Generalized topologies on 2^κ, Silver forcing, and $\diamondsuit_\kappa$

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29.05.2020

Tree forcing topologies

Fix a cardinal κ.
Let $\mathcal{P}(\kappa) \approx 2^\kappa = \{g \mid g\colon \kappa \to 2\}$, and let $2^{<\kappa} = \{g \mid \exists\alpha < \kappa\, g\colon \alpha \to 2\}$.

Definition 1

- A *κ-tree* is a subset of $2^{<\kappa}$ closed under initial segments.
- A *branch* through a κ-tree T is some $x \in 2^\kappa$ such that $x \restriction \alpha \in T$ for every $\alpha < \kappa$. $[T] \subseteq 2^\kappa$ denotes the set of all branches through T.
- A *tree forcing* notion P on κ is a notion of forcing in which conditions are κ-trees, including the full tree $2^{<\kappa}$, ordered by inclusion.
- Such a forcing notion P is *topological* if for any two $R, S \in P$ and any $x \in [R] \cap [S]$, there is $T \in P$ such that $x \in [T] \subseteq [R] \cap [S]$.
- If P is a topological notion of tree forcing on κ, we let the *P-topology* be the topology on 2^κ generated by the basic open sets of the form $[T]$, for conditions $T \in P$.

The conditions in κ-Cohen forcing are the elements of $2^{<\kappa}$, ordered by reverse inclusion. But we can also identify κ-Cohen forcing with a tree forcing notion: Given $s \in 2^{<\kappa}$, let

$$T_s = \{t \in 2^{<\kappa} \mid t \subseteq s \vee s \subseteq t\}.$$

It is easy to see that κ-Cohen forcing corresponds to the tree forcing notion consisting of conditions T_s for $s \in 2^{<\kappa}$, and that the topology generated by κ-Cohen forcing (when viewed as a tree forcing notion on κ) is the standard bounded topology on 2^κ.

Ideal Topologies

Let κ be a cardinal, and let $\mathcal{I}$ be an ideal on κ.
Given a partial function f from κ to 2, let $[f] = \{g \in 2^\kappa \mid f \subseteq g\}$.

Definition 2

The $\mathcal{I}$-*topology* is the topology on 2^κ with basic (cl)open sets of the form $[f]$ where $\mathrm{dom}(f) \in \mathcal{I}$.

Ideal topologies are in fact a special case of tree forcing topologies.

- We call open sets in the $\mathcal{I}$-topology $\mathcal{I}$-open sets, and similarly for other notions: $\mathcal{I}$-closed, ...
- We will also do so for forcing topologies: P-open, P-closed, ...
- In case $\mathcal{I} = \mathrm{NS}_\kappa$, we refer to the $\mathcal{I}$-topology as the *nonstationary topology*, in which the basic open sets are thus *induced* by functions with non-stationary domain.

Grigorieff forcing

Definition 3

Let κ be an infinite cardinal and let $\mathcal{I}$ be an ideal on κ. $\mathcal{G}_{\mathcal{I}}$, *Grigorieff forcing with the ideal* $\mathcal{I}$ is the notion of forcing consisting of conditions which are partial functions p from κ to 2 such that $\mathrm{dom}(p) \in \mathcal{I}$, ordered by inclusion.

We can view $\mathcal{G}_{\mathcal{I}}$ as a tree forcing by identifying a condition $p \in \mathcal{G}_{\mathcal{I}}$ with the tree T on $2^{<\kappa}$ which we inductively construct as follows:

$\emptyset \in T$. Given $t \in T$ of order-type α, let $t^\frown 0 \in T$ if $p(\alpha) \neq 1$, and let $t^\frown 1 \in T$ if $p(\alpha) \neq 0$ (these are both supposed to include the cases when α is not in the domain of p). At limit levels α, we extend every branch through the tree constructed so far.

