SMALL MODELS, LARGE CARDINALS, AND INDUCED IDEALS

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ABSTRACT. We show that many large cardinal notions up to measurability can be characterized through the existence of certain filters for small models of set theory. This correspondence will allow us to obtain a canonical way in which to assign ideals to many large cardinal notions. This assignment coincides with classical large cardinal ideals whenever such ideals had been defined before. Moreover, in many important cases, the relations between these ideals reflect the ordering of the corresponding large cardinal properties both under direct implication and consistency strength.

1. Introduction

The work presented in this paper is motivated by the aim to develop general frameworks for large cardinal properties and their ordering under both direct implication and consistency strength. We develop such a framework for large cardinal notions up to measurability, that is based on the existence of set-sized models and ultrafilters for these models satisfying certain degrees of amenability and normality. This will cover several classical large cardinal concepts like inaccessibility, weak compactness, ineffability, Ramseyness and measurability, and also many of the Ramsey-like cardinals that are an increasingly popular subject of recent set-theoretic research (see, for example, [1], [5], [9], [10], [15], [20], and [23]), but in addition, it also canonically yields a number of new large cardinal notions. We then use these large cardinal characterizations to canonically assign ideals to large cardinal notions, in a way that generalizes all such assignments previously considered in the set-theoretic literature, like the classical definition of the weakly compact ideal, the ineffable ideal, the completely ineffable ideal and the Ramsey ideal. In a great number of cases, we can show that the ordering of these ideals under inclusion directly corresponds to the ordering of the underlying large cardinal notions under direct implication. Similarly, the ordering of these large cardinal notions under consistency strength can usually be read off these ideals in a simple and canonical way. In combination, these results show that the framework developed in this paper provides a natural setting to study the lower reaches of the large cardinal hierarchy.

1.1. Characterization schemes. Starting from measurability upwards, many important large cardinal notions are defined through the existence of certain ultrafilters that can be used in ultrapower constructions in order to produce elementary embeddings $j : V \to M$ of the set-theoretic universe $V$ into some transitive class $M$ with the large cardinal in question as their critical point. A great variety of results shows that many prominent large cardinal properties below measurability can be characterized through the existence of filters that only measure sets contained in some set-sized model $M$ of set theory. For example, the equivalence of weak compactness to the filter property (see [1] Theorem 1.1.3) implies that an uncountable cardinal $\kappa$ is weakly compact if and only if for every model $M$ of $\text{ZFC}^-$ of cardinality at most $\kappa$ that contains $\kappa$, there exists an $M$-ultrafilter $U$ on $\kappa$ that is non-principal and $\kappa$-complete for $M$.

Isolating what was implicit in folklore results (see for example [23]), Gitman, Sharpe and Welch (see [9] Theorem 1.3 or [28] Theorem 1.5) showed that Ramseyness can be characterized through the existence of countably complete ultrafilters for transitive $\text{ZFC}^-$-models of cardinality $\kappa$. More examples of such characterizations are provided by results of Kunen [23], Kleinberg [22] and Abramson–Harrington–Kleinberg–Zwicker [11]. Their characterizations can be formulated through the following scheme, which is hinted at in the paragraph before [1] Lemma 3.5.1:

An uncountable cardinal $\kappa$ has the large cardinal property $\Phi(\kappa)$ if and only if for some (equivalently, for all) sufficiently large regular cardinals $\theta$ and for some (equivalently, for all) countable $M \prec \text{H}(\theta)$ with $\kappa \in M$, there exists a uniform $M$-ultrafilter $U$ on $\kappa$ with $\Psi(M, U)$. Their results show that this scheme holds true in the following cases:

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1In the following, we identify (not necessarily transitive) classes $M$ with the $\epsilon$-structure $\langle M, \epsilon \rangle$. In particular, given some theory $T$ in the language of set theory, we say that a class $M$ is a model of $T$ (and write $M \models T$) if and only if $\langle M, \epsilon \rangle \models T$.

2All relevant definitions can be found in Section 3.
\begin{itemize}
\item \(\Phi(\kappa) \equiv \text{"\(\kappa\) is inaccessible"}\) and \(\Psi(M,U) \equiv \text{"}\(U\) is \(<\kappa\)-amenable and \(<\kappa\)-complete for \(M\)"}\).
\item \(\Phi(\kappa) \equiv \text{"\(\kappa\) is weakly compact"}\) and \(\Psi(M,U) \equiv \text{"}\(U\) is \(\kappa\)-amenable and \(<\kappa\)-complete for \(M\)"}\).
\item \(\Phi(\kappa) \equiv \text{"\(\kappa\) is completely ineffable"}\) and \(\Psi(M,U) \equiv \text{"}\(U\) is \(\kappa\)-amenable for \(M\) and \(M\)-normal"}.
\end{itemize}

Generalizing the above scheme, our large cardinal characterizations will be based on three schemes that are introduced below. In order to phrase these schemes in a compact way, we introduce some terminology. As usual, we say that some statement \(\varphi(\beta)\) holds for sufficiently large ordinals \(\beta\) if there is an \(\alpha \in \text{Ord}\) such that \(\varphi(\beta)\) holds for all \(\alpha \leq \beta \in \text{Ord}\). Given infinite cardinals \(\lambda \leq \kappa\), a \(\text{ZFC}^\text{-model}\) \(M\) is a \((\lambda,\kappa)\)-model if it has cardinality \(\lambda\) and \((\lambda+1) \cup \{\kappa\} \subseteq M\). A \((\kappa,\kappa)\)-model is called a weak \(\kappa\)-model. A \(\kappa\)-model is a weak \(\kappa\)-model that is closed under \(<\kappa\)-sequences.\footnote{\(\text{Note that, here and in the following, in order to avoid mention of \(\kappa\) within \(\Psi(M,U)\), \(\kappa\) could be defined as being the least \(M\)-ordinal \(\eta\) satisfying \(\bigcup U \subseteq \eta\).} Moreover, given an infinite cardinal \(\theta\) and a class \(S\) of elementary submodels of \(H(\theta)\), we say that some statement \(\psi(\theta)\) holds for many models in \(S\) if for every \(x \in H(\theta)\), there exists an \(M \in S\) with \(x \in M\) for which \(\psi(M)\) holds. Finally, we say that a statement \(\psi(\theta)\) holds for many transitive weak \(\kappa\)-models \(M\) if for every \(\kappa \leq \theta\), there exists a transitive weak \(\kappa\)-model \(M\) with \(x \in M\) for which \(\psi(M)\) holds.

\textbf{Scheme A.} An uncountable cardinal \(\kappa\) has the large cardinal property \(\Phi(\kappa)\) if and only if for all sufficiently large regular cardinals \(\theta\) and all infinite cardinals \(\lambda < \kappa\), there are many \((\lambda, \kappa)\)-models \(M \prec H(\theta)\) for which there exists a uniform \(M\)-ultrafilter \(U\) on \(\kappa\) with \(\Psi(M,U)\).

\textbf{Scheme B.} An uncountable cardinal \(\kappa\) has the large cardinal property \(\Phi(\kappa)\) if and only if for all sufficiently large regular cardinals \(\theta\), there are many weak \(\kappa\)-models \(M \prec H(\theta)\) for which there exists a uniform \(M\)-ultrafilter \(U\) on \(\kappa\) with \(\Psi(M,U)\).

\textbf{Scheme C.} An uncountable cardinal \(\kappa\) has the large cardinal property \(\Phi(\kappa)\) if and only if for all sufficiently large regular cardinals \(\theta\), there are many weak \(\kappa\)-models \(M \prec H(\theta)\) for which there exists a uniform \(M\)-ultrafilter \(U\) on \(\kappa\) with \(\Psi(M,U)\).

Trivial examples of valid instances of the Schemes \(\text{A}\) and \(\text{C}\) can be obtained by considering the properties \(\Phi(\kappa) \equiv \Phi_{\text{ms}}(\kappa) \equiv \text{"\(\kappa\) is measurable"}\) and \(\Psi(M,U) \equiv \Psi_{\text{ms}}(M,U) \equiv \text{"}\(U\) is \(M\)-normal and \(U = F \cap M\) for some \(F \in M\)"}. In contrast, Scheme \(\text{B}\) cannot provably hold true for \(\Phi(\kappa) \equiv \Phi_{\text{ms}}(\kappa)\) and a property \(\Psi(M,U)\) of models \(M\) and \(M\)-ultrafilters \(U\) whose restriction to \(\kappa\)-models and filters on \(\kappa\) is definable by a \(\Pi^2_1\)-formula over \(V_\kappa\), because the statement that for many transitive weak \(\kappa\)-models \(M\) there exists a uniform \(M\)-ultrafilter \(U\) on \(\kappa\) with \(\Psi(M,U)\) can then again be formulated by a \(\Pi^2_1\)-sentence over \(V_\kappa\), and measurable cardinals are \(\Pi^2_1\)-indescribable (see \([21, \text{Proposition 6.5}]\)). Since the measurability of \(\kappa\) can be expressed by a \(\Sigma^2_1\)-formula over \(V_\kappa\), this shows that there is no reasonable\footnote{\(\text{Note that, unlike some authors, we do not require (weak) \(\kappa\)-models to be transitive.}\) characterization of measurability through Scheme \(\text{B}\). In order to have a trivial example for a valid instance of Scheme \(\text{B}\) available, we make the following definition:}

\textbf{Definition 1.1.} An uncountable cardinal \(\kappa\) is \textit{locally measurable} if and only if for many transitive weak \(\kappa\)-models \(M\) there exists a uniform \(M\)-normal ultrafilter \(U\) on \(\kappa\) with \(U \in M\).

