We will use the following line of argument.

1. The Class Forcing Theorem implies that
2. there is a truth predicate for $\mathcal{L}_{Ord,\omega}$-formulas, which in turn implies that
3. there is an *Ord*-iterated truth predicate for first order truth, which in turn implies that
4. ETR_{Ord} holds.

Extending the forcing theorem

In order to show that (1) implies (2), we will make use of the following theorem.

Theorem (Holy,Krapf,Lücke,Njegomir,Schlicht – generalized)

If a class forcing notion $\mathbb{P}$ admits a forcing relation for atomic formulas, then it admits a uniform forcing relation in the quantifier-free infinitary forcing language $\mathcal{L}_{Ord,0}(\in, V^{\mathbb{P}}, \dot{G})$.

Note: This is a theorem in *GBC* and does *not* use ETR_{Ord}.

The Forcing Notions

We now define the forcing notions that we will make use of. Let A be any proper class parameter. Let C be the class partial order having as conditions all finite injective partial functions f from ω to V, the usual forcing to add a bijection from ω to V. To form a larger partial order F_A, we augment C with additional conditions $\{e_{n,m} \mid n, m \in \omega\}$ and $\{a_n \mid n \in \omega\}$. For $f \in C$, we define

$$f \leq e_{n,m} \iff f(n) \in f(m), \text{ and}$$

$$f \leq a_n \iff f(n) \in A.$$

That is, $e_{n,m}$ is the supremum of the conditions f with $f(n) \in f(m)$, and a_n is the supremum of the conditions f with $f(n) \in A$. We can use these new conditions to form new (set-sized) names, that we would not have been able to form using only conditions from C.

New Names

$$\dot{\epsilon} = \{\langle\langle \check{n}, \check{m}\rangle, e_{n,m}\rangle \mid n, m \in \omega\}.$$

$\dot{\epsilon}$ is a name for a copy of the ground model $\in$-relation onto ω, making use of the generically added bijection.

$$\dot{A} = \{\langle \check{n}, a_n\rangle \mid n \in \omega\}.$$

$\dot{A}$ is a name for a copy of A on ω.

$$\text{For every } a \in \mathrm{V}, \text{ let } \dot{n}_a = \{\langle \check{k}, \{\langle n, a\rangle\}\rangle \mid k < n \in \omega\}.$$

This name does not use our additional conditions, and is the name of the number n that will get mapped to a by our generic bijection.

A first theorem

Theorem

If F_A satisfies the forcing theorem, then there is a truth predicate for $\mathcal{L}_{Ord,\omega}(\in, A)$.

Proof: By the above theorem, we may assume that F_A has a uniform forcing relation for quantifier-free infinitary formulas. We define a translation $\varphi \mapsto \varphi^*$ from $\mathcal{L}_{Ord,\omega}(\in, A)$ to quantifier-free infinitary formulas in the forcing language of F_A recursively.

- $(x \in y)^* = x \,\dot{\epsilon}\, y$,
- $(x = y)^* = (x = y)$,
- $(x \in A)^* = x \in \dot{A}$,
- $(\wedge)^*, (\neg)^*, (\bigwedge)^*$ are defined trivially,
- $(\forall x \varphi)^* = \bigwedge_{m \in \omega} \varphi^*(\check{m})$.

Idea

Our generic bijection makes $\langle \mathrm{V}, \in, A \rangle$ isomorphic to $\langle \omega, \dot{\epsilon}, \dot{A} \rangle$. In particular, universal quantifications in the ground model correspond to conjunctions over ω in the latter structure, which is the informal idea of how we get rid of quantifiers in our translation. One now would like to establish the following intuitive equivalence (which however is not formalizable).

$$\langle \mathrm{V}, \in, A \rangle \models \varphi(a) \iff \mathrm{V}[G] \models (\langle \omega, \dot{\epsilon}^G, \dot{A}^G \rangle \models \varphi^*(\dot{n}_a^G)).$$

What we actually do: For $\varphi \in \mathcal{L}_{Ord,\omega}(\in, A)$ and $\vec{a} = \langle a_0, \ldots, a_k \rangle \in \mathrm{V}$, we define a predicate Tr as follows.

$$Tr(\varphi, \vec{a}) \iff 1 \Vdash_{F_A} \varphi^*(\dot{n}_{a_0}, \ldots, \dot{n}_{a_k}).$$

What we actually do

For $\varphi \in \mathcal{L}_{Ord,\omega}(\in, A)$ and $\vec{a} = \langle a_0, \ldots, a_k \rangle \in \mathrm{V}$, we define a predicate Tr as follows.

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It can now be verified by induction on formula complexity, in a mostly straightforward way, that Tr is a truth predicate for $\mathcal{L}_{Ord,\omega}$ formulas. As a key step however, one needs to observe the following homogeneity property of F_A.

Lemma

For any formula $\varphi \in \mathcal{L}_{Ord,\omega}(\in, A)$, any sequence $\vec{a} = \langle a_0, \ldots, a_k \rangle$, and any condition p,

$$p \Vdash \varphi^*(\dot{n}_{a_0}, \ldots, \dot{n}_{a_k}) \iff 1 \Vdash \varphi^*(\dot{n}_{a_0}, \ldots, \dot{n}_{a_k}).$$

□

Iterated truth predicates

An *Ord-iterated truth predicate for first-order truth* with a class parameter A is a class Tr consisting of triples $\langle\beta,\varphi,\vec{a}\rangle$, with $\beta\in Ord$, φ a first order formula with additional predicates A and Tr, and $\vec{a}$ a valuation of the free variables of φ, such that

- Tr judes atomic formulas ($=$, $\in$, elements of A) correctly, and recursively performs Boolean logic and quantifier logic correctly.
- Tr judges atomic assertions of the truth predicate self-coherently, that is $Tr(\beta, Tr(x,y,z),\langle\alpha,\varphi,\vec{a}\rangle)$ if and only if $\alpha<\beta$ and $Tr(\alpha,\varphi,\vec{a})$.