It is easy to see that these two forcings are isomorphic. Then, if T is the tree on $2^{<\kappa}$ corresponding to the condition $p \in \mathcal{G}_{\mathcal{I}}$, we have $[T] = [p]$. Hence, the $\mathcal{G}_{\mathcal{I}}$-topology is exactly the $\mathcal{I}$-topology, and $\mathcal{G}_{\mathcal{I}}$ is topological.

κ-Silver forcing

Definition 4

Let κ be a regular uncountable cardinal. *κ-Silver forcing* (or *κ-club Silver forcing*) $\mathbb{V}_\kappa$ is the notion of forcing consisting of conditions p which are partial functions from κ to 2 such that the complement of the domain of p is a club subset of κ.

Note that $\mathbb{V}_\kappa$ is a dense subset of Grigorieff forcing with NS_κ. This yields that $\mathbb{V}_\kappa$ can be viewed as a κ-tree forcing notion. In fact, whenever p is a condition in $\mathcal{G}_{\mathrm{NS}_\kappa}$ and $x \in 2^\kappa$ is such that $p \subseteq x$, then p can be extended to a condition $q \subseteq x$ in $\mathbb{V}_\kappa$. This easily yields that those two notions of forcing generate the same topology, and hence that the $\mathbb{V}_\kappa$-topology is exactly the nonstationary topology.

Unsursprisingly, combinatorial properties of tree forcing notions P yield properties of their corresponding topologies. For example, if P is $<\kappa$-distributive, then the P-topology yields a κ-Baire space (i.e., the intersection of κ-many open dense sets of that space is nonempty).

Friedman, Khomskii and Kulikov (*Regularity Properties of the generalized Reals, Annals of Pure and Applied Logic, 2016*) investigated such consequences of a slight strengthening of Axiom A for κ-tree forcing notions. If κ is inaccessible, the classical proof that Silver forcing satisfies Axiom A also shows that $\mathbb{V}_\kappa$ satisfies this strong form of Axiom A. We are going to show that a more intricate argument yields the same result under the assumption of $\diamondsuit_\kappa$ – note that by results of Shelah, $\diamondsuit_\kappa$ holds whenever $\kappa > \omega_1$ is a successor cardinal for which $2^{<\kappa} = \kappa$. This will allow us to infer results on the nonstationary topology on 2^κ for many cardinals κ (namely, all regular cardinals $\kappa > \omega_1$ that satisfy $2^{<\kappa} = \kappa$, and also for $\kappa = \omega_1$ in case $\diamondsuit_{\omega_1}$ holds).

Axiom A^*

The following slight strengthening of Axiom A for κ-tree forcing notions was introduced by Friedman, Khomskii and Kulikov:

Definition 5

A notion $\langle P, \leq \rangle$ of tree forcing on κ satisfies Axiom A^* if there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$ with $\leq_0 = \leq$, satisfying:

1. $q \leq_\beta p$ implies $q \leq_\alpha p$ (i.e., $\leq_\beta \subseteq \leq_\alpha$) for all $\alpha \leq \beta$.
2. If $\langle p_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions in P and $\lambda \leq \kappa$, satisfying that $p_\beta \leq_\alpha p_\alpha$ for all $\alpha < \beta < \lambda$, then there is $q \in P$ such that $q \leq_\alpha p_\alpha$ for all $\alpha < \lambda$.
3. For all $p \in P$, all D that are dense below p in P, and all $\alpha < \kappa$, there exists $E \subseteq D$ of size at most κ, and $q \leq_\alpha p$ such that E is predense below q, and such that additionally $[q] \subseteq \bigcup \{[r] \mid r \in E\}$.

Note: Without the final condition on $[q]$, this is just the usual version of Axiom A at κ.

Friedman-Khomskii-Kulikov

Theorem 6 [Friedman-Khomskii-Kulikov]

If a tree forcing notion P satisfies Axiom A^*, then the nowhere dense sets in the P-topology are closed under κ-unions, i.e., all P-meager sets are P-nowhere dense.

