LARGE CARDINAL OPERATORS AND ELEMENTARY EMBEDDINGS

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Abstract. We introduce and investigate a uniform framework for large cardinal operators. This framework accomodates the Ramsey operator, and we will show that it also accommodates the subtle operator, the ineffability operator, and the pre-Ramsey operator. We use this framework to introduce a new large cardinal operator, and to show that operator to be strictly intermediate between the ineffability operator and the Ramsey operator. This new operator is closely connected to the notion of a $\mathbf{T}^\kappa_\omega$-Ramsey cardinal that was recently introduced by Philipp Lücke and the author. As a test application for our framework, we show that a strong form of the key results of James Baumgartner connecting ineffability to subtlety, and Ramseyness to pre-Ramseyness, generalizes to the context of our abstract large cardinal operators.

1. Introduction

In the set theoretic literature, some popular large cardinals have been connected to corresponding large cardinal ideals, and then also to operators on ideals, the earliest examples of the latter being the ineffability operator $\mathcal{I}$ due to James Baumgartner in [4], followed by the Ramsey operator $\mathcal{R}$ and the pre-Ramsey operator $\mathcal{R}_0$ that were introduced and extensively studied by Qi Feng in [7], while the subtle operator $\mathcal{I}_0$ was first made explicit in a recent paper by Brent Cody ([6]). In the present paper, inspired by the large cardinal framework based on the existence of certain ultrafilters for small models of set theory that was introduced in [11], we introduce such a framework for large cardinal operators. We show that the four operators mentioned above fit into this framework, provide some general results about these operators, and use this framework to introduce some new large cardinal operators. In particular, we introduce a large cardinal operator that is closely connected to the notion of a $\mathbf{T}^\kappa_\omega$-Ramsey cardinal that was introduced by Philipp Lücke and the author in [11], and show that, in a certain strong sense, this operator is strictly intermediate between the ineffability operator and the Ramsey operator. As a sidenote, we also make a small comment on a certain property of subtle cardinals that hints towards a negative answer for a question asked in [12]. Towards the end of the paper, as a kind of test application for our generalized operators, we show that one of the key results of Baumgartner ([3] and [4]) about the ineffable and the Ramsey operator, connecting them to the subtle operator and to the pre-Ramsey operator respectively, holds for our generalized operators (in a strong form, which

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is due to Cody for the Ramsey operator in [6]). Finally, we make some comments and ask some questions related to the notion of weak ineffability.

Without further mention, we will always require all ideals to be ideals on some regular and uncountable cardinal κ, and to be supersets of the bounded ideal on κ. For any ideal I, I^+ denotes the collection of I-positive sets, that is, those subsets of κ which are not in I, while I^* denotes the filter that is dual to I, that is, the collection of complements of sets in I. We will often introduce ideals by defining the collection of their positive sets when this is more convenient.

Let us start by introducing the two classical examples of ideal operators, the ineffability operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$. The definition of the Ramsey operator that is provided below is not the original definition from [7], but a version that was shown to be equivalent in [6, Proposition 2.8]. Recall that for any set A, an *A-list* is a sequence $\langle a_x \mid x \in A\rangle$ such that $a_x \subseteq x$ for any $x \in A$.

Definition 1.1. *Let I be an ideal on κ.*

- *Given a κ-list $\vec{a}$, we first define what we call a* local instance*, letting*
$$\mathcal{I}^{\vec{a}}(I)^+ = \{x \subseteq \kappa \mid \exists H \in I^+\ H \subseteq x \text{ is homogeneous for } \vec{a}\},$$
and we let $\mathcal{I}(I)^+ = \bigcap\{\mathcal{I}^{\vec{a}}(I)^+ \mid \vec{a} \text{ is a } \kappa\text{-list}\}$.
- *Given a regressive function $c\colon [\kappa]^{<\omega} \to \kappa$, we define a local instance, letting*
$$\mathcal{R}^c(I)^+ = \{x \subseteq \kappa \mid \exists H \in I^+\ H \subseteq x \text{ is homogeneous for } c\},$$
and we let $\mathcal{R}(I)^+ = \bigcap\{\mathcal{R}^c(I)^+ \mid c\colon [x]^{<\omega} \to \kappa \text{ regressive}\}$.

Fact 1.2.

(1) If κ is weakly ineffable, then $\mathcal{I}([\kappa]^{<\kappa})$ is the weakly ineffable ideal on κ.
(2) If κ is ineffable, then $\mathcal{I}(\mathrm{NS}_\kappa)$ is the ineffable ideal on κ.
(3) If κ is Ramsey, then $\mathcal{R}([\kappa]^{<\kappa})$ is the Ramsey ideal on κ.
(4) If κ is ineffably Ramsey, then $\mathcal{R}(\mathrm{NS}_\kappa)$ is the ineffably Ramsey ideal on κ.

Proof. (1) holds by the very definition of weak ineffability (with the original terminology by Baumgartner being *almost ineffable*) in [3]. (2) holds by the very definition of ineffability in [3]. (3) and (4) follow from [6, Proposition 2.8]. □

In [11], three schemes were proposed that allow for the characterization of a large number of large cardinals up to measurability in a uniform way. In the present paper, we will focus on one of these schemes, that was called *Scheme B*.

Scheme B: An uncountable cardinal κ has the large cardinal property $\Phi(\kappa)$ if and only if for any $y \subseteq \kappa$, there is a transitive weak κ-model M with $y \in M$, and a uniform M-ultrafilter U on κ for which $\Psi(M, U)$ holds.

Ineffability and Ramseyness are two instances of this scheme. By a minor adaptation (namely, by additionally asking whether $A \in U$), this scheme can be used to characterize not only large cardinals, but certain large subsets A of large cardinals (as was also extensively done in [11]). Regarding the above, these are the ineffable and the Ramsey sets respectively. Item (1) below extends a result of Abramson, Harrington, Kleinberg and Zwicker ([1, Corollary 1.3.1]), and is due to Philipp Lücke and the author ([11, Theorem 8.1]). Item (2) below is essentially due to William Mitchell ([14]), and was isolated by Ian Sharpe and Philip Welch in [16, Theorem 3.3] (see also [6, Theorem 2.10]). We say that M is a weak κ-model if

$M \supseteq \kappa+1$ is of size κ and a model of ZFC$^-$. Recall that an M-ultrafilter U on κ is *κ-amenable for M* if whenever $\mathcal{A} \in M$ is a κ-sized collection of subsets of κ in M, then $\mathcal{A} \cap U \in M$.

Theorem 1.3. *(1) (Holy, Lücke) $x \subseteq \kappa$ is ineffable if for every $y \subseteq \kappa$ there is a transitive weak κ-model M with $y \in M$ and an M-ultrafilter U on κ with $x \in U$ such that ΔU is stationary.*

(2) (Mitchell; Sharpe, Welch) $x \subseteq \kappa$ is Ramsey if for every $y \subseteq \kappa$ there is a transitive weak κ-model M with $y \in M$ and a countably complete and M-ultrafilter U on κ with $x \in U$ that is κ-amenable for M. Equivalently, we can additionally require U to be M-normal.

One of the goals of this paper is to extend these characterizations even further, namely to the corresponding large cardinal operators that are the ineffability operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$, [1] and to show that these characterizations naturally induce related characterizations of the subtle operator and of the pre-Ramsey operator.

2. The ineffability operator

In this section, we show that we can characterize the ineffability operator $\mathcal{I}$ via the existence of certain ultrafilters for small collections of sets (this characterization will be needed in Section 11), and then we show that for input values I containing the nonstationary ideal, we can characterize $\mathcal{I}$ also via the existence of certain ultrafilters for small models of set theory. As a reference to the notion of *flipping property* that was introduced in [1], if $\vec{x} = \langle x_\xi \mid \xi < \kappa\rangle$ is a sequence of subsets of κ, let us say that a set $U = \{u_\xi \mid \xi < \kappa\}$ *flips* $\vec{x}$ if $u_\xi \in \{x_\xi, \kappa \setminus x_\xi\}$ for every $\xi < \kappa$. Assume that we have fixed U to be such a *flip* of a sequence $\vec{x}$. Then, somewhat sloppily, we write ΔU to abbreviate $\Delta_{\xi<\kappa} u_\xi$.

Definition 2.1. *For any ideal I on κ and $\mathcal{A} \in [\mathcal{P}(\kappa)]^\kappa$, we first define a local instance, letting*

- *$x \in \mathcal{I}_C^{\mathcal{A}}(I)^+$ if $x \notin \mathcal{A}$, or for any κ-enumeration $\vec{x}$ of $\mathcal{A}$, there is a set U that flips $\vec{x}$ such that $x \in U$ and $\Delta U \in I^+$, and we let*
- *$\mathcal{I}_C(I)^+ = \bigcap_{\mathcal{A}\in[\mathcal{P}(\kappa)]^\kappa} \mathcal{I}_C^{\mathcal{A}}(I)^+$,*

The following result extends [1, Theorem 1.2.1 and Corollary 1.3.1].

Proposition 2.2. *Let I be an ideal on κ. Then, $\mathcal{I}(I) = \mathcal{I}_C(I)$.*

Proof. Assume first that $\vec{a}$ is a κ-list, and that $x \in \mathcal{I}_C(I)^+$. Define a sequence $\vec{r}$ by setting $r_0 = x$, and for every $\xi < \kappa$, $r_{1+\xi} = \{\alpha \in x \mid \xi \in a_\alpha\}$. Making use of our assumption, we may pick a set $U = \{u_\xi \mid \xi < \kappa\}$ that flips $\vec{r}$ such that $x = u_0 \in U$ and $\Delta_{\xi<\kappa} u_\xi \in I^+$, and therefore also $H := \Delta_{\xi<\kappa} u_\xi \setminus \omega \in I^+$. Fix $\alpha < \beta$ in H and $\xi < \alpha$. Then, since also $1+\xi < \alpha$, both α and β are elements of $u_{1+\xi}$. Thus, if $r_{1+\xi} = u_{1+\xi} \in U$, we have $\xi \in a_\alpha \cap a_\beta$. Otherwise, both α and β are elements of $\kappa \setminus r_{1+\xi}$, and hence $\xi \notin a_\alpha \cup a_\beta$. Together, this shows that $a_\alpha = a_\beta \cap \alpha$, and therefore that $H \in I^+$ is homogeneous for $\vec{a}$. Since $H \subseteq x$, we have $x \in \mathcal{I}^{\vec{a}}(I)^+$, as desired.

For the other direction, we assume that $x \in \mathcal{I}(I)^+$, and let $x \in \mathcal{A} \in [\mathcal{P}(\kappa)]^\kappa$. Pick an enumeration $\langle x_\xi \mid \xi < \kappa\rangle$ of $\mathcal{A}$, and let $\vec{a}$ be defined by setting, for every

[1] For $\mathcal{R}$, this was in fact already done by Sharpe and Welch in [16, Theorem 3.3].

$\alpha \in x$, $a_\alpha = \{\xi < \alpha \mid \alpha \in x_\xi\}$. By our assumption, there is $H \subseteq x$ in I^+ that is homogeneous for $\vec{a}$. We may thus pick $A \subseteq \kappa$ such that $a_\alpha = A \cap \alpha$ for every $\alpha \in H$. Given $\xi < \kappa$, let $u_\xi = x_\xi$ if $\xi \in A$, and let $u_\xi = \kappa \setminus x_\xi$ otherwise. Let $U = \{u_\xi \mid \xi < \kappa\}$. $x \in \mathcal{I}_C(I)^+$ is now a consequence of the following claim.

Claim 2.3. *$x \in U$, and $\Delta_{\xi<\kappa} u_\xi \in I^+$.*

Proof. $H \setminus (\xi + 1) \subseteq x_\xi$ for all $\xi \in A$, and $H \cap x_\xi \subseteq \xi + 1$ for $\xi \in \kappa \setminus A$. Hence $H \setminus (\xi + 1) \subseteq u_\xi$ for all $\xi < \kappa$, yielding that $x \in U$ and $H \subseteq \Delta_{\xi<\kappa} u_\xi \in I^+$. □

□

We next introduce what we call the *model version of the ineffability operator.*

Definition 2.4.
- *For any $y \subseteq \kappa$, we first define a local instance, letting $x \in \mathcal{I}_M^y(I)^+$ if there is a transitive weak κ-model M with $y \in M$, and an M-ultrafilter U on κ with $x \in U$, such that every diagonal intersection of U is in I^+ – somewhat sloppily, we want to abbreviate this latter property of U and of I again by simply stating that $\Delta U \in I^+$.*[2]
- *Let $\mathcal{I}_M(I)^+ = \bigcap_{y\subseteq\kappa} \mathcal{I}_M^y(I)^+$.*

The following result extends [11, Theorem 8.1].