1.2. \textbf{Large cardinal characterizations.} In combination with existing results, the work presented in this paper will yield a complete list of large cardinal properties that can be characterized through the above schemes by considering filters possessing various degrees of amenability and normality. In order to present these results in a compact way, we introduce abbreviations for the relevant properties of cardinals, models and filters. All relevant definitions will be provided in the later sections of our paper.

\begin{itemize}
\item \(\Phi_{\text{in}}(\kappa) \equiv \text{"\(\kappa\) is inaccessible"}\), \(\Psi_{\text{in}}(M,U) \equiv \text{"}\(U\) is \(<\kappa\)-amenable and \(<\kappa\)-complete for \(M\)"}\).
\item \(\Phi_{\text{wc}}(\kappa) \equiv \text{"\(\kappa\) is weakly compact"}\), \(\Psi_{\text{wc}}(M,U) \equiv \text{"}\(U\) is \(\kappa\)-amenable and \(<\kappa\)-complete for \(M\)"}\).
\item \(\Psi_{\text{s}}(M,U) \equiv \text{"}\(U\) is \(M\)-normal"}\).
\item \(\Phi_{\text{wic}}(\kappa) \equiv \text{"\(\kappa\) is weakly ineffable"}\), \(\Psi_{\text{wic}}(M,U) \equiv \text{"}\(U\) is genuine"}\).
\item \(\Phi_{\text{ie}}(\kappa) \equiv \text{"\(\kappa\) is ineffable"}\), \(\Psi_{\text{ie}}(M,U) \equiv \text{"}\(U\) is normal"}\).
\end{itemize}
through one of the above schemes, yielding the following trivial correspondences:

- $\Phi_{\text{c}M} (\kappa) \equiv \text{"$\kappa$ is completely ineffable"", } \Psi_{\text{c}M} (M,U) \equiv \text{"$M$ is $\kappa$-amenable for } M \text{ and } M\text{-normal"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is weakly Ramsey"},$
- $\Psi_{\text{w}R} (M,U) \equiv \text{"$U$ is $\kappa$-amenable for } M, \text{ $M$-normal, and } \text{Ult}(M,U) \text{ is well-founded"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is $\omega$-Ramsey"},$
- $\Phi_{\text{R}} (\kappa) \equiv \text{"$\kappa$ is Ramsey"}, \Psi_{\text{R}} (M,U) \equiv \text{"$U$ is $\kappa$-amenable for } M, \text{ $M$-normal and countably complete"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is ineffably Ramsey"},$
- $\Psi_{\text{R}} (M,U) \equiv \text{"$U$ is $\kappa$-amenable for } M, \text{ $M$-normal and stationary-complete"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is $\Delta^2_2$-Ramsey"}, \Psi_{\text{w}R} (M,U) \equiv \text{"$U$ is $\kappa$-amenable for } M \text{ and normal"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is strongly Ramsey"},$
- $\Psi_{\text{w}R} (M,U) \equiv \text{"$M$ is a $\kappa$-model, $U$ is $\kappa$-amenable for } M \text{ and normal"}.$
- $\Phi_{\text{w}R} (\kappa) \equiv \text{"$\kappa$ is super Ramsey"},$
- $\Psi_{\text{w}R} (M,U) \equiv \text{"$M \prec H(\kappa^+)$ is a $\kappa$-model, $U$ is $\kappa$-amenable for } M \text{ and } M\text{-normal"}.$

First, note that some of the large cardinal properties appearing in the above list are already defined among the above schemes, yielding the following trivial correspondences:

- Scheme A holds true in the following cases:
  - $\Phi(\kappa) \equiv \Theta_{\text{R}} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
  - $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
- Scheme B holds true in the following cases:
  - $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
  - $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
- Scheme C holds true in the following cases:
  - $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
  - $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$

The following theorem summarizes our results, together with a number of known results. Items (1), (2a) and (4) extend the classical results of Kunen, Kleinberg, and Abramson–Harrington–Kleinberg–Zwick in [11] mentioned above. Item (5) is the result from Gitman, Sharpe and Welch mentioned above. Items (5(a)), (5(b)) and (5c) are due to Abramson, Harrington, Kleinberg and Zwick in [11] Theorem 1.1.3, Theorem 1.2.1 and Corollary 1.3.1. Item (5(a)) is essentially due to Baumgartner in [3] Theorem 2.1.

**Theorem 1.2.** (1) Schemes A, B, and C hold true in case $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(2) Schemes A and C both hold true in the following cases:
(a) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(b) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(c) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(d) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(3) Scheme B holds true in the following cases:
(a) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(b) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Psi_{\text{w}R} (M,U).$
(4) Scheme A holds true in the following cases:
(a) $\Phi(\kappa) \equiv \text{"$\kappa$ is regular" and either}
(i) $\Psi(M,U) \equiv \text{"$U$ is $\lessdot \kappa$-complete for } M \text{ , or}
(ii) $\Psi(M,U) \equiv \text{"$U$ is normal".}
(b) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and either
(i) $\Psi(M,U) \equiv \Theta_{\text{w}R} (M,U) , or
(ii) $\Psi(M,U) \equiv \Theta_{\text{R}} (M,U).$
(5) Schemes B and C hold true in the following cases:
(a) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and either
(i) $\Psi(M,U) \equiv \text{"$U$ is $\lessdot \kappa$-complete for } M \text{ , or}
(ii) $\Psi(M,U) \equiv \Theta_{\text{w}R} (M,U) , or
(iii) $\Psi(M,U) \equiv \Theta_{\text{w}R} (M,U).$
(b) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Theta_{\text{w}R} (M,U).$
(c) $\Phi(\kappa) \equiv \Theta_{\text{w}R} (\kappa)$ and $\Psi(M,U) \equiv \Theta_{\text{w}R} (M,U).$

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These could of course have been stated in terms of the existence of many countable models $M \prec H(\theta)$ in [11]. Note that the arguments of [11] are strongly based on the existence of generic filters over countable models of set theory, hence we will need to follow a very different line of argument.
The above results are summarized in abbreviated form in Tables 1 and 2 below. The meaning of the tables should be clear to the reader when compared with the results presented in Theorem 1.2. All entries in Table 2 that are not mentioned within the statement of Theorem 1.2 will be immediate consequences of the definitions of the large cardinal notions that will be introduced later in our paper. Furthermore, let us remark that our results (some of which are mentioned already within Theorem 1.2) will show that the size of the models considered is irrelevant once we consider elementary submodels of (sufficiently large) $H(\theta)$’s in Table 2.

| $|M| < \kappa$ | $|M| = \kappa$ | $M < H(\theta)$ |
|----------------|----------------|-----------------|
| $<\kappa$-complete for $M$ | regular | inaccessible | weakly compact |
| $M$-normal | regular | inaccessible | weakly compact |
| $M$-normal and well-founded | regular | inaccessible | weakly compact |
| $M$-normal and countably complete | regular | inaccessible | weakly compact |
| $M$-normal and stationary-complete | regular | inaccessible | weakly compact |
| genuine | regular | inaccessible | weakly ineffable |
| normal | regular | inaccessible | ineffable |

Table 1. Characterizations without $\kappa$-amenability

| $\kappa$-amenable and ... | $|M| = \kappa$ | $M < H(\theta)$ |
|---------------------------|----------------|-----------------|
| $<\kappa$-complete for $M$ | weakly compact | weakly compact |
| $M$-normal | $\mathcal{T}_\omega^\kappa$-Ramsey | completely ineffable |
| $M$-normal and well-founded | weakly Ramsey | $\omega$-Ramsey |
| $M$-normal and countably complete | Ramsey | $\text{cc}_{\omega}^\kappa$-Ramsey |
| $M$-normal and stationary-complete | ineffably Ramsey | $\Delta^\kappa_\omega$-Ramsey |
| genuine | $\infty_\omega^\kappa$-Ramsey | $\Delta^\kappa_\omega$-Ramsey |
| normal | $\Delta^\kappa_\omega$-Ramsey | $\Delta^\kappa_\omega$-Ramsey |

Table 2. Characterizations with $\kappa$-amenability

1.3. Large cardinal ideals. We next want to study the large cardinal ideals that are canonically induced by our characterizations.

Definition 1.3. Let $\Psi(M,U)$ be a property of models $M$ and filters $U$, and let $\kappa$ be an uncountable cardinal.

1. We define $I^\kappa_\psi$ to be the collection of all $A \subseteq \kappa$ with the property that for all sufficiently large regular cardinals $\theta$, there exists a set $x \in H(\theta)$ such that for all infinite cardinals $\lambda < \kappa$, if $M \prec H(\theta)$ is a $(\lambda, \kappa)$-model with $x \in M$ and $U$ is a uniform $M$-ultrafilter on $\kappa$ with $\Psi(M,U)$, then $A \notin U$.

2. We define $I^\kappa_\psi$ to be the collection of all $A \subseteq \kappa$ with the property that there exists $x \subseteq \kappa$ such that if $M$ is a transitive weak $\kappa$-model with $x \in M$ and $U$ is a uniform $M$-ultrafilter on $\kappa$ with $\Psi(M,U)$, then $A \notin U$.

