Ramsey-like Operators

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Introduction

Starting from measurability upwards, larger large cardinals are usually characterized by the existence of certain elementary embeddings of the universe, or dually, the existence of certain ultrafilters. However, below measurability, we have a somewhat similar picture when we consider certain embeddings with set-sized domain, or ultrafilters for small collections of sets. I will present some new results, and also review some older ones, showing that not only large cardinals, but also several related concepts – in particular *large cardinal operators* and their associated *pre-operators* – can be characterized in such a way, supporting the usefulness of such characterizations. I will also provide a sample application of these characterizations.

Large Cardinals

Ramsey cardinals

κ is a *Ramsey* cardinal if every function $c\colon [\kappa]^{<\omega} \to 2$ has a homogeneous set H of size κ.

Theorem (Mitchell (70ies) / Gitman, Sharpe, Welch (2011))

κ is Ramsey iff for every $y \subseteq \kappa$ there is a weak κ-model $M \ni y$, and a κ-amenable, countably complete and M-normal M-ultrafilter U on κ.

- A *weak κ-model M* is a transitive model of ZFC^- with $|M| = \kappa$ and $\kappa + 1 \subseteq M$.
- An *M-ultrafilter U* on κ is a filter that measures all subset of κ in M. U is *M-normal* if it is closed under diagonal intersections in M.
- We require all our filters to be uniform: they only have elements of size κ.
- U is *countably complete* if any countable intersection (in V) of elements of U is nonempty (equivalently, unbounded in κ).
- U is *κ-amenable* if whenever X is a set of size κ in M, then $X \cap U \in M$.

Ineffable cardinals

- A κ-*list* is a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \kappa \rangle$ s.t. $a_\alpha \subseteq \alpha$ for every $\alpha < \kappa$.
- H is homogeneous for $\vec{a}$ if $a_\alpha = a_\beta \cap \alpha$ for $\alpha < \beta$ both in H.

κ is *ineffable* if every κ-list has a stationary homogeneous set.

Proposition (Abramson et al (70ies) / Holy, Lücke (2020))

κ is an ineffable cardinal iff for every $y \subseteq \kappa$ there is a weak κ-model $M \ni y$, and an M-ultrafilter U on κ such that any diagonal intersection of U is stationary – we write: ΔU is stationary.

Large Cardinal Ideals

The Ramsey ideal

Lemma (Baumgartner (70ies))

κ is a Ramsey cardinal iff every regressive function $f\colon [\kappa]^{<\omega} \to \kappa$ has a homogeneous set of size κ.

He used this to define Ramseyness of a subset A of κ:

$A \subseteq \kappa$ is *Ramsey* if every regressive function $f\colon [\kappa]^{<\omega} \to \kappa$ has a homogeneous set $H \subseteq A$ of size κ.

The *Ramsey ideal* on a cardinal κ is the collection of all subsets of κ that are not Ramsey. It is a normal ideal on κ.

Theorem (Mitchell (70ies) / Sharpe, Welch (2011))

$A \subseteq \kappa$ is Ramsey iff for every $y \subseteq \kappa$ there is a weak κ-model $M \ni y$, and a κ-amenable, countably complete and M-normal M-ultrafilter U on κ with $A \in U$.

The ineffable ideal

Baumgartner (70ies) also made the following definition:

$A \subseteq \kappa$ is *ineffable* if every κ-list has a stationary homogeneous set $H \subseteq A$.

The *ineffable ideal* on a cardinal κ is the collection of all subsets of κ that are not ineffable. It is a normal ideal on κ.

Proposition (Abramson et al (70ies) / Holy, Lücke (2020))

$A \subseteq \kappa$ is ineffable iff for every $y \subseteq \kappa$ there is a weak κ-model $M \ni y$, and an M-ultrafilter U on κ such that ΔU is stationary, with $A \in U$.