Proposition 2.5. *Let $I \supseteq \mathrm{NS}_\kappa$ be an ideal on κ. Then, $\mathcal{I}_M(I) = \mathcal{I}(I)$. We show this by verifying the following to hold.*
- *For any ideal I, if $\vec{a}$ is a κ-list, and $y \subseteq \kappa$ codes $\vec{a}$, then $\mathcal{I}_M^y(I) \supseteq \mathcal{I}^{\vec{a}}(I)$.*
- *If $I \supseteq \mathrm{NS}_\kappa$, then $\mathcal{I}_M(I) \subseteq \mathcal{I}(I)$.*

Proof.
- Let $\vec{a}$ be a κ-list, let $y \subseteq \kappa$ code $\vec{a}$, and let $x \in \mathcal{I}_M^y(I)^+$. We may pick a transitive weak κ-model M witnessing that $x \in \mathcal{I}_M^y(I)^+$, and an M-ultrafilter U on κ such that $x \in U$ and $\Delta U \in I^+$. Define, for every $\xi < \kappa$, $r_\xi = \{\alpha \in x \mid \xi \in a_\alpha\}$. Let $u_\xi = r_\xi$ if $r_\xi \in U$, and let $u_\xi = \kappa \setminus r_\xi$ otherwise, for every $\xi < \kappa$. Then, also $H := x \cap \Delta_{\xi<\kappa} u_\xi \in I^+$. Fix $\alpha < \beta$ in H and $\xi < \alpha$. Then, both α and β are elements of u_ξ. If $r_\xi = u_\xi \in U$, then $\xi \in a_\alpha \cap a_\beta$. Otherwise, both α and β are elements of $\kappa \setminus r_\xi$, and hence $\xi \notin a_\alpha \cup a_\beta$. Together, this shows that $a_\alpha = a_\beta \cap \alpha$, and therefore that $H \in I^+$ is homogeneous for $\vec{a}$. Since $H \subseteq x$, we have $x \in \mathcal{I}^{\vec{a}}(I)^+$, as desired.
- Let $y \subseteq \kappa$ and $x \in \mathcal{I}(I)^+$. Let M be a transitive weak κ-model such that $x, y \in M$. Pick an enumeration $\vec{x} = \langle x_\xi \mid \xi < \kappa\rangle$ of all subsets of κ in M. Using Proposition 2.2, we obtain a flip U of $\vec{x}$ such that $\Delta U \in I^+$. In the light of the comments made in Footnote 2, it suffices to observe that U being an M-ultrafilter on κ easily follows from $\Delta U \in I^+ \supseteq \mathrm{NS}_\kappa$.

□

In particular, if κ is ineffable, then $\mathcal{I}_M(\mathrm{NS}_\kappa) = \mathcal{I}_C(\mathrm{NS}_\kappa) = \mathcal{I}(\mathrm{NS}_\kappa)$ is the ineffable ideal on κ. We will discuss the question on what is the value of $\mathcal{I}_M([\kappa]^{<\kappa})$ briefly in Section 14.

[2]The meaning of $\Delta U \in I^+$ will thus depend on whether we have fixed an enumeration of U in the background, by having chosen U to flip a certain sequence. This should however not lead to any confusion. It is well-known that permuting the input of a diagonal intersection only changes its output by a non-stationary set (see [1, Lemma 1.3.3]). Hence, if $I \supseteq \mathrm{NS}_\kappa$, rather than requiring that every diagonal intersection of U be in I^+, it equivalently suffices to require one (arbitrary) diagonal intersection of U to be in I^+.

3. A short review of a notion of transfinite indescribability

Joan Bagaria ([2]) introduced a natural notion of Π^1_ξ formula for arbitrary ordinals ξ, that extends the hierarchy of Π^1_n-formulae for $n < \omega$. We will make use of this notion several times in the remainder of this paper, and we would like to shortly recall Bagaria's definitions in this section.

Definition 3.1 (Bagaria). *A formula is said to be $\Sigma^1_{\xi+1}$ if it is of the form $\exists X_0, \ldots, X_k\ \varphi(X_0, \ldots, X_k)$ for some Π^1_ξ-formula φ, and it is $\Pi^1_{\xi+1}$ if it is of the form $\forall X_0, \ldots, X_k\ \varphi(X_0, \ldots, X_k)$ for some Σ^1_ξ-formula φ, where all quantifiers displayed above are understood to be second order quantifiers.*

If ξ is a limit ordinal, we say that a formula is Π^1_ξ if it is a conjunction of the form $\bigwedge_{\zeta<\xi} \varphi_\zeta$, where each φ_ζ is a Π^1_ζ-formula, and the infinite conjunction has only finitely-many free variables. It is Σ^1_ξ if it is a disjunction of the form $\bigvee_{\zeta<\xi} \varphi_\zeta$, where each φ_ζ is a Σ^1_ζ-formula, and the infinite disjunction has only finitely-many free variables.

A corresponding notion of Π^1_ξ-indescribability has been introduced by Bagaria, and independently, an equivalent notion had been introduced by Sharpe and Welch in [16, Definition 3.21].

Definition 3.2 (Bagaria). *Suppose that κ is a regular cardinal, and that $\xi < \kappa$ is an ordinal. A set $A \subseteq \kappa$ is Π^1_ξ-indescribable if for all $y \subseteq V_\kappa$ and every Π^1_ξ-sentence φ, if $\langle V_\kappa, \in, y\rangle \models \varphi$, then there is $\alpha \in A$ such that $\langle V_\alpha, \in, y \cap V_\alpha\rangle \models \varphi$.*

We let $\Pi^1_\xi(\kappa)$ denote the Π^1_ξ-indescribable ideal – the collection of subsets of κ that are not Π^1_ξ-indescribable. We additionally let $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$, and remark that $\Pi^1_0(\kappa) = \mathrm{NS}_\kappa$. The following will be of relevance in Section 13.

Lemma 3.3. [2, Proposition 4.4] *The statement $A \in \Pi^1_\beta(\kappa)^+$ is expressible as a $\Pi^1_{\beta+1}$-property of A over V_κ.*

We will want to make use of the following result of Cody, that generalizes a classical result of Baumgartner from [3].

Lemma 3.4. [6, Lemma 2.20] *If $A \subseteq \kappa$, $\beta < \kappa$, and every A-list has a homogeneous set in $Q \subseteq \bigcap_{\xi\in\{-1\}\cup\beta} \Pi^1_\xi(\kappa)^+$, then A is $\Pi^1_{\beta+1}$-indescribable.*

The following minor generalization of folklore results will be of relevance in combination with Lemma 3.4.

Lemma 3.5. *Assume that M is a weak κ-model, U is an M-ultrafilter on κ, and I is an ideal on κ such that $U \subseteq I^+$ is κ-amenable for M and $A \in U$. Then, every A-list in M has a homogeneous set in I^+.*

Proof. A proof of this lemma can for example be extracted from the proof of [9, Lemma 3.6]. In that Lemma, Gitman makes the additional assumption that the ultrapower of M by U is well-founded, however this assumption is never made use of in her proof. □

4. The iterated ineffability operator

The iterated ineffability operator is defined in a natural way: For any ideal I, let $\mathcal{I}^0(I) = I$, and for any ordinal α, let $\mathcal{I}^{\alpha+1}(I) = \mathcal{I}(\mathcal{I}^\alpha(I))$, and let $\mathcal{I}^\alpha(I) = \bigcup_{\beta<\alpha} \mathcal{I}^\beta(I)$ in case α is a limit ordinal.

In [5, Remark 3.24], it is mentioned that there is a refinement of the ineffability hierarchy via indescribability that is highly analogous to the refinement of the Ramsey hierarchy via indescribability that is studied in detail in [6]. We will need a result on the ineffability hierarchy (namely, Lemma 4.2 below) later on in our paper, and for the benifit of our readers, we would like to provide a proof of this result. [3] Given his comments in [5], Cody was certainly aware of the possibility of this easy adaption of his material from [6], and hence the results in this section should be credited to him. They are the adaptions of [6, Lemma 3.1 and Lemma 3.2] to the ineffability operator.

Lemma 4.1 (Cody)**.** *Let $\alpha < \kappa$ and $\beta \in \{-1\} \cup \kappa$. Suppose $S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, and for each $\xi \in S$, let $S_\xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))^+$. Then, $\bigcup_{\xi \in S} S_\xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$.*

Proof. By induction on α. If $\alpha = 0$ or α is a limit ordinal, the argument is exactly as in [6, Lemma 3.1]. If α is a successor ordinal, fix a $\bigcup_{\xi \in S} S_\xi$-list $\vec{a}$. For each $\xi \in S$, there is some $H_\xi \subseteq S_\xi$ in $\mathcal{I}^{\alpha-1}(\Pi^1_\beta(\xi))^+$ that is homogeneous for $\vec{a} \restriction S_\xi$. Since $S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, there is a homogeneous set $H \subseteq S$ in $\mathcal{I}^{\alpha-1}(\Pi^1_\beta(\kappa))^+$ for the S-list $\langle H_\xi \mid \xi \in S \rangle$. By our inductive hypothesis, $\bigcup_{\xi \in H} H_\xi \in \mathcal{I}^{\alpha-1}(\Pi^1_\beta(\kappa))^+$, but clearly, $\bigcup_{\xi \in H} H_\xi$ is homogeneous for $\vec{a}$, and we are done. □

Lemma 4.2 (Cody)**.** *If $\kappa \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, $\alpha < \kappa$ and $\beta \in \{-1\} \cup \kappa$, then*

$$S = \{\xi < \kappa \mid \xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))\} \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+.$$

Proof. Assume that κ is the least counterexample to the statement of the lemma – that is, for some fixed α and β, κ is least such that $\kappa \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, while $S = \{\xi < \kappa \mid \xi \in \mathcal{I}^\alpha(\Pi^1_\beta(\xi))\} \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))$. Then, $\kappa \setminus S \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^*$, and hence also in $\mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$. For each $\zeta \in \kappa \setminus S$, by the minimality of κ, $S \cap \zeta \in \mathcal{I}^\alpha(\Pi^1_\beta(\zeta))^+$. Thus, by Lemma 4.1, $S = \bigcup_{\zeta \in \kappa \setminus S} S \cap \zeta \in \mathcal{I}^\alpha(\Pi^1_\beta(\kappa))^+$, contradicting our assumption on κ, as desired. □

5. The Ramsey operator

In this section, we want to present an argument showing that one can also characterize the Ramsey operator via the existence of certain ultrafilters for small models of set theory. This characterization is due to Sharpe and Welch ([16, Theorem 3.3]), however we would like to present a somewhat different proof based on the presentation of the proof of a somewhat less general result from [8], and we will need to take a closer look at some of these arguments in order to be able to adapt them later on in Section 9. The operator that we introduce below is implicit in the statement of [16, Theorem 3.3], and we want to call it the model version of the Ramsey operator.

Definition 5.1. *For any ideal I on κ, and $y \subseteq \kappa$, we first define a local instance, letting*

- *$x \in \mathcal{R}^y_M(I)^+$ if there is a transitive weak κ-model M with $y \in M$, and an M-normal M-ultrafilter U on κ with $x \in U$ that is κ-amenable for M, such that every countable intersection of elements of U is in I^+, and we let*

[3] In fact, we only need the below result for the ineffability operator rather than its iterations, however since treating also its iterations provides almost no additional effort, we would like to provide this more general result.

- $\mathcal{R}_M(I)^+ = \bigcap_{y\subseteq\kappa} \mathcal{R}^y_M(I)^+$.

The goal of this section will be to present an argument showing that the operators $\mathcal{R}$ and $\mathcal{R}_M$ are equal to each other. The first direction is an easy generalization of well-known results (see for example [9, Theorem 3.10]).

Lemma 5.2. *If $c\colon [\kappa]^{<\omega} \to \kappa$ is a regressive function, and $y \subseteq \kappa$ codes c, then $\mathcal{R}^y_M(I) \supseteq \mathcal{R}^c(I)$. In particular, $\mathcal{R}_M(I) \supseteq \mathcal{R}(I)$.*

Proof. Assume that $x \in \mathcal{R}^y_M(I)^+$, and let $c\colon [\kappa]^{<\omega} \to \kappa$ be a regressive function that is coded by $y \subseteq \kappa$. Pick a transitive weak κ-model M with $y \in M$, and an M-ultrafilter U on κ witnessing that $x \in \mathcal{I}^y_M(I)^+$. Using that $c \in M$, and following a line of well-known arguments, as for example in the proof of [6, Theorem 2.10], for every $n \in \omega$, we find a set $H_n \in U$ that is homogeneous for $c\restriction[x]^n$. But then, by the properties of U, we have $H := \bigcap_{n\in\omega} H_n \in I^+$ homogeneous for c, showing that $x \in \mathcal{R}(I)^+$. □

The other direction will be substantially more work, and we will need some preparatory results first. Let us start by recalling a standard definition.

Definition 5.3. *Suppose κ is a cardinal and $\mathcal{A} = \langle L_\kappa[A], A\rangle$ with $A \subseteq \kappa$. Then, $I \subseteq \kappa$ is a set of* good indiscernibles *for $\mathcal{A}$ if for all $\gamma \in I$, the following hold.*

- *$\langle L_\gamma[A], A\rangle \prec \langle L_\kappa[A], A\rangle$,*
- *γ is a cardinal, and*
- *$I \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi<\gamma}$.*

We will rely on the following, a proof of Item (1) of which can be found within the proof of [8, Lemma 2.43], [4] and Item (2) is obvious from the details given in that proof as well. The same argument is essentially contained in the proof of [16, Lemma 2.9]. In the present section, we will only need Item (1), but Item (2) will be of good use in Section 9 later on.

Lemma 5.4. *(1) Let κ be an inaccessible cardinal, and let $A \subseteq \kappa$. Then, there is a club $C \subseteq \kappa$ and a regressive function $h\colon [C]^{<\omega} \to \kappa$ such that any κ-sized homogeneous set for h is a set of good indiscernibles for $\langle L_\kappa[A], A\rangle$.* [5]

(2) For any inaccessible $\alpha \in C$, $C \cap \alpha$ and $h\restriction\alpha$ also have the above properties with respect to $A \cap \alpha$.

Proof. Since (2) is not mentioned anywhere in the literature, we would like to present the definition of C and h given κ and A, following [8], and then observe that this relationship is preserved under restrictions to inaccessible elements of C, thus yielding Item (2). For the complete proof of Item (1), the interested reader should consult [8].