3. We define $I^\kappa_\psi$ to be the collection of all $A \subseteq \kappa$ with the property that for all sufficiently large regular cardinals $\theta$, there exists a set $x \in H(\theta)$ such that if $M \prec H(\theta)$ is a weak $\kappa$-model with $x \in M$ and $U$ is a uniform $M$-ultrafilter on $\kappa$ with $\Psi(M,U)$, then $A \notin U$.

It is easy to see that the collections $I^\kappa_\psi$, $I^\kappa_\psi$ and $I^\kappa_\psi$ always form ideals on $\kappa$, and that, if Scheme $A$, $B$ or $C$ holds for some large cardinal property $\Phi(\kappa)$ and some property $\Psi(M,U)$ of models $M$ and filters $U$, then the statement that $\Phi(\kappa)$ holds for some uncountable cardinal $\kappa$ implies the properness of the ideal $I^\kappa_\psi$, $I^\kappa_\psi$, or $I^\kappa_\psi$ respectively. Moreover, in all cases covered by Theorem 1.2 (and also in most other natural situations), the converse of this implication also holds true. This is trivial for instances of Scheme $B$. For instances of Schemes $A$ and $C$, this is an easy consequence of the observation that all properties $\Psi$ listed in the theorem are restrictable, i.e. given uncountable cardinals $\bar{\theta} < \theta$, if $M \prec H(\theta)$ with $\bar{\theta} \in M$, $\kappa < \kappa \cap \bar{\theta}$ is a cardinal cardinal.

Note that we do not mention the cases when $|M| = \kappa$ and $U$ is $<\kappa$-amenable for $M$. This is because $<\kappa$-amenability trivializes in case $\kappa$ is inaccessible and when we consider many submodels of $H(\theta)$ of size $\kappa$, as we may always pick such submodels $M$ that contain all bounded subsets of $\kappa$, thus making any $M$-ultrafilter $<\kappa$-amenable for $M$ (that is, we would obtain another column identical to the last column of Table 1 containing four times weak compactness, followed by weak ineffability and then ineffability). Moreover, we would like to comment that if we were to consider models of size less than $\kappa$ in Table 2 without elementarity, we would clearly only get regularity at all levels of normality, as in Table 1.
Theorem 1.4. (1) If $\kappa$ is inaccessible, then $I^\kappa_{ms}$ is the bounded ideal on $\kappa$.
(2) If $\kappa$ is a regular and uncountable cardinal, then $I^\kappa_\delta$ is the non-stationary ideal on $\kappa$.
(3) If $\kappa$ is a weakly compact cardinal, then $I^\kappa_\delta$ is the weakly compact ideal on $\kappa$.
(4) If $\kappa$ is a weakly ineffable cardinal, then $I^\kappa_{wie}$ is the weakly ineffable ideal on $\kappa$.
(5) If $\kappa$ is an ineffable cardinal, then $I^\kappa_\omega$ is the ineffable ideal on $\kappa$.
(6) If $\kappa$ is a completely ineffable cardinal, then $I^\kappa_{cie}$ is the completely ineffable ideal on $\kappa$.
(7) If $\kappa$ is a Ramsey cardinal, then $I^\kappa_\delta$ is the Ramsey ideal on $\kappa$.
(8) If $\kappa$ is an ineffably Ramsey cardinal, then $I^\kappa_R$ is the ineffably Ramsey ideal on $\kappa$.
(9) If $\kappa$ is a measurable cardinal, then the complement of $I^\kappa_{ms}$ is the union of all normal ultrafilters on $\kappa$.

1.4. Structural properties of large cardinal ideals. We show that many aspects of the relationship between different large cardinal notions are reflected in the relationship of their corresponding ideals.

First, our results will show that, for many important examples, the ordering of large cardinal properties under direct implication turns out to be identical to the ordering of their corresponding ideals under inclusion.

Next, our approach to show that the ordering of large cardinal properties by their consistency strength can also be read off from the corresponding ideals is motivated by the fact that the well-foundedness of the Mitchell order (see [16, Lemma 19.32]) implies that for every measurable cardinal $\kappa$, there is a normal ultrafilter $U$ on $\kappa$ with the property that $\kappa$ is not measurable in $Ult(V, U)$. Translated into the context of our paper (using Theorem 1.4(9)) this shows that the set of all non-measurables below $\kappa$ is not contained in the ideal $I^\kappa_{ms}$. To generalize this to other large cardinal properties $\Phi$, if $\kappa$ is a cardinal, we let

$$N^\Phi_\kappa = \{\alpha < \kappa \mid \neg \Phi(\alpha)\}.$$ 

If $\Phi$ is an abbreviation for a large cardinal property, then we write $N^\Phi_\kappa$ instead of $N^\Phi_\kappa$. We show that the above result for measurable cardinals can be generalized to many other important large cardinal notions. More precisely, for various instances of our characterization schemes, we will show that the above set of ordinals without the given large cardinal property is not contained in the corresponding ideal. These results can be seen as indicators that the derived characterization and the associated ideal canonically describe the given large cardinal property, as one would expect these cardinals to lose some of their large cardinal properties in their ultrapowers. Moreover, our results also show that, in many important cases, the fact that some large cardinal property $\Phi^*$ has a strictly higher consistency strength than some other large cardinal property $\Phi$ is equivalent to the fact that $\Phi^*(\kappa)$ implies that the set $N^\Phi_\kappa$ is an element of the ideal on $\kappa$ corresponding to $\Phi^*$. This allows us to reconstruct the consistency strength ordering of these properties from structural properties of their corresponding ideals. Together with the correspondence described in the last paragraph, it also shows that, in many cases, the fact that some large cardinal property $\Phi^*$ provably implies a large cardinal property $\Phi$ of strictly lower consistency strength yields that $\Phi(\kappa)$ implies the ideal on $\kappa$ corresponding to $\Phi$ to be a proper subset of the ideal on $\kappa$ corresponding to $\Phi^*$.

Finally, we consider the question whether there are fundamental differences between the ideal $I^\kappa_{ms}$ induced by measurability and the ideals induced by weaker large cardinal notions. By classical results of Kunen (see [21, Theorem 20.10]), it is possible that there is a unique normal measure on a measurable cardinal $\kappa$. In this case, the ideal $I^\kappa_{ms}$ is equal to the complement of this measure and hence the induced partial order $P(\kappa)/I^\kappa_{ms}$ is trivial, hence in particular atomic. We study the question whether the partial orders induced by other large cardinal ideals can also be atomic, conjecturing that the possible atomicity of the quotient partial order is a property that separates measurability from all weaker large cardinal properties (this is motivated by Lemma 16.3 below). This conjecture turns out to be closely related to the previous topics, and we will verify it for many prominent large cardinal properties.

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*We will also provide an easy argument for this result that does not make any use of the Mitchell order in Lemma 16.2.

*For Ramsey, strongly Ramsey, and super Ramsey cardinals, this also follows from the results of [4], where the notion of Mitchell rank is generalized to apply to these large cardinal notions.
The following theorem provides some special cases of our results, namely some of their consequences for large cardinal notions that had already been introduced in the set theoretic literature. Item (1) and the statement that $I_{\kappa}^{c\kappa} \subseteq I_{\kappa}^{\kappa}$ in Item (2) below are of course trivial consequences of Theorem 1.4. The statement that $N_{\kappa}^c \in I_{\kappa}^{c\kappa}$ in Item (2) has been shown by Baumgartner in [3] Theorem 2.8. That $N_{\kappa}^c \in I_{\kappa}^{c\kappa}$ in Item (2) was shown by Johnson in [18] Corollary 4, however we will also provide an easy self-contained argument of this result later on in the benefit of our readers. Gitman has shown that weakly Ramsey cardinals (which are also known under the name of 1-iterable cardinals) are weakly ineffable limits of completely ineffable cardinals (see [9] Theorem 3.3 and Theorem 3.7). Her arguments in the proof of [9] Theorem 3.7 actually show that if $\kappa$ is a weakly Ramsey cardinal, then $N_{\kappa}^c \in I_{\kappa}^{c\kappa}$, as in Item (5). That $I_{\kappa}^{c\kappa} \subseteq I_{\kappa}^c$ in Item (7) is already immediate from our above definitions. The proof of [10] Theorem 4.1] shows that $N_{\kappa}^c R \in I_{\kappa}^{c\kappa}$ as in Item (7), and $I_{\kappa}^{c\kappa} \subseteq I_{\kappa}^c$ in Item (8) are due to Feng (see [8] Corollary 4.4 and Theorem 4.5]). That $N_{\kappa}^{sTR} \notin I_{\kappa}^{c\kappa}$ in Item (10) and that $N_{\kappa}^{sTR} \notin I_{\kappa}^{c\kappa}$ in Item (10) follows easily from the results of [4], and these statements will also be immediate consequences of fairly general results from our paper. The final statement of Item (12) is an immediate consequence of the above-mentioned result of Kunen.

**Theorem 1.5.** (1) If $\kappa$ is an inaccessible cardinal, then $N_{\kappa}^c \notin I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(2) If $\kappa$ is a weakly compact cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(3) If $\kappa$ is a weakly ineffable cardinal, then $I_{\kappa}^c \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(4) If $\kappa$ is an ineffable cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(5) If $\kappa$ is a completely ineffable cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(6) If $\kappa$ is a weakly Ramsey cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(7) If $\kappa$ is a Ramsey cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(8) If $\kappa$ is an ineffable Ramsey cardinal, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(9) If $\kappa$ is strongly Ramsey, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(10) If $\kappa$ is super Ramsey, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(11) If $\kappa$ is locally measurable, then $I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, $I_{\kappa}^{c\kappa} / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ is not atomic.