Corollary 7

If κ is inaccessible and $\mathcal{I} = \mathrm{NS}_\kappa$, then $\mathcal{I}$-meager $\equiv$ $\mathcal{I}$-nowhere dense.

Definition 8

$X \subseteq 2^\kappa$ satisfies the property of Baire in the P-topology in case X can be written in the form $X = \mathcal{O} \Delta M$, where $\mathcal{O}$ is P-open, and M is P-meager.

Theorem 9 [Friedman-Khomskii-Kulikov]

If κ is inaccessible and every Δ^1_1-subset of 2^κ satisfies the property of Baire (in the bounded topology) – which is consistent relative to ZFC – then it does so also in the $\mathbb{V}_\kappa$-topology, i.e., the nonstationary topology on 2^κ.

Axiom A^*, once again

Let me remind you once again about Axiom A^*:

Definition 5

A notion $\langle P, \leq \rangle$ of tree forcing on κ satisfies Axiom A^* if there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$ with $\leq_0 = \leq$, satisfying:

1. $q \leq_\beta p$ implies $q \leq_\alpha p$ (i.e., $\leq_\beta \subseteq \leq_\alpha$) for all $\alpha \leq \beta$.
2. If $\langle p_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions in P and $\lambda \leq \kappa$, satisfying that $p_\beta \leq_\alpha p_\alpha$ for all $\alpha < \beta < \lambda$, then there is $q \in P$ such that $q \leq_\alpha p_\alpha$ for all $\alpha < \lambda$.
3. For all $p \in P$, all D that are dense below p in P, and all $\alpha < \kappa$, there exists $E \subseteq D$ of size at most κ, and $q \leq_\alpha p$ such that E is predense below q, and such that additionally $[q] \subseteq \bigcup \{[r] \mid r \in E\}$.

κ-Silver forcing satisfies Axiom A^*

Theorem 10

If $\diamondsuit_\kappa$ holds, then $\mathbb{V} = \mathbb{V}_\kappa$ satisfies Axiom A^*.

Proof: For any $\alpha < \kappa$ and $p, q \in \mathbb{V}$, let $q \leq_\alpha p$ if $q \leq p$ and the first α-many elements of the complements of the domains of p and of q are the same. It is clear (or at least easy to check) that Items (1) and (2) in Definition 5 are thus satisfied, and we only have to verify Item (3).

Let $p \in \mathbb{V}$, let $\alpha < \kappa$, and let $D \subseteq \mathbb{V}$ be dense below p. We need to find $q \leq_\alpha p$ and $E \subseteq D$ of size at most κ such that E is predense below q. Fix a $\diamondsuit_\kappa$-sequence $\langle A_i \mid i < \kappa \rangle$: $\forall A \subseteq \kappa$ $\{i < \kappa \mid A \cap i = A_i\}$ is a stationary subset of κ.

We inductively construct a decreasing sequence $\langle p_i \mid i \leq \kappa \rangle$ of conditions in $\mathbb{V}$ with $p_i = p$ for $i \leq \alpha$, and a sequence $\langle \alpha_i \mid i < \kappa \rangle$ of ordinals with the property that $\langle \alpha_j \mid j \leq i \rangle$ enumerates the first $(i+1)$-many elements of $\kappa \setminus \mathrm{dom}(p_i)$ for every $i \leq \kappa$, as follows. Let $\langle \alpha_i \mid i \leq \alpha \rangle$ enumerate the first $\alpha + 1$-many elements of the complement of the domain of p.

Assume that we have constructed p_i for some $i \geq \alpha$, and also α_j for $j \leq i$.

Using that D is dense below p, let $q_i^0 \leq p_i$ be such that

- $q_i^0(\alpha_j) = A_i(j)$ for all $j < i$,
- $q_i^0(\alpha_i) = 0$, and
- $q_i^0 \in D$,

and let $q_i^1 \leq q_i^0 \upharpoonright (\mathrm{dom}(q_i^0) \setminus \{\alpha_i\})$ be such that

- $q_i^1(\alpha_i) = 1$, and
- $q_i^1 \in D$.