Why, for example, we should care about large cardinal ideals:

Two results of Baumgartner

Subtlety

First, I need to introduce even more notions.

Definition

A cardinal κ is *subtle* if for every club $C \subseteq \kappa$ and every κ-list $\vec{a}$, there are $\alpha < \beta$ in C such that $a_\alpha = a_\beta \cap \alpha$.

Definition

$A \subseteq \kappa$ is *subtle* if for every club $C \subseteq \kappa$ and every κ-list $\vec{a}$, there are $\alpha < \beta$ in $A \cap C$ such that $a_\alpha = a_\beta \cap \alpha$.

Lemma (Baumgartner (70ies))

$A \subseteq \kappa$ is subtle iff for every club $C \subseteq \kappa$ and every κ-list $\vec{a}$, there is $\alpha \in A$ and a stationary subset H of $C \cap A \cap \alpha$ such that H is homogeneous for $\vec{a}$.

The subtle ideal is the collection of all subsets of κ that are not subtle. It is a normal ideal on κ.

Pre-Ramseyness

Pre-Ramseyness is a sort of mix between Ramseyness and subtlety. It relates to Ramseyness as does subtlety to ineffability.

Reminder

$A \subseteq \kappa$ is subtle iff for every club $C \subseteq \kappa$ and every κ-list $\vec{a}$, there is $\alpha \in A$ and a stationary subset H of $C \cap A \cap \alpha$ such that H is homogeneous for $\vec{a}$.

Definition (Baumgartner)

$A \subseteq \kappa$ is *pre-Ramsey* if for every club $C \subseteq \kappa$ and every regressive function $f \colon [\kappa]^{<\omega} \to \kappa$, there is $\alpha \in A$ and an unbounded subset H of $C \cap A \cap \alpha$ such that H is homogeneous for f.

The pre-Ramsey ideal is the collection of all subsets of κ that are not pre-Ramsey. It is a normal ideal on κ.

Indescribability

A Σ^1_n-formula is a formula that starts with n alternating blocks of second order quantifiers $\exists$ and $\forall$, starting with $\exists$, followed by a formula with only first order quantifiers. Π^1_n-formulae are defined analogously, starting with $\forall$.

Definition (Levy, 70ies)

$A \subseteq \kappa$ is Π^1_n-indescribable if whenever $P \subseteq \kappa$ and φ is a Π^1_n-formula such that $\langle V_\kappa, \in, P\rangle \models \varphi$, then there is $\alpha \in A$ such that $\langle V_\alpha, \in, P \cap V_\alpha\rangle \models \varphi$.

The Π^1_n-indescribable ideal $\Pi^1_n(\kappa)$ on κ is the collection of all subsets of κ that are not Π^1_n-indescribable. It is a normal ideal on κ. Note that $\Pi^1_0(\kappa) = \mathrm{NS}_\kappa$, and that Π^1_1-indescribability $\equiv$ weak compactness.

There's an extension of this hierarchy, that allows one to consider Π^1_ξ-indescribability for arbitrary ordinals $\xi < \kappa$, independently due to Sharpe and Welch (2011), and Joan Bagaria (2019). In fact, extensions up to κ^+ have been developed by Sharpe and Welch (2011), and by Brent Cody (2020).

Baumgartner's results

We say that two ideals I and J on κ generate (an ideal) $K = \overline{I \cup J}$ on κ in case K consists of all unions $x \cup y$ with $x \in I$ and $y \in J$.

Theorem (Baumgartner, 70ies)

κ is Ramsey if the pre-Ramsey and the Π^1_1-indescribable ideal on κ generate a nontrivial ideal. This then is the Ramsey ideal on κ.

Ideals are necessary in this statement: the least cardinal that is pre-Ramsey and Π^1_1-indescribable is strictly below the least Ramsey cardinal.

Theorem (Baumgartner, 70ies)

κ is ineffable if the subtle and the Π^1_2-indescribable ideal on κ generate a nontrivial ideal. This then is the ineffable ideal on κ.