(12) If $\kappa$ is measurable, then $I_{\kappa}^{c\kappa} \cup I_{\kappa}^{c\kappa} \cup \{N_{\kappa}^c\} \subseteq I_{\kappa}^{c\kappa}$, $N_{\kappa}^c / I_{\kappa}^{c\kappa}$, and $P(\kappa)/I_{\kappa}^{c\kappa}$ may be atomic.

Note that the above statements show that the linear ordering of the mentioned large cardinal properties by their consistency strength can be read off from the containedness of sets of the form $N_{\kappa}^c$ in the induced ideals. Moreover, all known provable implications and consistent non-implications can be read from the ordering of the corresponding ideals under inclusion. Concerning the above-mentioned large cardinal notions, the question whether super Ramsayness implies complete ineffability remains open. For example, the fact that ineffiability and Ramseyess do not provably imply each other corresponds to the fact that $I_{\kappa}^{c\kappa} \subseteq I_{\kappa}^{c\kappa}$ holds whenever $\kappa$ is both ineffable and Ramsey, where the second non-inclusion is a consequence of $N_{\kappa}^{sTR} \subseteq N_{\kappa}^{sTR} \subseteq I_{\kappa}^{c\kappa}$ and $N_{\kappa}^{sTR} \notin I_{\kappa}^{c\kappa}$.

Figure 1 below summarizes the structural statements listed in Theorem 1.5. In this diagram, a provable inclusion $I_0 \subseteq I_0$ of large cardinal ideals is represented by a solid arrow $I_0 \Rightarrow I_1$. Moreover, if $I_1$ is an ideal induced by a large cardinal property $\Phi$, then a dashed arrow $I_0 \Rightarrow I_1$ represents the statement that $N_{\Phi} \in I_0$ provably holds.

**2. Some basic notions**

A key ingredient for our results will be the generalization of a number of standard notions to the context of non-transitive models, and, in the case of elementary embeddings, also to possibly non-well-founded target models. While most of these definitions are very much standard, we will take some care in order to present them in a way that makes them applicable also in these generalized settings. They clearly correspond to their usual counterparts in the case of transitive models $M$. In the following, we let $ZFC^-$ denote the collection of axioms of $ZFC$ without the powerset axiom (but, as usual, with the axiom scheme of Collection rather than the axiom scheme of Replacement). In order to avoid unnecessary technicalities, we restrict our attention to $\Sigma_0$-correct models, i.e., models that are $\Sigma_0$-elementary in $V$. Since every $\Sigma_0$-elementary submodel of a transitive class is $\Sigma_0$-correct, all models considered in the above schemes are $\Sigma_0$-correct. Note that if $M$ is $\Sigma_0$-correct, $\alpha \in M$ is an ordinal in $M$ and $f \in M$ is a function with domain $\alpha$ in $M$, then $\alpha$ is an ordinal and $f$ is a function with domain $\alpha$.

**Definition 2.1** (Properties of $\Sigma_0$-ultrafilters). Let $M$ be a class that is a $\Sigma_0$-correct model of $ZFC^-$ and let $\kappa$ be a cardinal of $M$. 

- For example, the fact that ineffability and Ramseyness do not provably imply each other corresponds to the fact that $I_{\kappa}^{c\kappa} \subseteq I_{\kappa}^{c\kappa}$ holds whenever $\kappa$ is both ineffable and Ramsey, where the second non-inclusion is a consequence of $N_{\kappa}^{sTR} \subseteq N_{\kappa}^{sTR} \subseteq I_{\kappa}^{c\kappa}$ and $N_{\kappa}^{sTR} \notin I_{\kappa}^{c\kappa}$.
A collection $U \subseteq M \cap \mathcal{P}(\kappa)$ is an $M$-ultrafilter on $\kappa$ if $\langle M, U \rangle \models "U \text{ is an ultrafilter on } \kappa".$

In the following, let $U$ denote an $M$-ultrafilter on $\kappa$.

- $U$ is non-principal if $\{\alpha\} \notin U$ for all $\alpha < \kappa$.
- $U$ contains all final segments of $\kappa$ in $M$ if $[\alpha, \kappa) \in U$ whenever $\alpha \in M \cap \kappa$.
- $U$ is uniform if $|\alpha|^M = \kappa$ for all $x \in U$.
- $U$ is $<\kappa$-amenable (respectively, $\kappa$-amenable) for $M$ if whenever $\alpha < \kappa$ (respectively, $\alpha = \kappa$) and $\langle x_\beta \mid \beta < \alpha \rangle$ is a sequence of subsets of $\kappa$ that is an element of $M$, then

$$\langle M, U \rangle \models "\exists x \forall \beta < \alpha \ [x_\beta \in U \iff \beta \in x]\".$$  

- Given $\alpha \leq \kappa$ in $M$, $U$ is $<\alpha$-complete for $M$ if $\langle M, U \rangle \models "U \text{ is } <\alpha\text{-complete}."$
- $U$ is $M$-normal if $\langle M, U \rangle \models "U \text{ is normal}."$
- $U$ is $M$-normal with respect to $\subseteq$-decreasing sequences if whenever $\langle x_\alpha \mid \alpha < \kappa \rangle$ is a sequence of subsets of $\kappa$ that is an element of $M$ and satisfies

$$\langle M, U \rangle \models "\forall \alpha \leq \beta < \kappa \ [x_\alpha \in U \land x_\beta \subseteq x_\alpha]\";$$

then $\langle M, U \rangle \models "\Delta_{\alpha<\kappa} x_\alpha \in U".$ \[10\]
- $U$ is countably complete if whenever $\langle x_n \mid n < \omega \rangle$ is a sequence of elements of $U$, then $\bigcap_{n<\omega} x_n \neq \emptyset$.
- $U$ is stationary-complete if whenever $\langle x_n \mid n < \omega \rangle$ is a sequence of elements of $U$, then $\bigcap_{n<\omega} x_n$ is a stationary subset of $\kappa$.
- $U$ is genuine if either $|U| = \kappa$ and $\Delta_{\alpha<\kappa} U_\alpha$ is unbounded in $\kappa$ for every enumeration $\langle U_\alpha \mid \alpha < \kappa \rangle$ of $U$, or $|U| < \kappa$ and $\bigcup U$ is unbounded in $\kappa$.
- $U$ is normal if either $\Delta_{\alpha<\kappa} U_\alpha$ is a stationary subset of $\kappa$ for every enumeration $\langle U_\alpha \mid \alpha < \kappa \rangle$ of $U$, or $|U| < \kappa$ and $\bigcup U$ is a stationary subset of $\kappa$.

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\[10\] Note that, from the point of view of $V$, given such a sequence $\langle x_\beta \mid \beta < \alpha \rangle$, a witness $x$ for amenability in $M$ need only satisfy $"x_\beta \in U \iff \beta \in x"$ whenever $\beta \in M \cap \alpha$.

\[11\] An easy argument shows that an $M$-ultrafilter on $\kappa$ that contains all final segments of $\kappa$ in $M \models \text{ZFC}^-$ is $M$-normal if and only if it is $<\kappa$-complete and $M$-normal with respect to $\subseteq$-decreasing sequences.
The following observation shows that the first columns of Table\ref{tab:1} and Table\ref{tab:2} list the above properties according to their strength.

**Lemma 2.2.** In the situation of Definition 2.7, if $U$ is genuine, then it is $M$-normal and stationary-complete.

**Proof.** Assume that $U$ is a genuine $M$-ultrafilter, and let $\langle x_\alpha \mid \alpha < \kappa \rangle$ be a sequence of elements of $U$ in $M$. If this sequence had diagonal intersection $x \notin U$, then the complement of $x$ would be an element of $U$, but then every diagonal intersection of $U$ would be non-stationary, contradicting our assumption on $U$.

Now, let $\langle x_\alpha \mid n < \omega \rangle$ be a sequence of elements of $U$. Let $C$ be a club subset of $\kappa$ with $\min(C) \geq \omega$ and $\kappa \setminus C$ is unbounded in $\kappa$. Let $\langle U_\alpha \mid U < \kappa \rangle$ be an enumeration of $U$ with $U_\kappa = \kappa \setminus \min(C \setminus (\alpha + 1))$ for every $\alpha \in C$ and $U_i = x_i$ for every $i < \omega$. It is easy to check that $\Delta_{\alpha < \kappa} U_i \subseteq C \cup (\min(C) + 1)$. Using that $U$ is genuine, it follows that $\Delta_{\alpha < \kappa} U_i$ is an unbounded subset of $\kappa$. In particular, we know that $\bigcap_{i < \omega} x_i \cap C \neq \emptyset$, showing that $U$ is stationary-complete.

**Lemma 2.3.** If $\Psi(M, U)$ implies that $U$ is $M$-normal, then the ideals $I^{<\kappa}_\Psi$, $I^\kappa_\Psi$ and $I^{<\kappa}_\Theta$ are all normal.