In her proof, Gitman makes use of an arbitrary bijection $f\colon \kappa \times \kappa \to \kappa \setminus \{\emptyset\}$, and for the sake of simplicity, we may take f to be defined using Gödel pairing, by setting $f(\langle\alpha, \beta\rangle) = 1 + \prec\alpha, \beta\succ$, yielding in particular that every cardinal is closed under f. C may then simply be defined as the set of all uncountable cardinals

[4]In her Lemma 2.43, Gitman assumes that κ is a Ramsey cardinal, and thus that certain homogeneous sets for colourings do exist. But the assumption of Ramseyness is otherwise not needed, and her proof is easily seen to verify the below lemma.

[5]In fact, it would suffice to require the homogeneous set to be of limit order type.

$\alpha < \kappa$ for which $\langle L_\alpha[A], A\rangle$ is an elementary substructure of $\langle L_\kappa[A], A\rangle$, which is clearly a club subset of κ.

Next, we fix an enumeration $\langle \varphi_m \mid m \in \omega\rangle$ of all formulas in the first order $\in$-language using the predicate A, and consider the following condition (*) on ordered tuples $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_{2n}\rangle$ of length $2n$ of elements of κ:

(*) $\exists \delta_1 < \ldots < \delta_k < \alpha_1$ and $m \in \omega$ such that:

$$\langle L_\kappa[A], A\rangle \not\models \varphi_m(\vec{\delta}, \alpha_1, \ldots, \alpha_n) \iff \varphi_m(\vec{\delta}, \alpha_{n+1}, \ldots, \alpha_{2n}).$$

If $\vec{\alpha}$ satisfies (*), let $w(\vec{\alpha})$ be the least λ that codes m and $\vec{\delta}$ by iteratively applying f, where m and $\vec{\delta}$ witness (*) for $\vec{\alpha}$, and let $w(\vec{\alpha}) = \emptyset$ otherwise. Now we define $h\colon [C]^{<\omega} \to \kappa$ by setting $h(\vec{\alpha}) = w(\vec{\alpha})$ if $\vec{\alpha}$ is of even length, and setting $h(\vec{\alpha}) = \emptyset$ for $\vec{\alpha}$ of odd length.

The remainder of the argument for Item (1), namely that C and h have the desired properties stated there, is pretty straightforward, and proceeds by showing that h has to take value 0 on any κ-sized homogeneous set for h, since any other value quickly leads to a contradiction. The interested reader may find the remaining details for this argument in [8].

Let us observe that Item (2) holds true: First note that any inaccessible cardinal $\alpha \in C$ is a limit point of C, and therefore $C \cap \alpha$ is a club subset of α. But now, using that $\langle L_\alpha[A], A\rangle \prec \langle L_\kappa[A], A\rangle$, it clearly follows that $w \restriction [\alpha]^{<\omega}$ as obtained above is the same as the function $\bar{w}$ that we would have obtained starting with α and $A \cap \alpha$ rather than with κ and with A. But then, it is immediate that if we restrict our function h to $[\alpha]^{<\omega}$, then this is the same as the function $\bar{h}$ that we would obtain from $\bar{w}$, thus yielding the statement of Item (2). $\square$

We also need the following characterization of the Ramsey operator (this is actually the original definition of the Ramsey operator in [7]):

Lemma 5.5. [6, Proposition 2.8] *For any ideal I, $\mathcal{R}(I)^+ = \{x \subseteq \kappa \mid \forall c\colon [x]^{<\omega} \to \kappa$ regressive $\forall C \subseteq \kappa$ club $\exists H \in I^+\ H \subseteq x \cap C$ is homogeneous for $c\}$.*

We are now ready to proceed with the main argument of this section, following the basic line of argument of [6, Theorem 2.10], which relies mostly on the arguments from [9, Section 4].

Theorem 5.6. *Let κ be an inaccessible cardinal. For any ideal I on κ,*

$$\mathcal{R}_M(I) = \mathcal{R}(I).$$

Proof. Having shown Lemma 5.2, it only remains to show that $\mathcal{R}(I) \supseteq \mathcal{R}_M(I)$. Assume thus that $x \in \mathcal{R}(I)^+$, and that $y \subseteq \kappa$, and let $A \subseteq \kappa$ code both x and y. Making use of Lemma 5.4, let C be a club subset of κ and let $h\colon [C]^{<\omega} \to \kappa$ be a regressive function such that any κ-sized homogeneous set for h is a set of good indiscernibles for $\mathcal{A} := \langle L_\kappa[A], A\rangle$. Making use of Lemma 5.5, let $H \subseteq x \cap C$ be homogeneous for h, with $H \in I^+$. Hence, H is a set of good indiscernibles for $\mathcal{A}$.

Now we proceed almost exactly as in [9, Section 4], and construct M and U witnessing that $x \in \mathcal{R}_M(I)^+$. There are only three differences to the argument required:

(1) Our homogeneous set H is in I^+ rather than just unbounded in κ, but this simply carries through the argument without requiring any modification.

(2) We need to show that $x \in U$, but this will be an easy consequence of having chosen $H \subseteq x$.
(3) We asked for U to be M-normal, which was omitted in [9]. [6] We have to show that the M-ultrafilter U that is constructed through the arguments of [9, Section 4] is actually already M-normal.

It might be justifiable to omit the remaining argument, but for the convenience of the reader, we would like to present a mostly self-contained outline of the proof, only refering to [9] for some short intermediate results. Another reason for providing this presentation is that in [9], the justification for one of the main points of the argument, namely that the final filter U is κ-amenable for M, is missing – Gitman only declares this to be *easy*, which to us, after figuring out the actual argument following a helpful conversation with Gitman, does perhaps not seem quite justified, for it seems to be one of the more intricate parts of the proof. Using quite different notation, slightly more detail concerning this argument is provided in [16], however some points also seem to only be touched there somewhat briefly, in particular the analogue of Lemma 5.12 below seems to be missing. Finally, and perhaps most importantly, we will need to refer to the proof of the present theorem in some detail within the proof of Theorem 9.2 later on, so it will be very convenient for the reader to at least have its essential structure available here for reference. Let us thus continue with the argument, which will closely follow the line of argument in (and also the notation from) [9, Section 4] for its most parts.

For every $\gamma \in H$ and $n \in \omega$, let $\vec{\gamma}_n$ denote the increasing sequence $\langle \gamma_1, \ldots, \gamma_n \rangle$ of the first n elements of H (strictly) above γ, and let $\tilde{\mathcal{M}}^n_\gamma = \langle \tilde{M}^n_\gamma, A \cap \tilde{M}^n_\gamma \rangle$ be the Skolem closure of $(\gamma+1) \cup \vec{\gamma}_n$ in $\mathcal{A}$, using the definable Skolem functions of $\mathcal{A}$. Since κ is inaccessible, $L_\kappa[A] \models \mathrm{ZFC}$, and hence in $L_\kappa[A]$, for every λ, $H(\lambda)$ exists and is a model of ZFC^-. Since $\tilde{\mathcal{M}}^n_\gamma \prec \mathcal{A}$ and $\gamma \in \tilde{M}^n_\gamma$, we have $H(\gamma^+)^{\mathcal{A}} \in \tilde{M}^n_\gamma$. Let $M^n_\gamma = \tilde{M}^n_\gamma \cap H(\gamma^+)^{\mathcal{A}}$, and let $\mathcal{M}^n_\gamma = \langle M^n_\gamma, A \cap M^n_\gamma \rangle$.

Lemma 5.7. [9, Lemma 4.2.1] *Each M^n_γ is a transitive model of* ZFC^-.

Lemma 5.8. [9, Lemma 4.2.2] *For every $\gamma \in H$ and $n \in \omega$, $\mathcal{M}^n_\gamma \prec \mathcal{M}^{n+1}_\gamma$.*

If $a \in \tilde{M}^n_\gamma$, then $a = S(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$, where S is a definable Skolem function of $\mathcal{A}$, and $\xi_i \in \gamma$ for $i \le m$. Given $\gamma < \delta$ in H and $n < \omega$, define $\tilde{f}^n_{\gamma\delta} \colon \tilde{M}^n_\gamma \to \tilde{M}^n_\delta$ by setting $\tilde{f}^n_{\gamma\delta}(a) = S(\xi_0, \ldots, \xi_m, \delta, \vec{\delta}_n)$ in case $a = S(\xi_0, \ldots, \xi_m, \gamma, \vec{\gamma}_n)$. Using that $H \setminus \gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi \rangle_{\xi \in \gamma}$, and that $\tilde{\mathcal{M}}^n_\gamma$ and $\tilde{\mathcal{M}}^n_\delta$ are both elementary substructures of $\mathcal{A}$, it easily follows that $\tilde{f}^n_{\gamma\delta} \colon \tilde{\mathcal{M}}^n_\gamma \to \tilde{\mathcal{M}}^n_\delta$ is a well-defined elementary embedding. Since $\tilde{f}^n_{\gamma\delta}(\xi) = \xi$ for all $\xi < \gamma$ and $\tilde{f}^n_{\gamma\delta}(\gamma) = \delta$, γ is the critical point of $\tilde{f}^n_{\gamma\delta}$. Moreover, different such embeddings commute, that is if $\gamma < \delta < \epsilon$ are in H, then $\tilde{f}_{\gamma\epsilon} = \tilde{f}_{\delta\epsilon} \circ \tilde{f}_{\gamma\delta}$.

Lemma 5.9. [9, Lemma 4.2.3] *For $\gamma < \delta$ in H and $n < \omega$, $f^n_{\gamma\delta} := \tilde{f}^n_{\gamma\delta} \restriction M^n_\gamma \colon \mathcal{M}^n_\gamma \to \mathcal{M}^n_\delta$ is an elementary embedding.*

For $\gamma \in H$, define $U^n_\gamma = \{X \in \mathcal{P}(\gamma)^{M^n_\gamma} \mid \gamma \in f^n_{\gamma\delta}(X) \text{ for some } \delta > \gamma \text{ in } H\}$. Equivalently, we could have used "for all $\delta > \gamma$" in this definition. The only nontrivial observation in the next lemma is that $U^n_\gamma \in M^{n+2}_\gamma$.

[6] We could have omitted the requirement of M-normality of U as well in our definition of $\mathcal{R}_M(I)$, and this would yield yet another (known) characterization of the Ramsey operator.

Lemma 5.10. [9, Lemma 4.2.5] *For any $\gamma \in H$ and $n < \omega$, $U^n_\gamma \in M^{n+2}_\gamma$ is an M^n_γ-normal M^n_γ-ultrafilter on γ.*

It is easy to check that for any $\gamma < \delta$ in H and $n < \omega$, $f^n_{\gamma\delta} = f^{n+1}_{\gamma\delta} \restriction M^n_\gamma$ and $U^n_\gamma = U^{n+1}_\gamma \cap M^n_\gamma$. Thus, let $M_\gamma = \bigcup_{n\in\omega} M^n_\gamma$, $U_\gamma = \bigcup_{n\in\omega} U^n_\gamma$ and $f_{\gamma\delta} = \bigcup_{n\in\omega} f^n_{\gamma\delta}$. Let $\mathcal{M}_\gamma = \langle M_\gamma, A \cap M_\gamma\rangle$. Using elementarity, it follows that M_γ is a transitive model of ZFC$^-$, and it is easy to check that $f_{\gamma\delta}$ is an elementary embedding from $\mathcal{M}_\gamma$ to $\mathcal{M}_\delta$ mapping its critical point γ to δ, and that U_γ is an M_γ-normal M_γ-ultrafilter on γ.

Lemma 5.11. [9, Lemma 4.2.6] *U_γ is γ-amenable for M_γ.*

We will need the following, which is not mentioned in [9], in order to be able to verify our final filter U to be κ-amenable for M:

Lemma 5.12. *For $\gamma < \delta$ in H and $n < \omega$, $f_{\gamma\delta}(U^n_\gamma) = U^n_\delta$.*

Proof. Fix $n < \omega$. By the argument for [9, Lemma 4.2.5], there is a first order formula φ, such that for any $\gamma \in H$, U^n_γ is definable in $\tilde{M}^{n+2}_\gamma$ using the formula φ and using γ and the first $n+1$ elements $\vec{\gamma}_{n+1}$ of H above γ as parameters. This implies that $\tilde{f}^{n+2}_{\gamma\delta}(U^n_\gamma) = U^n_\delta$. Since $U^n_\gamma \in M_\gamma$ by Lemma 5.10, this implies our desired statement. $\square$

Now, for every $\gamma \in H$, consider the structure $\langle M_\gamma, \in, A \cap M_\gamma, U_\gamma\rangle$ which extends $\mathcal{M}_\gamma$ by the predicate for the M_γ-ultrafilter U_γ on γ. If $\gamma < \delta$ are in H, we have an elementary embedding $f_{\gamma\delta}\colon \mathcal{M}_\gamma \to \mathcal{M}_\delta$ with critical point γ, such that also $X \in U_\gamma \iff f_{\gamma\delta}(X) \in U_\delta$. This is thus a directed system of embeddings between these structures, and we let $\langle B, E, A', W\rangle$ be its direct limit. Elements of B are functions t with domains $\{\xi \in H \mid \xi \geq \alpha\}$ for some $\alpha \in H$ satisfying that

(1) for $\gamma \in \operatorname{dom} t$, $t(\gamma) \in M_\gamma$,
(2) for $\gamma < \delta \in \operatorname{dom} t$, $t(\delta) = f_{\gamma\delta}(t(\gamma))$, and
(3) there is no $\xi \in H \cap \alpha$ for which there is $a \in M_\xi$ such that $f_{\xi\alpha}(a) = t(\alpha)$.