Ideals are necessary in this statement: the least cardinal that is subtle and Π^1_2-indescribable is strictly below the least ineffable cardinal.

Large Cardinal Operators

Large cardinal operators

Large cardinal operators are maps $\mathfrak{O}$ between ideals on (large) cardinals κ. While large cardinal ideals are collections of certain small subsets of large cardinals, given an ideal I on a large cardinal κ, $\mathfrak{O}(I)$ describes certain subsets that are small *relative to* I. This generalizes what we have seen so far:

- The Ramsey ideal consists of subsets of a Ramsey cardinal that are small with respect to the bounded ideal.
- The ineffable ideal consists of subsets of an ineffable cardinal that are small with respect to the nonstationary ideal.

Some basic properties of large cardinal operators:

- $\forall I\ \mathfrak{O}(I) \supseteq I$,
- $\forall I, J\ [I \subseteq J \to \mathfrak{O}(I) \subseteq \mathfrak{O}(J)]$.

Notation: Positivity

If I is an ideal on some cardinal κ, we let I^+ denote the collection of subsets of κ that are not in I, i.e. the complement of I. Sets in I^+ are usually also called *I-positive*.

We will often define certain ideals I by actually defining the collection of I-positive sets in the following.

The Ramsey operator

The Ramsey operator $\mathcal{R}$ was introduced by Qi Feng (1989).

Given an ideal I on κ, let $\mathcal{R}(I)^+$ be the set of all $A \subseteq \kappa$ such that any regressive function $f\colon [\kappa]^{<\omega} \to \kappa$ has a homogeneous set $H \subseteq A$ in I^+.

We next introduce what we want to call the *model version* of the Ramsey operator.

Let $\mathcal{R}_M(I)^+$ be the set of all $A \subseteq \kappa$ such that for any $y \subseteq \kappa$, there is a weak κ-model $M \ni y$, and a κ-amenable M-normal M-ultrafilter U on κ with $A \in U$, such that any countable intersection of elements of U is in I^+.

Theorem (Sharpe, Welch (2011))

For any ideal I on κ,

$$\mathcal{R}_M(I) = \mathcal{R}(I).$$

The ineffability operator

The ineffability operator $\mathcal{I}$ was introduced by Baumgartner (70ies).

Given an ideal I on κ, let $\mathcal{I}(I)^+$ be the set of all $A \subseteq \kappa$ such that any κ-list has a homogeneous set $H \subseteq A$ in I^+.

We also introduce a model version.

Let $\mathcal{I}_M(I)^+$ be the set of all $A \subseteq \kappa$ such that for any $y \subseteq \kappa$, there is a weak κ-model $M \ni y$, and an M-ultrafilter U on κ with $A \in U$ and $\Delta U \in I^+$.

Proposition

For any ideal $I \supseteq \mathrm{NS}_\kappa$ on κ,

$$\mathcal{I}_M(I) = \mathcal{I}(I).$$

Pre-operators

Pre-operators

We have seen that pre-Ramseyness relates to Ramseyness as does subtlety to ineffability. Hence, subtlety could perhaps be called *pre-ineffability*. This concept of *pre-versions* of large cardinals, and also their associated ideals and operators, can be generalized, in particular when we have suitable characterizations of these objects in terms of the existence of certain models and ultrafilters. For this, we need the (easy) concept of *local instances* of our operators.

Local Instances of our Operators

The ineffability operator

- $A \in \mathcal{I}(I)^+$ if any κ-list $\vec{a}$ has a homogeneous set $H \subseteq A$ in I^+.
- For any κ-list $\vec{a}$, $A \in \mathcal{I}^{\vec{a}}(I)^+$ if $\vec{a}$ has a homogeneous set $H \subseteq A$ in I^+.