**Proof.** We will only present the proof for the ideal $I^{<\kappa}_\Psi$, which requires the most difficult argument of the three. Assume thus that $\bar{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a sequence of elements of $I^{<\kappa}_\Psi$. Fix $\theta$ sufficiently large, and fix a sequence $\bar{x}$ such that for every $\alpha < \kappa$, $x_\alpha$ witnesses that $A_\alpha \in I^{<\kappa}_\Psi$ with respect to $\theta$. Pick $M \prec H(\theta)$ such that $\bar{x}, \bar{A} \in M$, $|\bar{M}| < \kappa$ and $\Psi(M, U)$ holds. Then $(M, U)$ thinks that $A_\alpha \notin U$ for every $\alpha < \kappa$, so since $\Psi(M, U)$ implies that $U$ is $M$-normal, it follows that $\nabla \bar{A} \notin U$, hence $\nabla \bar{A} \in I^{<\kappa}_\Psi$.

We now turn our attention to extended definitions regarding elementary embeddings. As mentioned above, we identify classes $M$ with the corresponding $\varepsilon$-structures ($M, \in$). Given transitive classes $M$ and $N$, the critical point of an elementary embedding $j : M \rightarrow N$ is simply defined as the least ordinal $\alpha \in M$ with $j(\alpha) > \alpha$. We need a generalization of this concept for elementary embeddings $j : M \rightarrow \langle N, \epsilon_N \rangle$ when the class $M$ is not necessarily transitive and the $\varepsilon$-structure $(N, \epsilon_N)$ is not necessarily well-founded. In the following, we let $j : M \rightarrow \langle N, \epsilon_N \rangle$ always denote an elementary embedding between $\varepsilon$-structures, whose domain is a $\Sigma_\varepsilon$-correct ZFC$^\ast$-model.

**Definition 2.4 (Jump).** Given $j : M \rightarrow \langle N, \epsilon_N \rangle$ and an ordinal $\alpha \in M$, we say that $j$ jumps at $\alpha$ if there exists an $N$-ordinal $\gamma$ with $\gamma \epsilon_N j(\alpha)$ and $j(\beta) \epsilon_N \gamma$ for all $\beta \in M \cap \alpha$.

Note that, in the above situation, for every $N$-ordinal $\gamma$, there is at most one ordinal $\alpha$ in $M$ such that $\gamma$ witnesses that $j$ jumps at $\alpha$.

**Definition 2.5 (Critical Point).** Given $j : M \rightarrow \langle N, \epsilon_N \rangle$, if there exists a minimal ordinal $\alpha$ such that $j$ jumps at $\alpha$, then we denote this ordinal by crit($j$), the critical point of $j$.

It is easy to see that if crit($j$) exists and $\alpha \in \text{Ord}$, then crit($j$) $> \alpha$ if and only if $j[\alpha] = \{ \beta \in N \mid \beta \epsilon_N j(\alpha) \}$. We will tacitly make use of this fact throughout this paper.

Next, we need to generalize the notions of $<\kappa$- and $\kappa$-powerset preservation to a non-transitive context. The notion of a $<\kappa$-powerset preserving elementary embedding is not of much interest in the context of transitive models (or weak $\kappa$-models); if we assume that $\kappa$ is a cardinal satisfying $\kappa = 2^{<\kappa}$, and that some subset of $\kappa$ coding $\mathcal{H}(\kappa)$ is an element of $M$, then any embedding $j : M \rightarrow N$ with crit($j$) $= \kappa$ is $<\kappa$-powerset preserving. However, as we will also see later on, it is an important notion in the study of embeddings between smaller models of set theory and it turns out to be closely connected to the accessibility of $\kappa$. The idea behind $j : M \rightarrow \langle N, \epsilon_N \rangle$ being $<\kappa$-powerset preserving (respectively, $\kappa$-powerset preserving) is that $M$ and $N$ contain the same subsets of ordinals below $\kappa$ (respectively, the same subsets of $\kappa$). Since the relevant subsets of $M$ are, in a sense made precise below, always contained in $N$, only one of those inclusions is part of the following definitions.

**Definition 2.6 ($<\kappa$-powerset preservation).** Given $j : M \rightarrow \langle N, \epsilon_N \rangle$ with crit($j$) $= \kappa$, the embedding $j$ is $<\kappa$-powerset preserving if

$$\forall y \in N \exists x \in M \ [\exists \alpha < M \cap \kappa \langle N, \epsilon_N \rangle \models \text{"y} \subseteq j(\alpha)\text{"} \rightarrow j(x) = y].$$

The following definition shows that there is still a useful notion of a $\kappa$-powerset preserving elementary embedding, even if we do not have a representative for $\kappa$ in the target model of our embedding.

**Definition 2.7 ($\kappa$-powerset preservation).** Given $j : M \rightarrow \langle N, \epsilon_N \rangle$ with crit($j$) $= \kappa$, the embedding $j$ is $\kappa$-powerset preserving if

$$\forall y \in N \exists x \in M \ [\langle N, \epsilon_N \rangle \models \text{"y} \subseteq j(\kappa)\text{"} \rightarrow M \cap x = \{ \alpha \in M \cap \kappa \mid j(\alpha) \epsilon_N y \}].$$

We close this section by isolating a property that implies the existence of a canonical representative for $\kappa$ in the target model of our elementary embedding.
**Definition 2.8** (κ-embedding). Given $j : M \rightarrow \langle N, \epsilon_N \rangle$ that jumps at $\kappa$, the embedding $j$ is a **κ-embedding** if there exists an $\epsilon_\kappa$-minimal $N$-ordinal $\gamma$ witnessing that $j$ jumps at $\kappa$. We denote this ordinal by $\kappa^N$.\footnote{If $\text{crit}(j) = \kappa$, then $\kappa^N$ is the unique $N$-ordinal on which the $\epsilon_N$-relation has order-type $\kappa$. Otherwise, $\kappa^N$ might also depend on the embedding $j$, which we nevertheless suppress in our notation.}

Note that if $M$ and $N$ are weak $\kappa$-models, then the usual notions of critical point and of $\kappa$-powerset preservation for an embedding $j : M \rightarrow N$ clearly coincide with our respective notions.

### 3. Correspondences between ultrapowers and elementary embeddings

The results of this section will allow us to interchangeably talk about ultrafilters or about embeddings for models of $\text{ZFC}^-$. If $M$ is a class that is a $\Sigma_\gamma$-correct model of $\text{ZFC}^-$, $\kappa$ is a cardinal of $M$, and $U$ is an $M$-ultrafilter on $\kappa$, then we can use the $\Sigma_\gamma$-correctness of $M[\kappa]$ to define the induced ultrapower embedding $j_U : M \rightarrow \langle \text{Ult}(M,U), \epsilon_U \rangle$ as usual: define an equivalence relation $\equiv_U$ on the class of all functions $f : \kappa \rightarrow M$ contained in $M$ by setting $f \equiv_U g$ if and only if $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$, let $\text{Ult}(M,U)$ consists of all sets $[f]_U$ of rank-minimal elements of $\equiv_U$-equivalence classes, define $[f]_U \epsilon_U [g]_U$ to hold if and only if $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U$ and set $j_U(x) = [c_x]_U$, where $c_\alpha \in M$ denotes the constant function with domain $\kappa$ and value $x$. It is easy to check that the assumption that $M \models \text{ZFC}^-$ implies that Los’ Theorem still holds true in our setting, i.e. we have

$$\text{Ult}(M,U) \models \varphi([f_0]_U, \ldots, [f_{n-1}]_U) \iff \langle M, U \rangle \models \exists x \in U \forall \alpha \in x. \varphi(f_0(\alpha), \ldots, f_{n-1}(\alpha))$$

for every first order $\epsilon$-formula $\varphi(v_0, \ldots, v_{n-1})$ and all functions $f_0, \ldots, f_{n-1} : \kappa \rightarrow M$.

Given an elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ that jumps at $\kappa$, let $\gamma$ be a witness for this, and let

$$U_\gamma = \{A \in M \cap P(\kappa) \mid \gamma \epsilon_N j(A)\}$$

denote the $M$-ultrafilter induced by $\gamma$ and by $j$. Since $\gamma$ is not in the range of $j$, $U_\gamma$ is non-principal. If $j$ is a $\kappa$-embedding and $\gamma = \kappa^N$, then we call $U_\gamma$ the canonical $M$-ultrafilter induced by $j$, or simply the $M$-ultrafilter induced by $j$.

In the following, we say that a property $\Psi(M,U)$ of $\Sigma_\gamma$-correct $\text{ZFC}^-$-models $M$ and $M$-ultrafilters $U$ corresponds to a property $\Theta(M,j)$ of such models $M$ and elementary embeddings $j : M \rightarrow \langle N, \epsilon_N \rangle$ if the following statements hold:

- If $\Psi(M,U)$ holds for an $M$-ultrafilter $U$, then $\Theta(M,j_U)$ holds.
- If $\Theta(M,j)$ holds for an elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ and $\gamma$ witnesses that $j$ jumps at some $\kappa \in M$, then $\Psi(M,U_\gamma)$ holds.

Most of the correspondences below are well-known, in a perhaps slightly less general setup.