Let $p_{i+1} = q_i^1 \upharpoonright (\mathrm{dom}(q_i^1) \setminus \{\alpha_j \mid j \leq i\})$, and note that $p_{i+1} \leq_i p_i$.
Let α_{i+1} be the least element of $\kappa \setminus \mathrm{dom}(p_{i+1})$ above α_i.

For limit ordinals $i \leq \kappa$, let $p_i = \bigcup_{j<i} p_j$, and if $i < \kappa$, let $\alpha_i = \bigcup_{j<i} \alpha_j$ be the least element of $\kappa \setminus \mathrm{dom}(p_i)$. Let $q = p_\kappa$, and let $E = \{q_i^0 \mid i < \kappa\} \cup \{q_i^1 \mid i < \kappa\}$. To verify Axiom A, we want to show that E is predense below q.

Thus, let $r \leq q$ be given. Using the properties of our diamond sequence, pick $i < \kappa$ such that $i \geq \alpha$, and such that for all $j < i$ with $\alpha_j \in \mathrm{dom}(r)$, $A_i(j) = r(\alpha_j)$. Pick $\delta \in \{0, 1\}$ such that $r(\alpha_i) = \delta$ in case $\alpha_i \in \mathrm{dom}(r)$. Then, q_i^δ is compatible to r, as desired.

In order to check the additional property for Axiom A^*, note that any extension s of q to a total function from κ to 2 can be treated in the same way as r above, yielding some $i < \kappa$ and $\delta \in \{0, 1\}$ such that $s \in [q_i^\delta]$. $\square$

So what does Axiom A^* have to do with meager sets?

In order to properly connect topics, let me present the following result:

Lemma 11 [Friedman-Khomskii-Kulikov]

If a κ-tree forcing notion P satisfies Axiom A^* (the proof uses quite a bit less), then every P-meager set is P-nowhere dense.

Proof: Let $\{A_i \mid i < \kappa\}$ be a collection of P-nowhere dense sets. We need to show that $\bigcup_{i<\kappa} A_i$ is P-nowhere dense. For every $i < \kappa$, let D_i be the dense subset $D_i = \{p \mid [p] \cap A_i = \emptyset\}$ of P, using that A_i is P-nowhere dense. Using Axiom A^*, construct $\langle p_i \mid i < \kappa \rangle$ and $\langle E_i \subseteq D_i \mid i < \kappa \rangle$, such that for all $i < j \leq \kappa$,

- $p_j \leq_i p_i$, and
- $[p_i] \subseteq \bigcup \{[p] \mid p \in E_i\}$.

Let $q \leq p_i$ for all i. Hence, for every $i < \kappa$, $[q] \subseteq \bigcup \{[p] \mid p \in D_i\}$. In particular, $[q] \cap A_i = \emptyset$ for all $i < \kappa$, hence $\bigcup_{i<\kappa} A_i$ is P-nowhere dense. $\square$

P-measurability

We will need the following, the forward direction of which is immediate:

Lemma 12 [Friedman-Khomskii-Kulikov]

If P is a topological notion of forcing that satisfies Axiom A^*, then $X \subset 2^\kappa$ satisfies the Baire property in the P-topology if and only if

$$\forall T \in P \; \exists S \leq T \; ([S] \subseteq X \vee [S] \cap X = \emptyset).$$

In particular, for $\mathcal{I} = \mathrm{NS}_\kappa$, $X \subseteq \kappa$ satisfies the $\mathcal{I}$-Baire property if every $\mathcal{I}$-basic open set $[f]$ contains an $\mathcal{I}$-basic open set $[g]$ such that either $[g] \subseteq X$ or $[g] \cap X = \emptyset$.