...and its model version

- $A \in \mathcal{I}_M(I)^+$ if for any $y \subseteq \kappa$, there is a weak κ-model $M \ni y$, and an M-ultrafilter U on κ with $A \in U$ and $\Delta U \in I^+$.
- For any $y \subseteq \kappa$, $A \in \mathcal{I}_M^y(I)^+$ if there is a weak κ-model $M \ni y$, and an M-ultrafilter U on κ with $A \in U$ and $\Delta U \in I^+$.

It seems that the local instances of the operators $\mathcal{I}$ and $\mathcal{I}_M$ do not agree. Similarly, this is also the case for the local instances for Ramseyness, where local instances for $\mathcal{R}$ are provided by regressive functions $f : [\kappa]^{<\omega} \to \kappa$, and local instances for $\mathcal{R}_M$ are provided by $y \subseteq \kappa$.

A little more notation: Sequences of Ideals

We refer to a sequence $\vec{I} = \langle I_\alpha \mid \alpha \leq \kappa \rangle$ such that each I_α is an ideal on α, and α ranges over inaccessible cardinals, as a sequence of ideals. Typical examples are when each $I_\alpha = [\alpha]^{<\alpha}$, each $I_\alpha = \mathrm{NS}_\alpha$, or for some fixed β, each $I_\alpha = \Pi^1_\beta(\kappa)$ (for $\alpha > \beta$, and trivial otherwise).

If $\vec{I}$ is uniformly defined (say for example $I_\alpha = NS_\alpha$ for every α), we sometimes identify $\vec{I}$ and I_κ.

Pre-Operators

Examples

The subtle operator is the operator $\mathcal{I}_0$, where

$$\mathcal{I}_0(\vec{I})^+ = \{A \subseteq \kappa \mid \forall \vec{a}\, \forall C \subseteq \kappa \text{ club } \exists \alpha \in A\ A \cap C \cap \alpha \in \mathcal{I}^{\vec{a} \restriction \alpha}(I_\alpha)^+\}.$$

The pre-Ramsey operator is the operator $\mathcal{R}_0$, where

$$\mathcal{R}_0(\vec{I})^+ = \{A \subseteq \kappa \mid \forall f\, \forall C \subseteq \kappa \text{ club } \exists \alpha \in A\ A \cap C \cap \alpha \in \mathcal{R}^{f \restriction \alpha}(I_\alpha)^+\}.$$

$\mathcal{I}_0(\mathrm{NS}_\kappa)$ is the subtle ideal on κ,
$\mathcal{R}_0([\kappa]^{<\kappa})$ is the pre-Ramsey ideal on κ.

General definition

Given an operator $\mathfrak{O}$ with local instances $\mathfrak{O}^p$, given by parameters p with restrictions $p \restriction \alpha$, and a sequence $\vec{I}$ of ideals, let $\mathfrak{O}_0(\vec{I})^+$ be defined as

$$\{A \subseteq \kappa \mid \forall p \forall C \subseteq \kappa \text{ club } \exists \alpha \in A\ A \cap C \cap \alpha \in \mathfrak{O}^{p \restriction \alpha}(I_\alpha)^+\}.$$

...and their model versions

As for the operators $\mathcal{I}$ and $\mathcal{R}$, the above also defines pre-operators $(\mathcal{I}_M)_0$ and $(\mathcal{R}_M)_0$ that correspond to the operators $\mathcal{I}_M$ and $\mathcal{R}_M$. As a strong indicator towards the usefulness of our model versions, we can show that they induce equivalent pre-operators.

Theorem

For any ideal I on κ, $(\mathcal{R}_M)_0(I) = R_0(I)$,

and if $I \supseteq \mathrm{NS}_\kappa$, then also $(\mathcal{I}_M)_0(I) = \mathcal{I}_0(I)$.

In particular, this gives us a way to characterize the subtle and the pre-Ramsey ideal using small models and ultrafilters.