**Proposition 3.1.** Let $\kappa$ be an ordinal.

1. “$U$ is an $M$-ultrafilter on $\kappa \in M$ that contains all final segments of $\kappa$ in $M$” corresponds to “$j$ jumps at $\kappa \in M$”.
2. Given $\alpha \leq \kappa$, “$\alpha \in M$ and $U$ is an $M$-ultrafilter on $\kappa \in M$ that is $<\alpha$-complete for $M$ and contains all final segments of $\kappa$ in $M$” corresponds to “$\text{crit}(j) \geq \alpha \in M$ and $j$ jumps at $\kappa \in M$”.
3. “$U$ is a non-principal $M$-ultrafilter on $\kappa \in M$ that is $<\kappa$-complete for $M$” corresponds to “$\text{crit}(j) = \kappa \in M$”.
4. “$U$ is a non-principal $M$-ultrafilter on $\kappa \in M$ that is $<\kappa$-complete for $M$ and $<\kappa$-amenable for $M$” corresponds to “$\text{crit}(j) = \kappa \in M$ and $j$ is $<\kappa$-powerset preserving”.
5. “$\kappa \in M$ and $U$ is $<\kappa$-complete for $M$, non-principal, and $\kappa$-amenable for $M$” corresponds to “$\text{crit}(j) = \kappa \in M$ and $j$ is $<\kappa$-powerset preserving”.

**Proof.** Throughout this proof, let $M$ denote a $\Sigma_\gamma$-correct $\text{ZFC}^-$-model with $\kappa \in M$.

1. Let $U$ be an $M$-ultrafilter on $\kappa$ that contains all final segments of $\kappa$ in $M$. Then $\text{id}_\kappa \in M$ and, given $\alpha < \kappa$ in $M$, we have $(\alpha, \kappa) \in U$ and hence $j_U(\alpha) = [\text{id}_\alpha]_U \epsilon_U j_U(\kappa)$. Hence $[\text{id}_\alpha]_U$ witnesses that $j_U$ jumps at $\alpha$. In the other direction, if $\gamma$ witnesses that $j : M \rightarrow \langle N, \epsilon_N \rangle$ jumps at $\kappa$ and $\alpha \in M \cap \kappa$, then we have $[\alpha, \kappa) \in M, \gamma \epsilon_N j(\alpha, j(\kappa))^N = j((\alpha, \kappa))$ and hence $[\alpha, \kappa) \in U_\gamma$, as desired.
2. Pick $\alpha \leq \kappa \in M$. Let $U$ be an $M$-ultrafilter on $\kappa$ that is $<\alpha$-complete for $M$ and contains all final segments of $\kappa$ in $M$. By (1), $\text{crit}(j_U) \exists$. Assume that $j_U$ jumps at $\beta < \alpha$. Then there is $f : \kappa \rightarrow \beta$ in $M$ such that $[\text{id}_\beta]_U \epsilon_U [f]_U \epsilon_U [\text{id}_\beta]_U$ holds for every $\delta \in M \cap \beta$. By our assumptions on $M$, there is a sequence $\langle x_\delta \mid \delta < \beta \rangle$ of subsets of $\kappa$ in $M$ such that $x_\delta = \{\xi < \kappa \mid \delta < f(\xi) < \beta\}$ for all $\delta < \beta$ and $x_\beta \in U$ for all $\delta \in M \cap \beta$. In this situation, the $<\alpha$-completeness of $U$ implies that $\bigcap_{\delta < \beta} x_\delta \in U$. Pick $\xi \in M \cap \bigcap_{\delta < \beta} x_\delta$. Then $f(\xi) \in M \cap \beta$ and $\xi \in f(\xi)$, a contradiction.

Note that, given a $\Sigma_\gamma$-correct $\text{ZFC}^-$-model $M$ and functions $f, g : \kappa \rightarrow M$ in $M$, then the set $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\}$ and $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\}$ are both contained in $M$ and satisfy the same defining properties in it.
In the other direction, assume that $\gamma$ witnesses $j : M \rightarrow (N, \epsilon_N^\gamma)$ to jump at $\kappa$, and that $\text{crit}(j) > \alpha$. Pick a sequence $\bar{x} = (x_{\beta} : \beta < \kappa) \in M$ with $\beta < \alpha$, and with $\gamma \epsilon_N j(x_{\delta})$ for all $\delta < \beta$. By our assumption, we have $j(\beta) = (\delta \in N | \delta \epsilon_N j(\beta))$, and hence elementarity implies that $\gamma \epsilon_N j(\bar{x}(\xi))$ for all $\xi \in N$ with $\xi \epsilon_N j(\beta)$. This allows us to conclude that $\gamma \epsilon_N j(\bigcap_{\beta < \alpha} x_{\beta})$ and hence $\bigcap_{\beta < \alpha} x_{\beta} \in U_j^\gamma$.

(3) This statement is a direct consequence of (2), because every non-principal $M$-ultrafilter on $\kappa$ that is $<\kappa$-complete for $M$ contains all final segments of $\kappa$ in $M$.

(4) Let $U$ be a non-principal $M$-ultrafilter on $\kappa$ that is $<\kappa$-complete for $M$ and $<\kappa$-amenable for $M$. Then (3) implies that $\text{crit}(j_U) = \kappa$. Fix a function $f : \kappa \rightarrow M$ and $\alpha \in M \cap \kappa$ such that $[f]_U$ is a subset of $j_U(\alpha)$ in $(\text{Ult}(M,U), \epsilon_U)$. Then the sequence $(x_\beta : \beta < \alpha)$ with $x_\beta = \{\xi \in \kappa | \beta \in f(\xi)\}$ for all $\beta < \alpha$ is an element of $M$. Given $\beta \in M \cap \alpha$, Los’ Theorem implies that $j_U(\beta) \epsilon_U [f]_U$ if and only if $x_\beta \in U$. The $<\kappa$-amenability of $U$ now yields an $x \in M$ with $M \cap x = \{\beta \in M \cap \alpha | j_U(\beta) \epsilon_U [f]_U\}$. Since $j_U(\alpha) = \{\gamma \in \text{Ult}(M,U) | \gamma \epsilon_U j_U(\alpha)\}$, extensionality allows us to conclude that $j_U(x) = [f]_U$.

In the other direction, let $j : M \rightarrow (N,\epsilon_N^\gamma)$ be a $<\kappa$-powerset preserving elementary embedding with $\text{crit}(j) = \kappa$ and let $\gamma$ be any witness that $\text{crit}(j) = \kappa$. By (3), we know that $U_j^\gamma$ is $<\kappa$-complete for $M$ and non-principal. Fix a sequence $\bar{x} = (x_\beta : \beta < \kappa)$ of subsets of $\kappa$ in $M$ with $\alpha \epsilon_N j(\bar{x}(\xi))$. By our assumption, there is $x \in M$ with $j(x) = y$ and

$$x_{\beta} \in U_j^\gamma \iff \gamma \epsilon_N x_{\beta} = j(\bar{x}(j(\beta))) \iff j(\beta) \epsilon_N y = j(x) \iff \beta \in x$$

for all $\beta \in M \cap \alpha$. This shows that $(M, U_j^\gamma) \models \langle y \subseteq j(\alpha) \wedge \forall \beta < j(\alpha) [\beta \in y \iff \gamma \epsilon_N j(\bar{x}(\beta)) \rangle$.

(5) Let $U$ be a $<\kappa$-complete, non-principal and $<\kappa$-amenable $M$-ultrafilter on $\kappa$. By (3), we have $\text{crit}(j) = \kappa$. Fix a function $f : \kappa \rightarrow M$ in $M$ with the property that $[f]_U$ is a subset of $j_U(\kappa)$ in $(\text{Ult}(M,U), \epsilon_U)$. Then $M$ contains the sequence $(x_\beta : \beta < \kappa)$ with $x_\beta = \{\xi < \kappa | \beta \in f(\xi)\}$ and $<\kappa$-amenability yields an $x \in M$ with $M \cap x = \{\beta \in M \cap \kappa | x_\beta \in U\}$. Given $\beta \in M \cap \kappa$, it is now easy to see that $\beta \in x$ if and only if $j_U(\beta) \epsilon_U [f]_U$.

In the other direction, let $j : M \rightarrow (N,\epsilon_N^\gamma)$ be $<\kappa$-powerset preserving with $\text{crit}(j) = \kappa$ and let $\beta$ be a witness that $j$ jumps at $\kappa$. Then (3) shows that $U$ is $<\kappa$-complete and non-principal. Fix a sequence $\bar{x}_{\beta} = (x_\beta : \beta < \kappa)$ of subsets of $\kappa$ in $M$ and $y \in N$ with

$$\langle N, \epsilon_N \rangle \models \langle y \subseteq j(\kappa) \wedge \forall \beta < j(\kappa) [\beta \in y \iff \gamma \epsilon_N j(\bar{x}(\beta)) \rangle$$

Then there is $x \in M$ with $M \cap x = \{\beta \in M \cap \kappa | j(\beta) \epsilon_N y\}$. Given $\beta \in M \cap \kappa$, we then have $\beta \in x$ if and only if $x_\beta \in U_j^\gamma$.

We next consider situations in which an elementary embedding $j : M \rightarrow (N,\epsilon_N^\gamma)$ induces a canonical $M$-ultrafilter $U$ on $\kappa$, i.e. situations in which $j$ is a $\kappa$-embedding. Given an ordinal $\alpha$, a property $\Psi(M,U)$ of $\Sigma_\alpha$-correct $\text{ZFC}^-$-models $M$ containing $\kappa$ and $M$-ultrafilters $U$ on $\kappa$ corresponds to a property $\Theta(M,j)$ of such models $M$ and elementary embeddings $j : M \rightarrow (N,\epsilon_N^\gamma)$ if the following statements hold:

- If $\Psi(M,U)$ holds for an $M$-ultrafilter $U$ on $\kappa$, then $\Theta(M,j_U)$ holds.
- If $\Theta(M,j)$ holds for an elementary embedding $j : M \rightarrow (N,\epsilon_N^\gamma)$, then $j$ is a $\kappa$-embedding and $\Psi(M,U_j)$ holds.