Note that any $t \in B$ is determined once $t(\xi)$ is known for any $\xi \in \operatorname{dom} t$.

Lemma 5.13. [9, Lemma 4.2.7] *The relation E on B is well-founded.*

We may therefore let $\langle M, \in, A^*, U\rangle$ be the Mostowski collapse of $\langle B, E, A', W\rangle$.

Lemma 5.14. [9, Lemma 4.2.8] *$\kappa \in M$.*

For any $\gamma \in H$, let $j_\gamma\colon M_\gamma \to M$ be defined such that for any $a \in M_\gamma$, $j_\gamma(a)$ is the collapse of the (unique) function $t \in B$ for which $t(\gamma) = a$. Then, j_γ is easily seen to be an elementary embedding of $\mathcal{M}_\gamma$ into $\langle M, \in, A^*\rangle$, and to also be elementary for atomic formulas in the language with the predicate for the ultrafilter. Observe that $j_\gamma(\xi) = \xi$ for all $\xi < \gamma$, $j_\gamma(\gamma) = \kappa$, and hence that $\operatorname{crit}(j_\gamma) = \gamma$. Moreover, if $\gamma < \delta$ are elements of H, then $j_\delta \circ f_{\gamma\delta} = j_\gamma$.

The proof of the following lemma is not contained in [9]: M-normality had not been considered, and the verification of κ-amenability is somewhat strangely missing there.

Lemma 5.15. *U is an M-normal M-ultrafilter on κ that is κ-amenable for M.*

Proof. It is easy to check that U is an M-ultrafilter on κ. For the M-normality of U, we show that every regressive function f on a set $x \in U$ is constant on a set in U. Let $\gamma \in H$ be such that there are g and y in M_γ with $f_\gamma(g) = f$ and $f_\gamma(y) = x$. By elementarity for atomic formulas using the predicate for the ultrafilter, $y \in U_\gamma$. By the M_γ-normality of U_γ, g is constant on a set $h \in U_\gamma$. It thus follows that f is constant on the set $f_\gamma(h) \in U$.

For the κ-amenability of U, let $\vec{x}$ be a κ-sequence of elements of $\mathcal{P}(\kappa)$ in M. Let $\gamma \in H$ be such that there is $\vec{a}$ in M_γ for which $j_\gamma(\vec{a}) = \vec{x}$. Using that $M_\gamma = \bigcup_{n<\omega} M_\gamma^n$, we may fix $n < \omega$ for which $\vec{a} \in M_\gamma^n$. Then, by Lemma 5.10,

$$b = \{\alpha < \gamma \mid a_\alpha \in U_\gamma\} = \{\alpha < \gamma \mid a_\alpha \in U_\gamma^n\} \in M_\gamma.$$

But then, making use of Lemma 5.12, for every $\delta > \gamma$ in H, we have

$$f_{\gamma\delta}(b) = \{\alpha < \delta \mid f_{\gamma\delta}(\vec{a})_\alpha \in U_\delta^n\} = \{\alpha < \delta \mid f_{\gamma\delta}(\vec{a})_\alpha \in U_\delta\}.$$

By the properties of the direct limit, it thus follows that $j_\gamma(b) = \{\alpha < \kappa \mid x_\alpha \in U\}$, showing that U is κ-amenable for M. □

Lemma 5.16. [9, Lemma 4.2.10] *For every $X \subseteq \kappa$, $X \in U$ if and only if there is $\alpha \in H$ such that $\{\xi \in H \mid \xi > \alpha\} \subseteq X$.*

As an easy consequence, one then obtains the following, that we would like to provide the short proof of for the convenience of our readers:

Lemma 5.17. [9, Lemma 4.2.11] *U is countably complete.*

Proof. Let $\langle A_n \mid n < \omega\rangle$ be a sequence of elements of U. For each $n < \omega$, there is $\gamma_n \in H$ for which $X_n = \{\xi \in H \mid \xi > \gamma_n\} \subseteq A_n$. Thus,

$$\emptyset \neq \bigcap_{n<\omega} X_n \subseteq \bigcap_{n<\omega} A_n.$$

□

Lemma 5.18. [9, Lemma 4.2.12] *$A^*\restriction\kappa = A$, and hence $A \in M$.*

This implies that both x and y are elements of M. Since we have chosen H to be a subset of x, it follows by Lemma 5.16 that $x \in U$, which concludes our argument. □

In particular, if κ is a Ramsey cardinal, then $\mathcal{R}_M([\kappa]^{<\kappa}) = \mathcal{R}([\kappa]^{<\kappa})$ is the Ramsey ideal on κ. If κ is an ineffably Ramsey cardinal, then $\mathcal{R}_M(\mathrm{NS}_\kappa) = \mathcal{R}(\mathrm{NS}_\kappa)$ is the ineffably Ramsey ideal on κ.

6. Pre-Operators

In his [7], Feng introduced the pre-Ramsey operator, which behaves with respect to the Ramsey operator as does the subtle operator with respect to the ineffability operator, and it is this notion that motivates the naming of our notion of *pre-operators*. We want to introduce some very simple and natural additional language, that will allow us to define pre-operators in a uniform way. Our ideal operators are all defined via local instances that are parametrized by certain objects. Given a cardinal κ, we would like to refer to the collection of all such objects on κ as the *object type at κ* of such an operator $\mathcal{O}$, and denote this by $\mathcal{T}(\mathcal{O}, \kappa)$. The object type $\mathcal{T}(\mathcal{I}, \kappa)$ of the ineffability operator at κ is the collection of all κ-lists, the object type $\mathcal{T}(\mathcal{R}, \kappa)$ of the Ramsey operator at κ is the collection of all regressive functions $c\colon [\kappa]^{<\omega} \to \kappa$, the object type of the operator $\mathcal{R}_\omega$ is the collection of all (ω,κ)-sequences, and the object type of our model based operators at κ is simply the powerset of κ.

Each object type $\mathcal{T}$ at κ comes with an associated restriction operator, which, given some $y \in \mathcal{T}$ and some $\alpha < \kappa$, outputs its natural restriction $y\restriction\alpha$. The following definition should not bear any surprises.

Definition 6.1.

- *If $\mathcal{T} = \mathcal{P}(\kappa)$ and $y \in \mathcal{T}$, then $y\restriction\alpha = y \cap \alpha$.*
- *If $\mathcal{T}$ is the collection of all κ-lists and $y \in \mathcal{T}$, then $y\restriction\alpha$ is the restriction of y to the domain α, i.e. the initial segment of length α of the κ-sequence y.*
- *If $\mathcal{T}$ is the collection of all functions $c\colon [\kappa]^{<\omega} \to 2$ and $y \in \mathcal{T}$, then $y\restriction\alpha$ is the restriction of y to the domain $[\alpha]^{<\omega}$.*
- *If $\mathcal{T}$ is the collection of all (ω,κ)-sequences and $y \in \mathcal{T}$, then $y\restriction\alpha$ is the restriction of y to the domain that is the space of all finite increasing sequences from α.*

Each ideal operator $\mathcal{O}$ with local instances $\mathcal{O}^y$ has what we would like to call its associated *pre-operator* $\mathcal{O}_0$. To define such an operator, we start with a sequence

$$\vec{I} = \langle I_\alpha \mid \alpha \leq \kappa \text{ is a regular uncountable cardinal}\rangle$$

for some inaccessible cardinal κ, such that each I_α is an ideal on α. We will somewhat sloppily refer to such a sequence as a *sequence of ideals* in the following.

Definition 6.2. *Given an ideal operator $\mathcal{O}$ together with its local instances $\mathcal{O}^y$, we define its associated pre-operator $\mathcal{O}_0$ as follows. Given a sequence $\vec{I}$ of ideals, let*

$$\mathcal{O}_0(\vec{I})^+ = \{x \subseteq \kappa \mid \forall y \in \mathcal{T}(\mathcal{O},\kappa)\, \forall C \subseteq \kappa \text{ club } \exists \alpha \in x\ x \cap C \cap \alpha \in \mathcal{O}^{y\restriction\alpha}(I_\alpha)^+\},$$

where α is understood to range over regular uncountable cardinals. If the I_α's are uniformly definable from α, we also write $\mathcal{O}_0(I_\kappa)$ rather than $\mathcal{O}_0(\vec{I})$.

7. The subtle operator

The subtle operator is the usual name for what could be called the pre-ineffable operator, which is implicit in [3], and explicit in [6]: it is the pre-operator $\mathcal{I}_0$ defined via the ineffability operator $\mathcal{I}$ (or rather, its local instances $\mathcal{I}^y$), and we let $\mathcal{I}_{M0} = (\mathcal{I}_M)_0$ be the *model version* of this operator, defined via the local instances of the model version of the ineffability operator. Note that while the operators $\mathcal{I}$ and $\mathcal{I}_M$ agree on ideals that contain the nonstationary ideal by Proposition 2.5, their local instances do not seem to do so, and it is therefore not immediate that $\mathcal{I}_0$ and $\mathcal{I}_{M0}$ actually agree on these ideals. We will however show this to be the case below. If κ is a subtle cardinal, then $\mathcal{I}_0(\mathrm{NS}_\kappa)$ is the subtle ideal on κ (see [3, Theorem 5.1]), which is classically defined as follows:

Definition 7.1. *$x \subseteq \kappa$ is* subtle *if for every x-list $\vec{a}$ and every club $C \subseteq \kappa$, there are $\alpha < \beta$ in C such that $a_\alpha = a_\beta \cap \alpha$. The* subtle ideal *on κ is the collection of all subsets of κ that are not subtle.*

Let us start with a simple observation.

Observation 7.2. *If $I_\kappa \supseteq \mathrm{NS}_\kappa$ induces a sequence $\vec{I}$ of ideals, then $\mathcal{I}_{M0}(\vec{I}) \supseteq \mathcal{I}_0(\vec{I})$.*

Proof. For every κ-list $\vec{a}$, there is $y \subseteq \kappa$ coding $\vec{a}$, and any reasonable choice of coding will have the property that for every cardinal $\alpha < \kappa$, $y \cap \alpha$ codes $\vec{a}\restriction\alpha$.

By Proposition 2.5 thus, for every $\alpha < \kappa$, $\mathcal{I}_M^{y\restriction\alpha}(I_\alpha) \supseteq \mathcal{I}^{\vec{a}\restriction\alpha}(I_\alpha)$, and hence the observation immediately follows from the definition of the operators $\mathcal{I}_{M0}$ and $\mathcal{I}_0$. $\square$

By a careful adaptation of the arguments for Proposition 2.5 (2), it is in fact possible to verify equality.

Theorem 7.3. *If $I_\kappa \supseteq \mathrm{NS}_\kappa$ induces a sequence $\vec{I}$ of ideals, then $\mathcal{I}_{M0}(\vec{I}) = \mathcal{I}_0(\vec{I})$.*

Proof. Having Observation 7.2 available, it only remains to show that $\mathcal{I}_{M0}(\vec{I}) \subseteq \mathcal{I}_0(\vec{I})$. We may also assume that κ is a subtle cardinal, for otherwise $\mathcal{I}_0(\vec{I}) = \mathcal{P}(\kappa)$, and we are thus done. Assume that $x \in \mathcal{I}_0(\vec{I})^+$. We want to show that $x \in \mathcal{I}_{M0}(\vec{I})^+$. Let $y \subseteq \kappa$ and let C be a club subset of κ. We need to find $\alpha \in x$ such that $x \cap C \cap \alpha \in \mathcal{I}_M^{y\cap\alpha}(I_\alpha)^+$.

Fix a set of Skolem functions for $H(\kappa^+)$, and let M be the Skolem hull of $(\kappa+1) \cup \{x, y, C\}$ in $H(\kappa^+)$. Pick an enumeration $\langle x_\xi \mid \xi < \kappa\rangle$ of all subsets of κ in M, and let $\vec{a}$ be the κ-list defined by setting $a_\beta = \{\xi < \beta \mid \beta \in x_\xi\}$ for every $\beta < \kappa$. Let D be the club set of cardinals γ below κ such that if M_γ denotes the Skolem hull of $\gamma \cup \{\kappa, x, y, C\}$ in M, then

(1) $M_\gamma \cap \kappa = \gamma$, and
(2) $\langle x_\xi \mid \xi < \gamma\rangle$ enumerates all subsets of κ in M_γ.

Making use of our assumption that $x \in \mathcal{I}_0(\vec{I})^+$, there is $\alpha \in x$ such that

$$x \cap C \cap D \cap \alpha \in I^{\vec{a}\restriction\alpha}(I_\alpha)^+.$$

Let $\bar{M}$ be the transitive collapse of M_α. Then, $\bar{M}$ is a transitive weak α-model with $x\cap\alpha, y\cap\alpha, C\cap\alpha \in \bar{M}$, and by (1) and (2) above, $\langle x_\xi \cap \alpha \mid \xi < \alpha\rangle$ enumerates all subsets of α in $\bar{M}$. Moreover, $\vec{a}\restriction\alpha$ satisfies that $a_\beta = \{\xi < \beta \mid \beta \in x_\xi \cap \alpha\}$ for every $\beta < \alpha$. We now proceed exactly as in the proof of Proposition 2.5: By our choice of α, there is $H \subseteq x \cap C \cap D \cap \alpha$ in I_α^+ that is homogeneous for $\vec{a}\restriction\alpha$. We may thus pick $A \subseteq \alpha$ such that $a_\beta = A \cap \beta$ for every $\beta \in H$. Given $\xi < \alpha$, let $u_\xi = x_\xi \cap \alpha$ if $\xi \in A$, and let $u_\xi = \alpha \setminus x_\xi$ otherwise. Let $U = \{u_\xi \mid \xi < \alpha\}$. By the corresponding version at α of the claim within the proof of Proposition 2.5, U is an $\bar{M}$-ultrafilter on α with $x \cap C \cap \alpha \in U$, such that $\Delta_{\xi<\alpha} u_\xi \in I_\alpha^+$. This shows that $x \cap C \cap \alpha \in \mathcal{I}_M^{y\cap\alpha}(I_\alpha)^+$, as desired. $\square$

8. A small embedding characterization of subtlety using an anti-correctness property

In this short section, we want to place a sidenote that doesn't really use the techiques developed in this paper, but is somewhat closely related to them, and gives a strong hint towards a possible negative answer for a question from [12]. That is, in that paper, many types of large cardinals, including subtle cardinals, were characterized using so-called *small embedding characterizations*, stating that there exists an embedding $j\colon M \to H(\theta)$ with $j(\operatorname{crit} j) = \kappa$ and such that certain additional properties hold true. All of these characterizations except for the one for subtle cardinals were based on what we called *correctness properties*, that is properties that were either provable in V or in M, and that were ascertained to also hold in M or V respectively by our characterization. We want to show here that for subtlety, we can provide a small embedding characterization that is rather based on an *anti-correctness property*, i.e. a property that at least in some cases is provably non-absolute between M and V.