On the Baire property

Quite similar arguments as for P-meager $\equiv$ P-nowhere dense (without the intermediate principle of Axiom A^*) show the following, where the case of inaccessible κ is implicit in Friedman-Khomskii-Kulikov:

Theorem 13

If κ is inaccessible or $\diamondsuit_\kappa$ holds, then every κ-intersection of open dense subsets of 2^κ (in the bounded topology) contains a dense set, that is additionally open in the nonstationary topology.

This allows us to show the following, again due to Friedman et al. in the case of inaccessible κ (and the proof below is essentially theirs):

Theorem 14

If κ is inaccessible or $\diamondsuit_\kappa$ holds, and every Δ^1_1-subset of 2^κ has the Baire property (which is known to be consistent relative to ZFC), then it does so also in the nonstationary topology.

Proof of Theorem 14:

Let P denote κ-Silver forcing, let $\mathcal{I} = \mathrm{NS}_\kappa$. Let $A \in \Delta^1_1$, and let $f \in P$. We need to find $g \leq f$ such that either $[g] \subseteq A$ or $[g] \cap A = \emptyset$. Let C denote the club subset of κ that is the complement of the domain of f, and enumerate C in increasing order as $\langle c_\gamma \mid \gamma < \kappa \rangle$. Let φ denote the natural order-preserving bijection between $2^{<\kappa}$ and extensions of f by bounded functions: Given $s \in 2^\alpha$ with $\alpha < \kappa$, let $\varphi(s)$ be the $\subseteq$-minimal $g \in P$ such that g extends f and $g(c_\gamma) = s(\gamma)$ for every $\gamma < \alpha$. Let φ^* be the induced homeomorphism between 2^κ and $[f]$. Let $A' = \varphi^*[A]$, which is again a Δ^1_1-subset of 2^κ, using that Δ^1_1 is closed under continuous preimages. Hence, A' has the Baire property, by our assumption. This means that either A' is meager, or it is comeager in some basic open set $[s]$ of the bounded topology on 2^κ. If A' is meager, Theorem 12 yields an $\mathcal{I}$-open set $[t]$ that is disjoint from A'. If A' is comeager in $[s]$, applying Theorem 12 relativized to $[s]$, we find an $\mathcal{I}$-open set $[t] \subseteq A' \cap [s]$. But then, in either case, $[g] := (\varphi^*)^{-1}[[t]] \subseteq [f]$ is an $\mathcal{I}$-open set that is either disjoint from or contained in A, as desired. □

A further result – Comparing notions of meagerness

Let $\mathcal{I} = \mathrm{NS}_\kappa$.

Observation 15

If $[f]$ is an $\mathcal{I}$-basic open set, with $\mathrm{dom}(f)$ of size κ, then $[f]$ is meager (in fact, nowhere dense) in the bounded topology. Thus, there is always a meager set that is not $\mathcal{I}$-meager.

Observation 16

Every set of size less than 2^κ is $\mathcal{I}$-meager. Hence, if $\mathrm{non}(\mathcal{M}_\kappa) < 2^\kappa$, then there is an $\mathcal{I}$-meager set that is not meager.

Theorem 17

If κ is inaccessible or $\diamondsuit_\kappa$ holds, and the reaping number $\mathfrak{r}(\kappa) = 2^\kappa$, then there is an $\mathcal{I}$-meager set which does not have the Baire property (and thus in particular is not meager) in the bounded topology.

Open Questions

We have answered the following positively whenever κ is inaccessible or $\diamondsuit_\kappa$ holds.

Question 18

- Does κ-Silver forcing satisfy Axiom A^* whenever κ is regular and uncountable?
- If κ is regular and uncountable, and $\mathcal{I} = \mathrm{NS}_\kappa$, are $\mathcal{I}$-meager sets always $\mathcal{I}$-nowhere dense?

We know the following holds for many κ, at least under certain assumptions on generalized cardinal invariants.

Question 19

Let $\mathcal{I} = \mathrm{NS}_\kappa$. Is there always an $\mathcal{I}$-meager set that is not meager?