A general framework for large cardinal operators

General Framework

One strong benefit of our model characterizations is that we can talk about, and prove theorems about a number of large cardinal operators at once, in particular for both the ineffability and the Ramsey operator. I am not going to tell you what this framework looks like exactly (basically, we are quite free to vary our requirements on M and on U, as long as they can be described as Δ^1_1-properties), but I want to introduce a new large cardinal operator which fits into this framework, and which is different from both the ineffability and the Ramsey operator.

A new operator

A new operator

The following large cardinal notion came up in recent joint work with Philipp Lücke. It is *Ramseyness without countable completeness*.

Definition

A cardinal κ is $\mathrm{T}^{\kappa}_{\omega}$-Ramsey if for every $y \subseteq \kappa$ there is a weak κ-model $M \ni y$, and an M-normal M-ultrafilter U on κ that is κ-amenable for M.

Note that since we require all our filters to be uniform, we implicitly require that $U \subseteq ([\kappa]^{<\kappa})^+$ in the above. This naturally induces a weak version of the Ramsey operator.

Definition

$A \in \mathrm{T}(I)^+$ if for every $y \subseteq \kappa$, there is a weak κ-model $M \ni y$, and an M-normal M-ultrafilter U on κ with $A \in U$ that is κ-amenable for M, and such that $U \subseteq I^+$.

So, the difference to the Ramsey operator is that we only ask that $U \subseteq I^+$, rather than that all countable intersections from U be in I^+.

It is actually new!

In above-mentioned joint work with Philipp Lücke, we didn't consider large cardinal operators, however our results show that $\mathcal{I}([\kappa]^{<\kappa}) \subsetneq \mathsf{T}([\kappa]^{<\kappa}) \subsetneq \mathcal{R}([\kappa]^{<\kappa})$.

We can extend this to indescribability ideals (remember: $\Pi^1_0(\kappa) = \mathrm{NS}_\kappa$).

Theorem

For any $\beta < \kappa$, $\mathcal{I}(\Pi^1_\beta(\kappa)) \subsetneq \mathsf{T}(\Pi^1_\beta(\kappa)) \subsetneq \mathcal{R}(\Pi^1_\beta(\kappa))$.

We can't hope to obtain properness as above with respect to any ideal I. For example, if κ is measurable and I is the complement of any normal ultrafilter on κ, then $I \subseteq \mathcal{I}(I) \subseteq \mathsf{T}(I) \subseteq \mathcal{R}(I) = I$.

A test application for large cardinal operators: Baumgartner's result

By a uniform argument, we obtain the following. For $\mathcal{I}$ and $\mathcal{R}$, our argument proceeds using the model versions $\mathcal{I}_M$ and $\mathcal{R}_M$.

Theorem (for $\mathcal{I}$ and $\mathcal{R}$: Brent Cody (2020))

For many operators $\mathfrak{O}$, in particular also for $\mathfrak{O} \in \{\mathcal{I}, \mathsf{T}, \mathcal{R}\}$, and all $\beta < \kappa$, we have

$$\mathfrak{O}(\Pi^1_\beta(\kappa)) = \overline{\mathfrak{O}_0(\Pi^1_\beta(\kappa)) \cup \Pi^1_{\beta+2}(\kappa)}.$$

Ideals are necessary in this statement: the least cardinal κ such that $\kappa \in \mathfrak{O}_0(\Pi^1_\beta(\kappa))^+$ and $\kappa \in \Pi^1_{\beta+2}(\kappa)^+$ is strictly below the least cardinal κ such that $\kappa \in \mathfrak{O}(\Pi^1_\beta(\kappa))^+$.

In most, but not all cases, letting $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$, the above also holds for $\beta = -1$. In fact, many further results on the ineffability operator and the Ramsey operator can be shown to carry over to a larger class of large cardinal operators, that includes the operators $\mathcal{I}$, T, and $\mathcal{R}$, and potentially many other operators defined by the existence of ultrafilters for weak κ-models, by uniform arguments.