Definition 8.1. [12, Definition 1.1] *Given cardinals $\kappa < \theta$, we say that a nontrivial elementary embedding $j\colon M \to H(\theta)$ is a* small embedding *for κ if $M \in H(\theta)$ is transitive, and $j(\operatorname{crit} j) = \kappa$ holds.*

It is immediate that the property $\kappa \in \mathcal{I}_0(\mathrm{NS}_\kappa)^+$ can be rewritten to yield a small embedding characterization of the subtlety of κ that is different to the one provided in [Holy-Lücke-Njegomir].

Observation 8.2. *A cardinal κ is subtle if for every cardinal $\theta > \kappa$, every κ-list $\vec{a}$ and every club $C \subseteq \kappa$, there is a small embedding $j\colon M \to H(\theta)$ for κ such that $\vec{a}, C \in \operatorname{range} j$ and $C \cap \operatorname{crit} j \in \mathcal{I}^{\vec{a}\restriction\alpha}(\mathrm{NS}_{\operatorname{crit} j})^+$.* □

However, the property used to characterize subtlety in the above is easily seen to be an anti-correctness property in many circumstances:

Observation 8.3. *Assume that κ is subtle, but not ineffable. Then, there are a κ-list $\vec{a}$ and a club subset C of κ such that for every cardinal $\theta > \kappa$ and every small embedding $j\colon M \to H(\theta)$ for κ with $\vec{a}$ and C both in the range of j, letting $\bar{C} = C \cap \operatorname{crit} j = j^{-1}(C)$ and $\bar{a} = \vec{a}\restriction\operatorname{crit} j = j^{-1}(\vec{a})$, M thinks that $\bar{C} \in \mathcal{I}^{\bar{a}}(\mathrm{NS}_{\operatorname{crit} j})$.*

Proof. If this weren't the case, then κ would be ineffable by the elementarity of the small embeddings. □

9. The pre-Ramsey operator

The pre-Ramsey operator is implicit in [4], and was explicitly introduced by Feng in [7]: it is the pre-operator $\mathcal{R}_0$ defined via the Ramsey operator $\mathcal{R}$ (or rather, its local instances $\mathcal{R}^y$), and we let $\mathcal{R}_{M0} = (\mathcal{R}_M)_0$ be the *model version* of this operator, defined via the local instances of the model version of the Ramsey operator. Note that as for the ineffability operator, while the Ramsey operator and its model version agree, their local instances do not seem to do so, and it is therefore not immediate that the above two operators actually agree. We will however again show this to be the case below. A cardinal κ is a *pre-Ramsey* cardinal if $\kappa \in \mathcal{R}_0([\kappa]^{<\kappa})^+$. If κ is a pre-Ramsey cardinal, then $\mathcal{R}_0([\kappa]^{<\kappa})$ is the pre-Ramsey ideal on κ (see [3]). Let us start with a simple observation.

Observation 9.1. *For any sequence $\vec{I}$ of ideals, $\mathcal{R}_{M0}(\vec{I}) \supseteq \mathcal{R}_0(\vec{I})$.*

Proof. For every regressive function $f\colon [\kappa]^{<\omega} \to \kappa$, there is $y \subseteq \kappa$ coding f, and any reasonable choice of coding will have the property that for every cardinal $\alpha < \kappa$, $y \cap \alpha$ codes $f\restriction\alpha$. By Lemma 5.2 thus, for every $\alpha < \kappa$, $\mathcal{R}_M^{y\restriction\alpha}(I_\alpha) \supseteq \mathcal{R}^{\vec{a}\restriction\alpha}(I_\alpha)$, and thus the observation immediately follows from the definition of the operators $\mathcal{R}_{M0}$ and $\mathcal{R}_0$. □

By a careful adaptation of the arguments for Theorem 5.6, very much as for the subtle operator in Theorem 7.3, it is in fact possible to verify equality.

Theorem 9.2. *For any sequence $\vec{I}$ of ideals, $\mathcal{R}_{M0}(\vec{I}) = \mathcal{R}_0(\vec{I})$.*

Proof. Having Observation 9.1 available, it only remains to show that $\mathcal{R}_{M0}(\vec{I}) \subseteq \mathcal{R}_0(\vec{I})$. We may also assume that κ is a pre-Ramsey cardinal, for otherwise $\mathcal{R}_0(\vec{I}) = \mathcal{P}(\kappa)$, and we are thus done. Assume that $x \in \mathcal{R}_0(\vec{I})^+$. We want to show that

$x \in \mathcal{R}_{M0}(\vec{I})^+$. Let $y \subseteq \kappa$ and let C be a club subset of κ. We need to find $\alpha \in x$ such that $x \cap C \cap \alpha \in \mathcal{R}_M^{y\cap\alpha}(I_\alpha)^+$.

Let $A \subseteq \kappa$ code x on the even ordinals, and y on the odd ordinals. Let $h\colon [\kappa]^{<\omega} \to \kappa$ be the regressive function and let $D \subseteq \kappa$ be the club set of cardinals obtained from an application of Lemma 5.4 (1) for $A \subseteq \kappa$. Making use of our assumption that $x \in \mathcal{R}_0(\vec{I})^+$, there is $\alpha \in x$ such that $x \cap C \cap D \cap \alpha \in \mathcal{R}^{h\restriction\alpha}(I_\alpha)^+$.

Let $\mathcal{A} = \langle L_\alpha[A], A \cap \alpha\rangle$, which is an elementary substructure of $\langle L_\kappa[A], A\rangle$, since $\alpha \in D$. By Lemma 5.4 (2), $D \cap \alpha$ and $h\restriction\alpha$ witness that Lemma 5.4 (1) holds for $A \cap \alpha$. By our choice of α and by Lemma 5.5, there is $H \subseteq x \cap C \cap D \cap \alpha$ in I_α^+ that is homogeneous for $h\restriction\alpha$, and by Lemma 5.4 (1), H is a set of good indiscernibles for $\mathcal{A}$. We now proceed exactly as in the proof of Theorem 5.6, constructing a weak α-model M with $y \in M$ and an M-normal M-ultrafilter U on α that is κ-amenable for M and countably complete with $x \cap C \cap D \cap \alpha \in U$, thus showing that $x \cap C \cap D \cap \alpha \in \mathcal{R}_M^{y\cap\alpha}(I_\alpha)^+$. Since $x \cap C \cap D \cap \alpha \subseteq x \cap C \cap \alpha$, and the latter set is easily (somewhat cumbersome, but straightforward, by proceeding along the model construction in the proof of Theorem 5.6) checked to be an element of M, and thus also of U, this implies that $x \cap C \cap \alpha \in \mathcal{R}_M^{y\cap\alpha}(I_\alpha)^+$, as desired. $\square$

10. An abstract notion of large cardinal operator

In this section, we present an abstract notion of large cardinal operator, which has both (the model versions of) the ineffability operator and the Ramsey operator as special instances, and which – unlike those two operators – easily lends itself to generalizations. We also provide a first few basic results for such operators.

Definition 10.1. *Let $\Psi(M, U)$ and $\Omega(U, I)$ be parameter-free first order formulae such that* ZFC *proves the following:* [7]

- *For any ideal I, $\Omega(U, I)$ implies that $U \subseteq I^+$, and*
- *For any ideals I and J on κ,*

$$[I \supseteq J \wedge \Omega(U, I)] \to \Omega(U, J).$$

We define an operator $\mathfrak{O}\Psi\Omega$ on ideals on κ as follows.

- *For any $y \subseteq \kappa$, we first define a local instance by letting $x \in \mathfrak{O}\Psi\Omega^y(I)^+$ if there exists a transitive weak κ-model M with $y \in M$ and an M-ultrafilter U on κ with $x \in U$ such that $\Psi(M, U)$ and $\Omega(U, I)$ hold, and we let*
- $\mathfrak{O}\Psi\Omega(I)^+ = \bigcap_{y\subseteq\kappa} \mathfrak{O}\Psi\Omega^y(I)^+$.

Let us check how the examples we saw so far fit into these schemes:

- If $\Psi(M, U)$ is trivial, and $\Omega(U, I)$ denotes the property that $\Delta U \in I^+$, then $\mathfrak{O}\Psi\Omega$ is the model version $\mathcal{I}_M$ of the ineffability operator.
- If $\Psi(M, U)$ denotes the property that U is M-normal and κ-amenable for M, and $\Omega(U, I)$ denotes the property that every countable intersection of elements of U is in I^+, then $\mathfrak{O}\Psi\Omega$ is (the model version $\mathcal{R}_M$ of) the Ramsey operator.

Proposition 10.2. *Assume that Ψ and Ω are as in Definition 10.1, and that $I \supseteq J$ are ideals on κ. Then, the following hold.*

- *$\mathfrak{O}\Psi\Omega(I) \supseteq I$ is an ideal on κ.*

[7] We will always implicitly assume $\Psi(M, U) \wedge \Omega(U, I)$ to imply M to be a transitive weak κ-model, and U to be an M-ultrafilter on κ.

- $\mathfrak{D}\Psi\Omega(I) \supseteq \mathfrak{D}\Psi\Omega(J)$.
- *If $\Psi(M,U) \wedge \Omega(U,I)$ implies that U is M-normal, then $\mathfrak{D}\Psi\Omega(I)$ is normal.*
- *In particular, if $I \supseteq \mathrm{NS}_\kappa$, then $\Delta U \in I^+$ implies that U is M-normal.*
- *If Ψ' and Ω' are as in Definition 10.1 as well, and $\Psi'(M,U) \wedge \Omega'(U,I)$ implies $\Psi(M,U) \wedge \Omega(U,I)$, then $\mathfrak{D}\Psi'\Omega'(I) \supseteq \mathfrak{D}\Psi\Omega(I)$.*

Proof. Assume that I is an ideal on a cardinal κ, that $A \in \mathfrak{D}\Psi\Omega(I)$ and that $B \subseteq A$. We want to show that also $B \in \mathfrak{D}\Psi\Omega(I)$. Let $y \subseteq \kappa$ be such that $A \in \mathfrak{D}\Psi\Omega^y(I)$. Now if M is a transitive weak κ-model with $y \in M$ and U is an M-ultrafilter on κ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold, then A is not an element of U, and hence also $B \subseteq A$ is not an element of U, showing that $B \in \mathfrak{D}\Psi\Omega^y(I) \subseteq \mathfrak{D}\Psi\Omega(I)$, as desired.

Now assume that both A and B are in $\mathfrak{D}\Psi\Omega(I)$. We want to show that also $A \cup B \in \mathfrak{D}\Psi\Omega(I)$. Let y_A and y_B be such that $A \in \mathfrak{D}\Psi\Omega^{y_A}(I)$ and $B \in \mathfrak{D}\Psi\Omega^{y_B}(I)$. Let $y \subseteq \kappa$ code all of A, B, y_A and y_B. Now if M is a transitive weak κ-model with $y \in M$ and U is an M-ultrafilter on κ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold, it follows that both A and B are in $(\mathcal{P}(\kappa) \cap M) \setminus U$, and hence also that $A \cup B$ is not in U, showing that $A \cup B \in \mathfrak{D}\Psi\Omega^y(I) \subseteq \mathfrak{D}\Psi\Omega(I)$, as desired.

Clearly, using that $\Omega(U,I)$ implies that $U \subseteq I^+$, if $x \in \mathfrak{D}\Psi\Omega(I)^+$, then $x \in I^+$, thus finishing the argument for the first item of the proposition.

The monotonicity statement in the second item is immediate from our monotonicity requirement on Ω.

For the third item, assume that $\Psi(M,U) \wedge \Omega(U,I)$ implies that U is M-normal. Let $A \in \mathfrak{D}\Psi\Omega(I)^+$, and let $f\colon A \to \kappa$ be a regressive function. Assume for a contradiction that $f^{-1}(\{\alpha\}) \in \mathfrak{D}\Psi\Omega(I)$ for every $\alpha < \kappa$. We may thus pick a sequence $\vec{y} = \langle y_\alpha \mid \alpha < \kappa\rangle$ such that $f^{-1}(\{\alpha\}) \in \mathfrak{D}\Psi\Omega^{y_\alpha}(I)$ for every $\alpha < \kappa$. Let $y \subseteq \kappa$ code both f and $\vec{y}$. Let M be a transitive weak κ-model with $y \in M$, and let U be an M-ultrafilter on κ with $A \in U$ such that both $\Psi(M,U)$ and $\Omega(U,I)$ hold. Since $y_\alpha \in M$ for every $\alpha < \kappa$, it follows that for no $\alpha < \kappa$ we have $f^{-1}(\{\alpha\}) \in U$. On the other hand, since U is M-normal, there is some $B \in U$ that is homogeneous for f, which is clearly contradicting the above, as desired.