**Proposition 3.2.** “$U$ is an $M$-ultrafilter on $\kappa$ that is $M$-normal with respect to $\subseteq$-decreasing sequences and contains all final segments of $\kappa$ in $M$ corresponds to $j$ is a $\kappa$-embedding”.

**Proof.** Let $M$ denote a $\Sigma_\alpha$-correct $\text{ZFC}^-$-model with $\kappa \in M$.

First, assume that $U$ is $M$-normal with respect to $\subseteq$-decreasing sequences and contains all final segments of $\kappa$ in $M$. Then, the proof of Proposition 3.3.1 shows that $[\text{id}_{\Delta} ]_U$ witnesses that $j_U$ jumps at $\kappa$. Assume that there is an $f : \kappa \rightarrow M$ in $M$ with $[f]_U \epsilon_U [\text{id} ]_U$ and $j_U(\beta) \epsilon_U [f]_U$ for all $\beta < \kappa$. Then the sequence $\langle x_\beta : \beta < \kappa \rangle$ with $x_\beta = \{\xi < \kappa | \beta < f(\xi) < \xi\}$ for all $\beta < \kappa$ is an element of $M$, and we have $x_\beta \in U$ for all $\beta \in M \cap \kappa$. Since this sequence is $\subseteq$-decreasing, we know that $\Delta_{\beta < \kappa} x_\beta \in U$. But then, there is $\xi \in M \cap \Delta_{\beta < \kappa} x_\beta$ with $\xi \in j_U(\xi)$, a contradiction. This shows that $[\text{id}_\Delta ]_U$ witnesses that $j_U$ is a $\kappa$-embedding.

Now, assume that $j : M \rightarrow (N,\epsilon_N^\gamma)$ is a $\kappa$-embedding. Then, Proposition 3.3.1 shows that $U$ contains all final segments of $\kappa$ in $M$. Let $\bar{x} = (x_\beta : \beta < \kappa)$ be a $\subseteq$-decreasing sequence of subsets of $\kappa$ in $M$ with $x_\beta \in U_j$ for all $\beta \in M \cap \kappa$. Pick $\gamma \in N$ with $\gamma \epsilon_N x_\kappa$. Then the minimality of $\kappa_N^U$ yields $\beta \in M \cap \kappa$ with $\gamma \epsilon_N j(\bar{x}(\beta)) \epsilon_N x_\kappa$. Since $(N,\epsilon_N^\gamma)$ believes that $j(\bar{x})$ is $\subseteq$-decreasing and $x_\beta \in U_j$ implies that $\kappa_N \epsilon_N j(\bar{x}(\beta))$, this shows that $\kappa_N \epsilon_N j(\bar{x}(\gamma))$. But this shows that $\kappa_N \epsilon_N j(\Delta_{\beta < \kappa} x_\beta)$ and hence $\Delta_{\beta < \kappa} x_\beta \in U_j$.

If $j : M \rightarrow (N,\epsilon_N^\gamma)$ is a $\kappa$-embedding that is induced by an $M$-ultrafilter $U$, we may also write $U^\kappa$ rather than $\kappa^N$. We can now add the assumptions from Proposition 3.2 to each item in Proposition 3.1. For example, Clause (5) in Proposition 3.1 together with the observation following Definition 2.1 yields the following:
Corollary 3.3. “U is an M-ultrafilter on κ that is uniform, M-normal and κ-amenable for M ” κ-corresponds to “j is a κ-poawser preserving κ-embedding”. □

Remark 3.4. Using the above results, one could easily rephrase the results from [1], [22] and [23] cited in the introduction in order to obtain characterizations of inaccessible, of weakly compact, and of completely ineffable cardinals in terms of the existence of certain elementary embeddings on countable elementary submodels of structures of the form H(θ). We leave this – given the above results, straightforward – task to the interested reader (for inaccessible cardinals, this was done in [27]).

The following lemma will be useful later on.

Lemma 3.5. Let κ be an inaccessible cardinal, let s : κ → Vκ be a surjection, let M be a Σ0-correct model of ZFC− with (κ + 1) ∪ {s} ⊆ M and let j : M → ⟨N, ϵN⟩ be a κ-poawser preserving κ-embedding with crit(j) = κ.

(1) The map

\[ j_* : V_{κ+1}^M \rightarrow \{ (y \in N \mid y ϵ N V_{κ+1}^N), ϵ_N \}; x \mapsto (j(x) \cap V_N)^N \]

is an ϵ-isomorphism extending j \upharpoonright (Vκ ∩ M).

(2) There is an ϵ-isomorphism

\[ j^* : H(κ^+)^M \rightarrow \{ (y \in N \mid y ϵ N H((κ^+)^N)), ϵ_N \} \]

extending j∗.

Proof. (1) First, note that, using Σ0-correctness, one can show that ran(s)M = ran(s) = Vκ = Vκ^M ∈ M and V_{κ+1}^M = V_{κ+1} ∩ M. Now, if x ∈ Vκ, then there is α < κ with x ⊆ Vα and, since j(α) < κN holds, we have j∗(x) = j(x) ϵ N V_{κ+1}^N. This shows that j∗(x) ϵ N V_{κ+1}^N for all x ∈ Vκ and j∗ \upharpoonright (Vκ) is an ϵ-isomorphism, i.e. given x, y ∈ Vκ, we have x = y if and only if j∗(x) = j∗(y).

Claim. If γ ∈ N with γ ϵ N κN, then there is β ∈ κ with j(β) = γ.

Proof of the Claim. Assume that the statement fails and pick β ∈ M with M ∩ β = {α ∈ M ∩ κ | j(α) ϵ N γ}. By elementarity, β is transitive, well-ordered by the ϵ-relation and a proper subset of κ. Hence β is an element of κ and a subset of M. By our assumptions and the choice of β, we have γ ϵ N j(β). Since j does not jump at β, there is an α < β with γ ϵ N j(α). This shows that j(α) ϵ N j(α), a contradiction. □

Next, if z ∈ V_{κ+1}^M \setminus Vκ, then j∗(z) ϵ N (V_{κ+1}^N)^N and the above claim shows that j∗(z) ϵ N VκN. In combination with the above observation, this shows that j∗ is an ϵ-isomorphism and, by Extensionality, this also shows that j∗ is injective. Next, pick a club subset C of κ in M with s[α] = Vα for all α ∈ C. By the above claim, elementarity implies that κN ϵ N j(C). In this situation, the above claim and elementarity imply that

\[ j(Vκ) = (j \circ s)[κ] = \{ y \in N \mid y ϵ N j(s)[κN]^N \} = \{ y \in N \mid y ϵ N VκN^N \}. \]

Finally, pick z ∈ N with z ∈ N VκN^N. By elementarity and the above computations, there is y ∈ N with ⟨N, ϵN⟩ ≃ “y ⊆ κN ∧ j(s)[y] = z”. Pick x ∈ P(κ)^M with M ∩ x = {α < κ | j(α) ϵ N y}. Then s[x] ∈ V_{κ+1}^M.

Claim. j∗(s[x]) = z.

Proof of the Claim. In one direction, fix u ∈ N with u ϵ N j∗(s[x]) = (j(s)[j(x)] ∩ VκN)^N. Then there is γ ∈ N with γ ϵ N j(x) and ⟨N, ϵN⟩ ≃ “u = j(s)[γ]”. Since u ϵ N VκN and κN ϵ N j(C), we know that γ ϵ N κN. By the above claim, there is β ∈ x with j(β) = γ. But then γ ϵ N y and u ϵ N z. In the other direction, fix u ∈ N with u ϵ N z. Then there is γ ∈ N with γ ϵ N y and ⟨N, ϵN⟩ ≃ “j(s)[γ] = u”. By the above claim, there is β ∈ x with j(β) = γ and hence u = j(s[β]) ϵ N j(s[x]). By Extensionality, these computations yield the desired equality. □

Since the above claim shows that j∗ is surjective, we now know that this map is an ϵ-isomorphism.

(2) There are first order ϵ-formulas \( \varphi_0(κ), \varphi_1(κ, v_1), \varphi_2(v_0, v_1), \varphi_3(v_0, v_1, v_2) \) and \( \varphi_4(v_0, v_1, v_2) \) with the property that the axioms of ZFC− prove that whenever \( s : κ → Vκ \) is a surjection for some inaccessible cardinal κ, then

- the formula \( \varphi_0 \) defines a subset \( D_κ \in V_{κ+1}^κ \),
- the formula \( \varphi_1 \) defines an equivalence relation \( ≡κ \) on \( D_κ \),
- the formula \( \varphi_2 \) defines an \( ≡κ \)-invariant binary relation \( E_κ \) on \( D_κ \),
- the formula \( \varphi_3 \) defines an epimorphism \( π_κ : (D_κ, ≡κ, E_κ) → (H(κ^+), =, ∈) \), and
- the formula \( \varphi_4 \) and the parameter s define a function \( s_κ : V_{κ+1}^κ → D_κ \) with \( π_κ \circ s_κ = id_{V_{κ+1}^κ} \).
By (1), we now obtain a map $j^*: H(\kappa^+)^M \rightarrow \{y \in N \mid y \in_N H((\kappa^N)^+)^N\}$ extending $j_*$ with
\[ (N, \epsilon_N) \models \"\pi_{\kappa^N}(j_*(x)) = j^*(\pi_{\kappa}(x))\" \]
for all $x \in D_\kappa$. By the above assumptions on the uniform definability of $D_\kappa$, $\equiv_\kappa$ and $E_\kappa$, we can conclude that $j^*$ is an $\epsilon$-isomorphism extending $j_*$.