That is, there is an Ord-sequence of truth predicates, with the predicates in the sequence in particular also evaluating statements involving the earlier truth predicates correctly.

A second theorem

We now want to establish the implication from (2) to (3), that is starting from a truth predicate for $\mathcal{L}_{Ord,\omega}$-formulas, we want to obtain an *Ord*-iterated first order truth predicate.

Theorem

For any class A, if there is an $\mathcal{L}_{Ord,\omega}(\in, A)$-truth predicate T, then there is an Ord-iterated truth predicate I for the first order language with the class predicate A.

Proof: We define another syntactic translation, this time inductively taking pairs $\langle\beta,\varphi\rangle$ to $\mathcal{L}_{Ord,\omega}$-formulas φ^*_β, where β is an ordinal and $\varphi \in \mathcal{L}_\in(Tr, A)$. This translation is trivial for formulas not mentioning the truth predicate. $Tr(x,y,z)^*_\beta$ is the assertion

$$\bigvee_{\xi<\beta,\psi\in\mathcal{L}_\in(Tr,A)} \left[x=\xi, y=\psi, z=\vec{a} \text{ and } \psi^*_\xi(\vec{a})\right].$$

Finishing the proof

$Tr(x, y, z)^*_\beta$ is the assertion

$$\bigvee_{\xi<\beta,\psi\in\mathcal{L}_\in(Tr,A)} \left[x = \xi, y = \psi, z = \vec{a} \text{ and } \psi^*_\xi(\vec{a})\right].$$

One needs to observe that $Tr(x, y, z)^*_\beta$ is indeed an $\mathcal{L}_{Ord,\omega}(\in, A)$-formula. This is easy, using that every set is $\mathcal{L}_{Ord,\omega}$-definable, and so in particular the statement $y = \psi$ can be replaced by a formula that holds for y exactly if y represents the formula ψ. We can now define our proposed iterated truth predicate $I(\beta, \varphi, \vec{a})$ to hold if and only if $T(\varphi^*_\beta, \vec{a})$. It is straightforward to check that this this relation fulfills the requirements of an iterated first order truth predicate. □

A third theorem

We now want to establish the final implication from (3) to (4).

Theorem

ETR_{Ord} is equivalent to the existence of an Ord-iterated first order truth predicate, allowing any class parameter in each case.

The proof of the forward direction is pretty immediate, as the *Ord*-iterated truth predicate can be defined by a class recursion of length *Ord*. The proof of the converse (that is the implication from (3) to (4)) is a refinement of an argument by Kentaro Fujimoto, that is due to Joel Hamkins and Victoria Gitman. The idea is that using the truth predicate, one can define a solution to any relevant recursion. As the class parameter plays no role in the proof to come, we will omit it.

Proof pt. 1

Proof: Assume that we have an iterated truth predicate Tr of length Ord for first order formulas. Suppose further that we have an instance of ETR_{Ord}, iterating a formula $\varphi(x, X)$, where we seek a solution S, that is a class $S \subseteq Ord \times V$ for which

$$S_\alpha := \{x \mid \langle \alpha, x\rangle \in S\} = \{x \mid \varphi(x, S \restriction \alpha)\}$$

for every ordinal α. We claim that using the iterated truth predicate as a class parameter, we may define such a solution S.

Proof pt. 2

To do this, we first claim that there is a formula $\bar{\varphi} \in \mathcal{L}(Tr)$ such that if one extracts from Tr the class defined by $\bar{\varphi}$, namely

$$(*)\ S = \{\langle \alpha, x \rangle \mid Tr(\alpha, \bar{\varphi}, x)\},$$

then S is our required solution. $\bar{\varphi}$ should be chosen so that

$$\bar{\varphi}(x, Tr \restriction \alpha) \iff \varphi(x, S \restriction \alpha)$$

holds for all x and all $\alpha \in Ord$, where $S \restriction \alpha$ is defined from $Tr \restriction \alpha$ according to (*). It follows inductively that for every $\alpha \in Ord$, $\{x \mid \varphi(x, S \restriction \alpha)\} = \{x \mid \bar{\varphi}(x, Tr \restriction \alpha)\} = \{x \mid Tr(\alpha, \bar{\varphi}, x)\} = S_\alpha$, as desired.

Proof pt. 3

$\bar{\varphi}$ should be chosen so that

$$\bar{\varphi}(x, Tr \restriction \alpha) \iff \varphi(x, S \restriction \alpha).$$

The formula $\bar{\varphi}$ exists by the Gödel-Carnap fixed point lemma (GCF). For any $e \in \mathcal{L}(Tr)$ and $\alpha \in Ord$, let

$$Tr_e \restriction \alpha = \{\langle \beta, x \rangle \mid \beta < \alpha \wedge Tr(\beta, e, x)\},$$

and let $\psi(e, x, Tr \restriction \alpha)$ be the assertion $\varphi(x, Tr_e \restriction \alpha)$, noting that $Tr_e \restriction \alpha$ is definable from $Tr \restriction \alpha$ for any e. By the GCF, find $\bar{\varphi}$ such that

$$\psi(\bar{\varphi}, x, Tr \restriction \alpha) \iff \bar{\varphi}(x, Tr \restriction \alpha).$$

This implies our desired equivalence, for S is defined to be $Tr_{\bar{\varphi}}$. $\square$