For the fourth item, assume that $I \supseteq \mathrm{NS}_\kappa$ and $\Delta U \in I^+$, however that U is not M-normal. Then, there is a κ-sequence of elements of M with a diagonal intersection that is not in U, and hence the complement of this diagonal intersection is in U. But now, using this, it is easy to find an enumeration of U with nonstationary diagonal intersection, which contradicts that $\Delta U \in I^+$.

The statement of the fifth item is immediate from the definitions involved. □

Definition 10.3.
- *A pair of formulas $\langle\Psi,\Omega\rangle$ is* ineffable *in case* ZFC *proves that for any ideal I, $\Psi(M,U) \wedge \Omega(U,I)$ implies that for every $A \in U$, every A-list $\vec{a} \in M$ has a homogeneous set in I^+.*
- *Let $\mathcal{O}$ be an ideal operator. We say that $\mathcal{O}$ is* ineffable *in case* ZFC *proves that whenever $A \in \mathcal{O}(I)^+$ and $\vec{a}$ is an A-list, then $\vec{a}$ has a homogeneous set in I^+.*

Note that by the above, the ineffability operator $\mathcal{I}$ is ineffable. But also, if $\mathcal{O}$ can be characterized in the form $\mathcal{O} = \mathfrak{D}\Psi\Omega$ for some ineffable pair of formulas $\langle\Psi,\Omega\rangle$, then $\mathcal{O}$ is ineffable.

Observation 10.4. *Let $\mathcal{O}$ be an ideal operator of the form $\mathfrak{D}\Psi\Omega$. Then,*

- *If* $\Psi(M,U)$ ZFC*-provably implies that* U *is* κ*-amenable for* M*, then* $\mathcal{O}$ *is ineffable.*
- *If* $\Omega(U,I)$ ZFC*-provably implies that* $\Delta U\in I^+$*, then* $\mathcal{O}$ *is ineffable.*
- *If* $\mathcal{O}$ *is ineffable, then for any ideal* I*,* $\mathcal{O}(I)\supseteq\mathcal{I}(I)\supseteq\mathrm{NS}_\kappa$*.*
- *If* ZFC *proves that for any ideal* I*,* $\mathcal{O}(I)\supseteq\mathcal{I}(I)$*, then* $\mathcal{O}$ *is ineffable.*

□

Proof. The first item follows from Lemma 3.5, and the second item follows from the first item of Proposition 2.5. The third and fourth item are both immediate from the definition of the ineffability operator $\mathcal{I}$, and the fact that $\mathcal{I}(I)\supseteq\mathcal{I}([\kappa]^{<\kappa})\supseteq\mathrm{NS}_\kappa$, where the easy argument for the latter inclusion can be found in the proof of [3, Theorem 2.3]. □

In particular, the first item above implies that the operator $\mathcal{R}$ is ineffable.

The following important observation is now immediate from Lemma 3.4.

Corollary 10.5. *If* $\mathcal{O}$ *is ineffable,* $I\supseteq\bigcup_{\xi\in\{-1\}\cup\beta}\Pi^1_\xi(\kappa)$*, and* $\beta<\kappa$ *is an ordinal, then*

$$\mathcal{O}\Psi\Omega(I)\supseteq\Pi^1_{\beta+1}(\kappa).$$

□

11. New large cardinal operators

In this section, we want to introduce two new large cardinal operators: One that is naturally derived from the notion of a $\mathbf{T}^\kappa_\omega$-Ramsey cardinal that was introduced in [11, Section 9], and one that is naturally derived from the notion of a weakly Ramsey cardinal that was introduced in [9]. These operators fit into the framework that we introduced in Section 10. We will argue that these two operators are strictly intermediate between the operators $\mathcal{I}$ and $\mathcal{R}$, and that they are different from each other in a strong sense.

Definition 11.1.

- *Let* $\mathbf{T}$ *denote the* $\mathbf{T}^\kappa_\omega$-Ramsey subset operator – *the operator* $\mathfrak{O}\Psi\Omega$ *for which* $\Psi(M,U)$ *denotes the statement that* U *is* M*-normal and* κ*-amenable for* M*, and* $\Omega(U,I)$ *denotes the statement that* $U\subseteq I^+$*.*
- *Let* $\mathcal{W}$ *denote the* weakly Ramsey subset operator – *the operator* $\mathfrak{O}\Psi\Omega$ *for* $\Psi(M,U)$ *denoting the statement that* U *is* M*-normal,* κ*-amenable for* M*, and the ultrapower of* M *by* U *is well-founded, and* $\Omega(U,I)$ *denotes the statement that* $U\subseteq I^+$*.*

Clearly, $\mathcal{I}(I)\subseteq\mathbf{T}(I)\subseteq\mathcal{W}(I)\subseteq\mathcal{R}(I)$ for any ideal I, where the first inclusion follows from the fact that the operator $\mathbf{T}$ is ineffable by Observation 10.4, and the remaining inclusions follow from the final statement of Proposition 10.2. $\mathbf{T}^\kappa_\omega$-Ramsey cardinals are exactly those cardinals κ for which $\kappa\in\mathbf{T}([\kappa]^{<\kappa})^+$. Weakly Ramsey cardinals are exactly those cardinals κ for which $\kappa\in\mathcal{W}([\kappa]^{<\kappa})^+$.

It was shown in [11, Lemma 10.1], using different notation, that $\mathcal{I}([\kappa]^{<\kappa})\subsetneq\mathbf{T}([\kappa]^{<\kappa})$. We first want to argue that this result can be extended to the nonstationary ideal, essentially by the same argument, which we would like to provide here for the convenience of our readers. [8]

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[8]This should be seen as a sample result indicating that these two operators are different. We cannot expect to show that $\mathcal{I}(I)\subsetneq\mathbf{T}(I)$ for any ideal I, by an argument as in the footnote to

Proposition 11.2. *If $I \in \{[\kappa]^{<\kappa}, \mathrm{NS}_\kappa\}$, then*

$$\mathcal{I}(I) \subsetneq \mathbf{T}(I).$$

Proof. The inclusion itself has already been discussed above, and we are only left to verify its properness. By Lemma 4.2, $S = \{\xi < \kappa \mid \xi \in \mathcal{I}(I)\} \notin \mathcal{I}(I)$. We will thus be done if we can show that $S \in \mathbf{T}(I)$. Assume for a contradiction that this is not the case. Then, there is a transitive weak κ-model M with $b \in M$ for some bijection $b\colon \kappa \to V_\kappa$, and there is a κ-amenable, M-normal M-ultrafilter $U \subseteq I^+$ on κ such that $S \in U$. Note that since $V_\kappa \subseteq M$, S satisfies the same definition in M that it satisfies in V. Let $j\colon \langle M, \in\rangle \to \langle N, \in_N\rangle$ be the ultrapower embedding induced by U. Note that ϵ_N may not be well-founded. However, by [11, Lemma 3.5], there is an ordinal κ^N of N and an isomorphism

$$j^*\colon \langle H(\kappa^+)^M, \in\rangle \to \langle H((\kappa^N)^+)^N, \in_N\rangle$$

extending $j{\restriction}V_\kappa$, which satisfies $j^*(\kappa) = \kappa^N$. Therefore, we may identify the elements of $H((\kappa^N)^+)^N$ with the corresponding elements of $H(\kappa^+)^M$, with κ^N corresponding to κ, and have $j{\restriction}V_\kappa = \mathrm{id}$, modulo this identification. Using this, it follows that $N \models \kappa \in_N j(S)$, and hence that $N \models \kappa \in_N \mathcal{I}(I)$.

On the other hand, since being either unbounded or stationary in κ is downwards absolute to $M \supseteq V_\kappa$, it follows that every element of U is either an unbounded or stationary subset of κ in M respectively. But then, given a collection $\mathcal{A} \in ([\mathcal{P}(\kappa)]^\kappa)^M$, $U \cap \mathcal{A} \in M$ by the κ-amenability of U, and $\Delta(U \cap \mathcal{A}) \in U$ by the M-normality of U. Since $U \subseteq I^+$, this shows that $M \models \kappa \in \mathcal{I}_C(I)^+ = \mathcal{I}(I)^+$, where the equality follows by Proposition 2.2. [9] Using the above identification once again, it follows that also $N \models \kappa \in_N \mathcal{I}(I)^+$, which is clearly a contradiction. □

We next want to separate $\mathbf{T}$ from $\mathcal{W}$ in a strong sense, and also make an attempt at separating $\mathcal{W}$ from $\mathcal{R}$. In order to do so, we will need to introduce two further operators, one that is closely linked to completely ineffable cardinals, and another one that is closely linked to the notion of ω-Ramsey cardinals that was introduced by Philipp Schlicht and the author in [13]. Neither of those operators fits into the framework that is used within this paper, for they make use of weak κ-models that are elementary in structures of the form $H(\theta)$. [10]

Definition 11.3.

- *We define the* completely ineffable subset operator $\mathcal{C}$ *as follows: Given any ideal I on κ, let $A \in \mathcal{C}(I)^+$ if for all $y \subseteq \kappa$ and all regular cardinals $\theta > \kappa$, there is a weak κ-model $M \prec H(\theta)$ with $y \in M$, and an M-normal ultrafilter U on κ that is κ-amenable for M, with $A \in U$ and $U \subseteq I^+$.*
- *We define the* ω-Ramsey subset operator $\mathbf{wf}^\forall$ *as follows: Given any ideal I on κ, let $A \in \mathbf{wf}^\forall(I)^+$ if for all $y \subseteq \kappa$ and all regular cardinals $\theta > \kappa$, there is a weak κ-model $M \prec H(\theta)$ with $y \in M$, and an M-normal ultrafilter U on κ that is κ-amenable for M, with $A \in U$ and $U \subseteq I^+$, such that the ultrapower of M by U is well-founded.*

Theorem 11.8 below. It would however be interesting to find a larger family of ideals on which these two operators disagree.

[9] In fact, we need to observe that Proposition 2.2 is a theorem of ZFC^-.

[10] So far, there wouldn't be any reason to not extend our framework to also accomodate these operators. The reason why we focus our attention on operators that relate to Scheme B from [11] is that they correspond to second order properties of V_κ, as we will argue in Section 12, and that this allows for applications of our concepts: we provide an example of this in Section 13.

Note that by the results of [11, Section 11], a set $A \subseteq \kappa$ is completely ineffable if and only if $\kappa \in \mathcal{C}([\kappa]^{<\kappa})^+$, and moreover that $\mathcal{C}([\kappa]^{<\kappa})$ is the completely ineffable ideal on κ. By its very definition, a cardinal κ is ω-Ramsey if and only if $\kappa \in \mathbf{wf}^\forall([\kappa]^{<\kappa})^+$. We will need the following, which in particular yields an easy argument that starting from their above characterization, completely ineffable subsets of a cardinal κ are Π^1_β-indescribable for every $\beta < \kappa$.

Lemma 11.4.
- *If $\theta > \kappa$ is a regular cardinal, $M \prec H(\theta)$ is a weak κ-model with $V_\kappa \subseteq M$, and U is an M-normal M-ultrafilter on κ that is κ-amenable for M, then, for every $\beta < \kappa$, $U \subseteq \Pi^1_\beta(\kappa)^+$.*
- *In particular, for every $\beta < \kappa$,*
  - *$\mathcal{C}(\Pi^1_\beta(\kappa)) = \mathcal{C}([\kappa]^{<\kappa})$, and*
  - *$\mathbf{wf}^\forall(\Pi^1_\beta(\kappa)) = \mathbf{wf}^\forall([\kappa]^{<\kappa})$.*

Proof. For the first item, assume for a contradiction that for some $\beta < \kappa$, an M-ultrafilter U as in the statement of the lemma contains a set X that is Π^1_β-describable. Thus, there is a Π^1_β-formula φ and $Q \subseteq V_\kappa$ such that $\langle V_\kappa, Q\rangle \models \varphi$, however, for every $\alpha \in X$, $\langle V_\alpha, Q \cap V_\alpha\rangle \models \neg\varphi$. Since $V_\kappa \subseteq M$, the code for the formula φ is an element of M. Thus, by elementarity, the above statement holds true also in M. But then, in the ultrapower of M by U, using the identification from [11, Lemma 3.5], $\langle V_\kappa, Q\rangle \models \neg\varphi$. This however contradicts that U is κ-amenable for M, and hence that the ultrapower embedding induced by U is κ-powerset preserving.

The second item is immediate from the first item. □

Before we can finally separate the operators $\mathbf{T}$ and $\mathcal{W}$, we need to introduce two more operators that do not fit within the framework of operators that is used within this paper, because they are making use of weak κ-models that are elementary in structures of the form $H(\kappa^+)$. These operators are closely linked to the notions of $\mathbf{T}^{\kappa^+}_\omega$-Ramsey cardinals and $\mathbf{wf}^{\kappa^+}_\omega$-Ramsey cardinals that were introduced in [11, Section 9].

Definition 11.5.
- *We define the $\mathbf{T}^+$-Ramsey subset operator $\mathbf{T}^+$ as follows: Given any ideal I on κ, let $A \in \mathbf{T}^+(I)^+$ if for all $y \subseteq \kappa$, there is a weak κ-model $M \prec H(\kappa^+)$ with $y \in M$, and an M-normal M-ultrafilter U on κ that is κ-amenable for M, with $A \in U$ and $U \subseteq I^+$.*
- *We define the $\mathbf{wf}^+$-Ramsey subset operator $\mathbf{wf}^+$ as follows: Given any ideal I on κ, let $A \in \mathbf{wf}^+(I)^+$ if for all $y \subseteq \kappa$, there is a weak κ-model $M \prec H(\kappa^+)$ with $y \in M$, and an M-normal M-ultrafilter U on κ that is κ-amenable for M, with $A \in U$ and $U \subseteq I^+$, such that the ultrapower of M by U is well-founded.*

Note that by the very definitions of the operators involved, for any ideal I on κ, we trivially have $\mathbf{T}(I) \subseteq \mathbf{T}^+(I) \subseteq \mathcal{C}(I)$, and $\mathcal{W}(I) \subseteq \mathbf{wf}^+(I) \subseteq \mathbf{wf}^\forall(I)$. [11]

Lemma 11.6 (Holy,Lücke)**.**
- $\{\alpha < \kappa \mid \alpha \in \mathbf{T}^+(\Pi^1_\beta(\alpha))\} \notin \mathbf{T}(\Pi^1_\beta(\kappa))$.
- $\{\alpha < \kappa \mid \alpha \in \mathbf{wf}^+(\Pi^1_\beta(\alpha))\} \notin \mathcal{W}(\Pi^1_\beta(\kappa))$.

[11]Moreover, we observe that the proof of Lemma 11.4 shows that the statements about the operators $\mathcal{C}$ and $\mathbf{wf}^\forall$ in Lemma 11.4 actually hold true already for the operators $\mathbf{T}^+$ and $\mathbf{wf}^+$. However, we will not need this for our present purposes.

Proof. By [11, Lemma 9.13 (2)], taking $\Psi(M,U)$ to be the statement that $U \subseteq \Pi^1_\beta(\kappa)^+$ for the first item, and taking $\Psi(M,U)$ to be the conjunction of that statment with the statement that the ultrapower of M by U is well-founded for the second item. $\square$

Proposition 11.7.
- *(Holy, Lücke)* $\mathbf{T}([\kappa]^{<\kappa}) \subsetneq \mathcal{W}([\kappa]^{<\kappa})$.
- *(Gitman, Welch)* $\mathcal{W}([\kappa]^{<\kappa}) \subsetneq \mathcal{R}([\kappa]^{<\kappa})$.

Proof. The first item follows from a combination of results from [11] together with [9, Theorem 3.7], as in the proof of Theorem 11.8 below, except that we do not need to make use of Lemma 11.4 in the end. The second item is shown within the proof of [10, Theorem 4.1] (see also [11, Theorem 1.5]), noting that weakly Ramsey cardinals are exactly the 1-iterable cardinals from that paper. $\square$

For the inclusion between $\mathbf{T}$ and $\mathcal{W}$, we can extend this result to the indescribability ideals.

Theorem 11.8. *If $\kappa \in \mathbf{T}(\Pi^1_\beta(\kappa))^+$ and $\beta \in \{-1\} \cup \kappa$, then*

$$\mathbf{T}(\Pi^1_\beta(\kappa)) \subsetneq \mathcal{W}(\Pi^1_\beta(\kappa)).^{12}$$

Proof. Clearly, by the very definitions involved, $\mathbf{T}(I) \subseteq \mathcal{W}(I)$ for any ideal I on κ, and it only remains to verify inequality in the above. Since this is trivial otherwise, we may as well assume that $\kappa \in \mathcal{W}(\Pi^1_\beta(\kappa))^+$. By Lemma 11.6,

$$X = \{\alpha < \kappa \mid \alpha \in \mathbf{T}^+(\Pi^1_\beta(\alpha))\} \notin \mathbf{T}(\Pi^1_\beta(\kappa)).$$

On the other hand, the proof of Gitman's [9, Theorem 3.7] shows (see also [11, Theorem 1.5]) that if κ is a weakly Ramsey cardinal, then

$$Y = \{\alpha < \kappa \mid \alpha \text{ is not completely ineffable}\} \in w\mathcal{R}(\kappa).$$

Since $w\mathcal{R}(\kappa) \subseteq w\mathcal{R}(\Pi^1_\beta(\kappa))$, we will be done if we show that $X \subseteq Y$, which amounts to showing that completely ineffable cardinals α satisfy $\alpha \in \mathbf{T}^+(\Pi^1_\beta(\alpha))^+$. But clearly, $\mathbf{T}^+(\Pi^1_\beta(\alpha)) \subseteq \mathcal{C}(\Pi^1_\beta(\alpha)) = \mathcal{C}([\alpha]^{<\alpha})$ by Lemma 11.4, and thus we are done. $\square$

If the answer to the following question were positive, using ω-Ramseyness rather than complete ineffability in the proof of Theorem 11.8, we could analogously separate the operators $\mathcal{W}$ and $\mathcal{R}$ in a strong sense. This question is a particular instance of [11, Question 17.4].

Question 11.9. *Assume that κ is a Ramsey cardinal. Does it follow that the set of ω-Ramsey cardinals below κ is a Ramsey subset of κ?*

12. Some coding apparatus

In order to be able to present a sample result for our generalized operators in the final section of this paper, we will need to code weak κ-models M and M-ultrafilters U on κ as subsets of V_κ in some simple way, and since we are only really interested in the case when κ is an inaccessible cardinal, we may assume this to be the case whenever necessary. Our definition will be tailored so that any transitive weak

[12] Let us remark that we cannot expect this subset relation to be proper for any ideal I, since for example if κ is a completely Ramsey cardinal (see [7, Section 3]), and I is the completely Ramsey ideal on κ, then $I \subseteq \mathbf{T}(I) \subseteq \mathcal{R}(I) = I$, and hence all of these ideals have to be equal.

κ-model that can be coded will have to be a superset of V_κ, with elements x of V_κ being coded as ordered pairs of the form $\langle 0, x\rangle$, and we code κ by 0.

Definition 12.1. *We say that $\mathcal{M} \subseteq V_\kappa$ is a* code for a transitive weak κ-model *if $\mathcal{M} \subseteq V_\kappa$ with the following properties:*

- *$\mathcal{M}$ is a binary relation on V_κ, such that* $\mathrm{dom}(\mathcal{M}) = V_\kappa$,
- *for all $x, y \in V_\kappa$, $\langle 0, x\rangle\mathcal{M}\langle 0, y\rangle$ if and only if $x \in y$,*
- *for all x, $x\,\mathcal{M}\,0$ if and only if $x \in \kappa$,*
- *$\mathcal{M}$ is well-founded and extensional, and*
- $\langle V_\kappa, \mathcal{M}\rangle \models \mathrm{ZFC}^-$.

Note that the weak κ-model that is coded here is the model M such that $\langle M, \in\rangle$ is the transitive collapse of $\langle V_\kappa, \mathcal{M}\rangle$. Let π denote the transitive collapsing map. If $X \in M$, we let $\mathcal{M}(X) = x$ in case $X = \pi(x)$, and we say that x is the code of X *(within $\mathcal{M}$).*

Lemma 12.2. *The property that $\mathcal{M}$ is a code for a transitive weak κ-model is a Δ^1_1-property over $\langle V_\kappa, \mathcal{M}\rangle$.*

Proof. All but the final item in the above list can easily be phrased as first order properties within $\langle V_\kappa, \mathcal{M}\rangle$. The final item can be seen to be a Δ^1_1 property, for we need to say that either there is a satisfaction relation (coded as a subset of V_κ in some obvious way) for $\langle V_\kappa, \mathcal{M}\rangle$ that contains all axioms of ZFC^-, or that this is the case for all satisfaction relations, and being a satisfaction relation for $\langle V_\kappa, \mathcal{M}\rangle$ is again a first order property, which is seen as usual. $\square$

Note that we can easily shift between subsets of V_κ and their codes within $\mathcal{M}$ – For $X \subseteq V_\kappa$ in M and $x \in V_\kappa$, the property $\mathcal{M}(X) = x$ is a first order property over $\langle V_\kappa, \in, \mathcal{M}, X\rangle$: $\mathcal{M}(X) = x$ in case $\forall y\ [\langle 0, y\rangle\mathcal{M}x \iff y \in X]$.

Next, we want to define what it means to code an M-ultrafilter on κ, which will easily be seen to be a first order property.

Definition 12.3. *Given a code $\mathcal{M}$ for a transitive weak κ-model, we say that $\mathcal{U} \subseteq V_\kappa$ is a* code for an M-ultrafilter on κ *if $\langle V_\kappa, \mathcal{M}, \mathcal{U}\rangle$ thinks that $\mathcal{U}$ is an ultrafilter on 0 (note that our setup is so that 0 codes κ).*

For our desired applications, we will need our operators to satisfy some properties of simple definability.

Definition 12.4.

- *A pair of formulas $\langle \Psi, \Omega\rangle$ is* simple *in case* ZFC *proves the following:*
  - *whenever M is a transitive weak κ-model, and U is an M-ultrafilter on κ, then $\Psi(M, U)$ translates to a Δ^1_1-property of the pair of codes $\langle \mathcal{M}, \mathcal{U}\rangle$ for $\langle M, U\rangle$ over V_κ, and*
  - *whenever the property $X \in I^+$ is definable over V_κ by a Π^1_β-formula $\varphi(X)$ for some $0 < \beta < \kappa$, then $\Omega(U, I)$ translates to a Π^1_β-property of the code $\mathcal{U}$ of U over V_κ.*
- *$\langle \Psi, \Omega\rangle$ is* always simple *in case* ZFC *additionally proves that if in the second item above, the property $X \in I^+$ is first order definable over V_κ, then $\Omega(U, I)$ translates to a Δ^1_1-property of $\mathcal{U}$ over V_κ.*
- *Let $\mathcal{O}$ be an ideal operator. We say that $\mathcal{O}$ is* simple *or* always simple *in case* ZFC *proves that $\mathcal{O}$ can be characterized in the form $\mathcal{O} = \mathfrak{O}\Psi\Omega$ for some pair of formulas $\langle \Psi, \Omega\rangle$ that is simple or always simple respectively.*

The first item above is immediate if Ψ can be expressed as a first order property of the structure $\langle M,\in,U\rangle$, for example if $\Psi(M,U)$ denotes the statement that U is κ-amenable for M. If $\Psi(M,U)$ denotes the property that U is countably complete, this translates to the first order statement that for any countable sequence $\langle u_i \mid i<\omega\rangle$ of elements of $\mathcal{U}$, [13] there is x such that $x\mathcal{M}u_i$ for every $i<\omega$. Let us treat the case when $\Psi(M,U)$ denotes the statement that the ultrapower of M by U is well-founded. Let us say that $\langle N,R\rangle$ *represents an ultrapower* of M by U if $N\subseteq V_\kappa$ consists of codes for functions f with domain κ such that

- for every $g\in M^*$ for which $\mathcal{M}(g)$ is a function with domain κ, there is $f\in N$ such that $\{\alpha<\kappa \mid \mathcal{M}(f)(\alpha)=\mathcal{M}(g)(\alpha)\})$ is coded by a set in $\mathcal{U}$,
- for all $f,g\in N$, $\{\alpha<\kappa \mid \mathcal{M}(f)(\alpha)=\mathcal{M}(g)(\alpha)\}$ is not coded by a set in $\mathcal{U}$, and
- if $f,g\in N$, $f\,R\,g$ iff $\{\alpha<\kappa \mid \mathcal{M}(f)(\alpha)\in\mathcal{M}(g)(\alpha)\}$ is coded by a set in $\mathcal{U}$.

That is, essentially, a class $[f]_U$ in a usual ultrapower of M by U is taken to be represented by one of its elements. But now, asking that $\langle N,R\rangle$ is well-founded is clearly equivalent to asking the ultrapower of M by U to be well-founded, and moreover, as for the case of countable completeness above, this translates to a first order statement of V_κ, using that κ is regular and uncountable: it requires asking that no countable sequence of elements of N is decreasing with respect to R.

Let us look at some examples regarding the second item in Definition 12.4.

- If $\Omega(U,I)$ denotes the statement that $U\subseteq I^+$, then this translates to the statement that $\forall x\,\mathcal{M}\,\mathcal{U}\,\forall X\ [\mathcal{M}(X)=x\rightarrow\varphi(X)]$.
- If $\Omega(U,I)$ denotes the property that countable intersections from U are in I^+, then this translates to the statement that for any countable sequence $\langle u_\beta \mid \beta<\omega\rangle$ of $\mathcal{M}$-elements of $\mathcal{U}$,
$$\varphi(\{\alpha<\kappa \mid \forall\beta<\omega\ \langle 0,\alpha\rangle\,\mathcal{M}\,u_\beta\}).$$
- If $\Omega(U,I)$ denotes the property that $\Delta U\in I^+$, then this translates to the statement that for any κ-enumeration $\langle u_\beta \mid \beta<\kappa\rangle$ of the $\mathcal{M}$-elements of $\mathcal{U}$,
$$\varphi(\{\alpha<\kappa \mid \forall\beta<\alpha\ \langle 0,\alpha\rangle\,\mathcal{M}\,u_\beta\}).$$

If the property $X\in I^+$ is first order definable, observe that we obtain a Δ^1_1-statement in the first two cases above, for we can equivalently rephrase the above to use existential rather than universal second order quantifiers. However this does not work in the third case (see the remarks made in Footnote 2). In particular, this means that the operators $\mathbf{T}$, $\mathcal{W}$ and $\mathcal{R}$ are always simple, while $\mathcal{I}_M$ is simple.

The following lemma extracts what we will actually need in the next section.

Lemma 12.5. *If $\langle\Omega,\Psi\rangle$ is simple, then the following hold.*

- *If $y\subseteq\kappa$ and $\beta<\kappa$, then the statement that $X\in\mathfrak{O}^y(\Pi^1_\beta(\kappa))$ can be expressed as a $\Pi^1_{\beta+2}$-property of X and y over V_κ.*
- *If $\beta<\kappa$, then the statement that $X\in\mathfrak{O}\Psi\Omega(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property of X over V_κ.*
- *If $\langle\Omega,\Psi\rangle$ is always simple, then the above also hold in case $\beta=-1$.*

Proof. Immediate from Lemma 3.3 together with the above. □

[13] If κ is inaccessible (regular and uncountable suffices), then these countable sequences are elements of V_κ.

13. A test application: Generalized Pre-Operators

In this section, we want to provide a sample result, showing that simple ineffable operators are structurally well-behaved, by providing a basic theorem about their relationship to their corresponding pre-operators. This result generalizes the case when $\alpha = 1$ of [6, Theorem 6.1], and shows that our pre-operators have the same key role with respect to their corresponding operators as does the subtle operator with respect to the ineffability operator, and the pre-Ramsey operator with respect to the Ramsey operator by classical results of Baumgartner from his [3] and [4] [14] – these results are instances of our general result below, which in particular also provides valid new instances for the operators $\mathbf{T}$ and $\mathcal{W}$. The proof follows the proof in [6] for its most parts, but there are some subtleties involved in how to make use of the machinery that we developed in Section 12 above, and therefore we would like to provide the complete argument. If $\mathcal{A}$ is a collection of subsets of a cardinal κ, we write $\overline{\mathcal{A}}$ to denote the ideal on κ that is generated by $\mathcal{A}$: This is the collection of all subsets of κ that are contained in some finite union of elements of $\mathcal{A}$.

Theorem 13.1. *If $\mathcal{O}$ is ineffable and simple, and $\beta < \kappa$, then*

$$\mathcal{O}(\Pi^1_\beta(\kappa)) = \overline{\mathcal{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa)}.$$

If $\mathcal{O}$ is ineffable and always simple, then the above also holds in case $\beta = -1$.

Proof. Fix some β, and let $J = \overline{\mathcal{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa)}$. We show that $X \in J^+$ if and only if $X \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$.

Suppose for a contradiction that $X \in J^+$, however $X \in \mathcal{O}(\Pi^1_\beta(\kappa))$. Let $y \subseteq \kappa$ be such that $X \in \mathcal{O}^y(\Pi^1_\beta(\kappa))$, that is whenever M is a transitive weak κ-model with $y \in M$ and U is an M-ultrafilter on κ with $X \in U$ and with $\Psi(M, U)$, then $\Omega(U, \Pi^1_\beta(\kappa))$ fails. By Lemma 12.5, this can be expressed by a $\Pi^1_{\beta+2}$-sentence φ over V_κ, and thus

$$C = \{\xi < \kappa \mid V_\xi \models \varphi(X \cap \xi, y \cap \xi)\} \in \Pi^1_{\beta+2}(\kappa)^*.$$

Since $X \notin J$, X is not the union of a set in $\mathcal{O}_0(\Pi^1_\beta(\kappa))$ and a set in $\Pi^1_{\beta+2}(\kappa)$, and since $X = (X \cap C) \cup (X \setminus C)$, it follows that $X \cap C \notin \mathcal{O}_0(\Pi^1_\beta(\kappa))$. Thus we may find $\xi \in (X \cap C) \setminus (\beta + 1)$, a weak ξ-model $\bar{M}$ with $y \cap \xi \in \bar{M}$ and an $\bar{M}$-ultrafilter $\bar{U}$ on ξ with $X \cap C \cap \xi \in \bar{U}$ such that $\Psi(\bar{M}, \bar{U})$ and $\Omega(\bar{U}, \Pi^1_\beta(\xi))$ hold, contradicting the above.

Now suppose $X \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$. By [6, Remark 2.1], it suffices to show that $X \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$ and $X \in \Pi^1_{\beta+2}(\kappa)^+$, where the latter is immediate from Corollary 10.5. We are thus left to show that $X \in \mathcal{O}_0(\Pi^1_\beta(\kappa))^+$. Fix $y \subseteq \kappa$ and a club $C \subseteq \kappa$. By the third item in Observation 10.4, it follows that $X \cap C \in \mathcal{O}(\Pi^1_\beta(\kappa))^+$. Thus, there are $\mathcal{M}$ and $\mathcal{U}$ such that the following $\Pi^1_{\beta+1}$-sentence φ holds over the structure $\langle V_\kappa, \in, y, X \cap C, M, U\rangle$: $\mathcal{M}$ is (a code for) a transitive weak κ-model M with $y \in M$ and $\mathcal{U}$ is (a code for) an M-ultrafilter U on κ with $X \cap C \in U$ such that $\Psi(M, U)$

[14] Baumgartner has also verified a version of Theorem 13.1 for the weakly ineffable ideal in [3, Section 7]. Namely, he has shown that the weakly ineffable ideal on a cardinal κ is generated by the subtle ideal together with the Π^1_1-indescribable ideal. We do not know whether this result could also be obtained via Theorem 13.1, by using the operator $\mathcal{I}_M$ or perhaps some slight variant. Some strongly related issues will be discussed in Section 14.

and $\Omega(U,\Pi^1_\beta(\kappa))$ hold. Since $X\cap C\in\Pi^1_{\beta+2}(\kappa)^+$, there is $\xi\in(X\cap C)\setminus(\beta+1)$ such that

$$\langle V_\xi,\in,y\cap\xi,X\cap C\cap\xi,M\cap\xi,U\cap\xi\rangle\models\varphi,$$

and hence $X\cap C\cap\xi\in\mathcal{O}^y(\Pi^1_\beta(\xi))^+$, as witnessed by the code $\mathcal{M}\cap\xi$ for a weak ξ-model and the code $\mathcal{U}\cap\xi$ for an M-ultrafilter on ξ, yielding that $X\in\mathcal{O}_0(\Pi^1_\beta(\kappa))^+$. □

By similar means as in Theorem 13.1, many of the results from [6] for the Ramsey operator can be extended to our generalized operators in a fairly straightforward way. Amongst other things, these further generalizations are planned to be included in a follow-up paper. Let us present one final easy sample result here, namely that applications of our operators to different indescribability ideals give rise to a proper hierarchy of large cardinal notions (this is a very weak generalized analogue of results from [6]).

Proposition 13.2. *If $\mathcal{O}$ is ineffable and simple, $\beta<\kappa$, and $\kappa\in\mathcal{O}(\Pi^1_{\beta+1}(\kappa))^+$, then κ is a stationary limit of cardinals α for which $\alpha\in\mathcal{O}(\Pi^1_\beta(\alpha))^+$. If $\mathcal{O}$ is ineffable and always simple, then the above also holds in case $\beta=-1$.*

Proof. Assume that $\kappa\in\mathcal{O}(\Pi^1_{\beta+1}(\kappa))^+$. By the monotonicity of $\mathcal{O}$, we thus also have $\kappa\in\mathcal{O}(\Pi^1_\beta(\kappa))^+$. But by Corollary 10.5, our assumption implies that $\kappa\in\Pi^1_{\beta+3}(\kappa)^+$, and by Lemma 12.5, $\kappa\in\mathcal{O}(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property over V_κ. Hence, the set of cardinals α for which $\alpha\in\mathcal{O}(\Pi^1_\beta(\alpha))^+$ is contained in the $\Pi^1_{\beta+3}$-indescribable filter on κ, and hence in particular this set is stationary in κ. □

Let us close this section by providing some additional information about the relationship between the operators $\mathcal{I}$ and $\mathcal{R}$.

Observation 13.3. *Assume that $\beta\in\{-1\}\cup\mathrm{Ord}$, and that κ is least such that $\kappa\in\mathcal{R}(\Pi^1_\beta(\kappa))^+$. Then $\kappa\notin\mathcal{I}(\Pi^1_{\beta+1}(\kappa))^+$.*

Proof. Assume for a contradiction that $\kappa\in\mathcal{I}(\Pi^1_{\beta+1}(\kappa))^+$. Then, by Theorem 13.1, $\kappa\in\Pi^1_{\beta+3}(\kappa)^+$. By Lemma 12.5, $\kappa\in\mathcal{R}(\Pi^1_\beta(\kappa))^+$ can be expressed as a $\Pi^1_{\beta+3}$-property over V_κ. Combining these, we find some $\alpha<\kappa$ such that $\alpha\in\mathcal{R}(\Pi^1_\beta(\alpha))^+$, contradicting the leastness of κ. □

14. Some remarks on weak ineffability

In this final section, we want to treat the seemingly problematic case of applying operators of the form $\mathcal{O}\Psi\Omega$ with $\Omega(U,I)$ being the statement that $\Delta U\in I^+$ to ideals of the form $[\kappa]^{<\kappa}$. Note that by the convention from Section 2, $\Delta U\in I^+$ abbreviates the statement that every diagonal intersection of U is in I^+. If we strengthen this a tiny bit, we can show the following, which is based on the observation from [11] that the notions of *genuinity* and *normality* from [15] coincide for weak κ-models.

Observation 14.1. *If $\Omega(U,I)$ denotes the property that every diagonal intersection of elements of U is in I^+, and $\mathcal{O}=\mathcal{O}\Psi\Omega$, then for any ideal I on κ and $y\subseteq\kappa$, $\mathcal{O}^y(I)=\mathcal{O}^y(\overline{I\cup\mathrm{NS}_\kappa})$. In particular, this implies that $\mathcal{O}(I)=\mathcal{O}(\overline{I\cup\mathrm{NS}_\kappa})$, and hence that $\mathcal{O}([\kappa]^{<\kappa})=\mathcal{O}(\mathrm{NS}_\kappa)$. Moreover, given a sequence $\vec{I}$ of ideals, this also implies that $\mathcal{O}_0(\vec{I})=\mathcal{O}_0(\langle\overline{I_\alpha\cup\mathrm{NS}_\alpha}\mid\alpha\leq\kappa\rangle)$, and hence that $\mathcal{O}_0([\kappa]^{<\kappa})=\mathcal{O}_0(\mathrm{NS}_\kappa)$.*

Proof. It is immediate that $\mathcal{O}(I) \subseteq \mathcal{O}(\overline{I \cup \mathrm{NS}_\kappa})$. Assume thus that $y \subseteq \kappa$, and that $A \in \mathcal{O}^y(I)^+$. That is, there is a transitive weak κ-model M with $y \in M$ and an M-ultrafilter U on κ with $A \in U$ such that $\Psi(M, U)$ holds and every diagonal intersection of elements of U is in I^+, and thus in particular an unbounded subset of κ. In the notation of [15], this means that U is a *genuine* M-ultrafilter. But by [11, Proposition 17.2], this implies that U is in fact a *normal* M-ultrafilter, meaning that ΔU is a stationary subset of κ. Making use of [6, Remark 2.1], this shows that $A \in \mathcal{O}^y(\overline{I \cup \mathrm{NS}_\kappa})^+$, as desired. The remaining statements follow by the very definitions of the operators involved. □

The above tells us for example that we cannot characterize the weakly ineffable ideal $\mathcal{I}([\kappa]^{<\kappa})$ by an operator of the form $\Psi\Omega$, where $\Psi(M, U)$ is trivial and $\Omega(U, I)$ denotes the slight strengthening of $\Delta U \in I^+$ from Observation 14.1, for applying such an operator to ideals of the form $[\kappa]^{<\kappa}$ would already yield the ineffable ideal rather than the weakly ineffable ideal on κ whenever κ is ineffable. As was already observed in [11, Section 17], such a characterization (of weak ineffability only) was wrongly claimed in [15, Theorem 3.2 (ii)]. Concerning our original operator $\mathcal{I}_M$, we do not know as to whether $\mathcal{I}_M([\kappa]^{<\kappa})$ is the weakly ineffable ideal on κ, however it seems unlikely to us. Let us introduce yet another variant of the operator $\mathcal{I}_M$.

Definition 14.2.
- *For any $y \subseteq \kappa$, let $x \in \mathcal{I}_M^{*y}(I)^+$ if there is a transitive weak κ-model M with $y \in M$ such that for any κ-enumeration $\vec{x}$ of $\mathcal{P}(\kappa) \cap M$, there is an M-ultrafilter U on κ that flips $\vec{x}$ such that $x \in U$ and $\Delta U \in I^+$, and let*
- *$\mathcal{I}_M^*(I)^+ = \bigcap_{y \subseteq \kappa} \mathcal{I}_M^{*y}(I)^+$.*

Assume that κ is weakly ineffable. Then, by the very definitions of the operators involved, $I_M^*([\kappa]^{<\kappa}) \subseteq I_C([\kappa]^{<\kappa})$, and we have shown the latter to be equal to the weakly ineffable ideal on κ in Proposition 2.2. However, the reverse inclusion seems to be potentially problematic, and thus we ask the following, which we conjecture to be wrong.

Question 14.3. *Is $\mathcal{I}_M^*([\kappa]^{<\kappa})$ the weakly ineffable ideal on κ?*

A positive answer to Question 14.3 would show that a cardinal κ is weakly ineffable if and only if $\kappa \in I_M^*([\kappa]^{<\kappa})^+$, and this was in fact claimed in [11, Paragraph after Proposition 17.2] without proof, however we do not know how to verify this claim, and would like to pose it as an open question. [15] It seems likely that a positive answer to Question 14.4 would also yield a positive answer for Question 14.3.

Question 14.4. *Does $\kappa \in I_M^*([\kappa]^{<\kappa})^+$ imply that κ is weakly ineffable?*

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