We introduce one further type of correspondence between ultrafilters and elementary embeddings by saying that, given an ordinal $\kappa$, a property $\Psi(M, U)$ of $\Sigma_0^\kappa$-correct ZFC$^-\gamma$-models $M$ containing $\kappa$ and $M$-ultrafilters $U$ on $\kappa$ weakly $\kappa$-corresponds to a property $\Theta(M, j)$ of such models $M$ and elementary embeddings $j : M \rightarrow (N, \epsilon_N)$ if the following properties hold:

- Whenever $\Psi(M, U)$ holds for an $M$-ultrafilter $U$ on $\kappa$, then $\Theta(j_U, M)$ holds.
- $\Theta(M, j)$ implies that $j$ jumps at $\kappa$.
- Whenever $\Theta(j, M)$ holds for an elementary embedding $j : M \rightarrow (N, \epsilon_N)$, then $\Psi(M, U)$ holds for some $N$-ordinal $\gamma$ witnessing that $j$ jumps at $\kappa$.

Note that if $\Psi(M, U)$ corresponds to $\Theta(M, j)$, then these properties weakly $\kappa$-correspond for some ordinal $\kappa$. Moreover, if $\Psi(M, U)$ weakly $\kappa$-corresponds to $\Theta(M, j)$ and $\Theta(M, j)$ implies that $j$ is a $\kappa$-embedding, then these properties also $\kappa$-correspond. Finally, if $\Psi(M, U)$ and $\Theta(M, j)$ $\kappa$-correspond, then they also weakly $\kappa$-correspond. Our next result is an easy consequence of Los’ theorem, and is a most frequently used standard result in a less general setup.

**Lemma 3.6.** Given $A \subseteq \kappa$, “$A \in U$ and $U$ contains all final segments of $\kappa$ in $M$” weakly $\kappa$-corresponds to “$A \in M$ and there is $\gamma \in N$ with $\gamma \in_N j(A)$ witnessing that $j$ jumps at $\kappa$”.

**Proof.** First, assume that $M$ is a $\Sigma_0^\kappa$-correct ZFC$^-\gamma$-model with $\kappa \in M$ and $U$ is an $M$-ultrafilters on $\kappa$ such that $A \in U$ and $U$ contains all final segments of $\kappa$ in $M$. Set $\gamma = [id_{\kappa^U}](A)$. Then $\gamma \in_U j_U(A)$ holds by Los’ theorem and $\gamma$ witnesses that $j_U$ jumps at $\kappa$. In the other direction, if $j : M \rightarrow (N, \epsilon_N)$ is an elementary embedding, $\gamma \in N$ witnesses that $j$ jumps at $\kappa$ and $\gamma \in_N j(A)$, then $A \in U^j$ and Proposition 3.1(1) shows that $U$ contains all final segments of $\kappa$ in $M$.

Combining earlier observations with arguments from the proofs of Proposition 3.2 and Lemma 3.6 immediately yields the following correspondence, which will be of use later on:

**Corollary 3.7.** Given $A \subseteq \kappa$, “$U$ is $\kappa$-normal with respect to $\subseteq$-decreasing sequences, $U$ contains all final segments of $\kappa$ in $M$, and $A \in U$” $\kappa$-corresponds to “$A \in M$ and $j$ is a $\kappa$-embedding with $\kappa^N \in_N j(A)$”.

We want to close this section with two $\kappa$-correspondences, which may seem somewhat trivial, but which will be useful to have available later on.

**Lemma 3.8.** “$U$ contains all final segments of $\kappa$ in $M$ and $(\Ult(M, U), \epsilon_U)$ is well-founded” $\kappa$-corresponds to “$j$ jumps at $\kappa$ and $(N, \epsilon_N)$ is well-founded”.

**Proof.** The forward direction is immediate from Proposition 3.1(1). On the other hand, assume that $j : M \rightarrow (N, \epsilon_N)$ is such that $\crit(j) = \kappa$ and $(N, \epsilon_N)$ is well-founded. Then the well-foundedness of $\epsilon_N$ directly implies that $j$ is a $\kappa$-embedding and Proposition 3.1(1) shows that $U_j$ contains all final segments of $\kappa$ in $M$. As in the standard setting, we can now define a map $k$ from $\Ult(M, U_j)$ to $N$ that sends $[f]_{U_j}$ to $(\langle f(\gamma)^N \rangle)^N$. Since this map satisfies $k \circ j_{U_j} = j$, the well-foundedness of $(N, \epsilon_N)$ implies the well-foundedness of $(\Ult(M, U_j), \epsilon_{U_j})$.

**Lemma 3.9.** Given a first order $\epsilon$-formula $\varphi(v_0, v_1)$, “$U$ is $\kappa$-normal with respect to $\subseteq$-decreasing sequences, $U$ contains all final segments of $\kappa$ in $M$, and $\varphi(M, U)$ holds” $\kappa$-corresponds to “$j$ is a $\kappa$-embedding with $\varphi(M, U_j)$”.

**Proof.** For the forward direction, we know by Proposition 3.2 and its proof that $[\id_U](U)$ witnesses that $j_U$ is a $\kappa$-embedding. This directly implies that $U = U_{jU}$ and hence $\varphi(M, U_{jU})$ holds. The backward direction is a direct consequence of Proposition 3.2.

### 4. Inaccessible cardinals and the bounded ideal

In this section, we characterize inaccessible limits of certain types of ordinals through the existence of $<\kappa$-amenable filters for small models $M$. We then use these characterizations to determine the corresponding ideals, which turn out to be the bounded ideals on the corresponding cardinals. The following easy lemma will be crucial for these characterizations.

**Lemma 4.1.** If $\kappa$ is an inaccessible cardinal and $j : M \rightarrow (N, \epsilon_N)$ is an elementary embedding with $\crit(j) = \kappa$ and $\kappa \in M \prec H(\theta)$ for some regular $\theta > \kappa$, then $j$ is $<\kappa$-powerset preserving.
Proof. Let $\alpha \in M \cap \kappa$ and pick $y \in N$ with $\langle N, \epsilon_N \rangle \models "y \subseteq j(\alpha)"$. Since $\kappa$ is inaccessible in $M$, there is $\delta < \kappa$ in $M$ and a bijection $b : \delta \rightarrow \mathcal{P}(\alpha)$ that is an element of $M$. By elementarity, $j(b)$ is a bijection between $j(\delta)$ and the powerset of $j(\alpha)$ in $N$. Then there is $\gamma \in N$ with $\gamma \epsilon_N j(\delta)$ and $\langle N, \epsilon_N \rangle \models "y = j(b)(\gamma)"$. Since $\alpha < \text{crit}(j)$, there is $\beta < \alpha$ with $j(\beta) = \gamma$ and elementarity implies that $y = j(b(\beta))$. \qed

We will also make use of a classical characterization of inaccessible cardinals. Following [15] Definition 2.2], our formulation of this result uses slightly generalized notions of filters on arbitrary collections of subsets of $\kappa$. It is easy to see that these notions correspond well with our already defined notions of $M$-ultrafilters.

**Definition 4.2.** (1) A uniform filter on $\kappa$ is a subset $F$ of $\mathcal{P}(\kappa)$ such that $|\bigcap_{i<n} A_i| = \kappa$ whenever $n < \omega$ and $(A_i | i < n)$ is a sequence of elements of $F$.
(2) A uniform filter $F$ on $\kappa$ measures a subset $A$ of $\kappa$ if $A \in F$ or $\kappa \setminus A \in F$ and it measures a subset $X$ of $\mathcal{P}(\kappa)$ if it measures every element of $X$. (3) A uniform filter $F$ on $\kappa$ is $<\kappa$-complete if $|\bigcap_{i<n} A_i| = \kappa$ holds for every $\gamma < \kappa$ and every sequence $(A_i | i < \gamma)$ of elements of $F$.

**Lemma 4.3 (\cite{11} Corollary 1.1.2).** An uncountable cardinal $\kappa$ is inaccessible if and only if it has the $<\kappa$-filter extension property, i.e. whenever $F$ is a uniform, $<\kappa$-complete filter on $\kappa$, of size less than $\kappa$, and $X$ is a collection of subsets of $\kappa$ with $X$ of size less than $\kappa$, then there exists a uniform, $<\kappa$-complete filter $F' \supseteq F$ that measures $X$.

We are now ready to state the key proposition of this section. These results will directly yield the statements of Theorems 1.4(1) and 1.5(1).

**Proposition 4.4.** Let $\kappa$ be an inaccessible cardinal.
(1) If $A \subseteq \kappa$ has size $\kappa$, $\theta > \kappa$ is a regular cardinal and $M < \mathcal{H}(\theta)$ with $|M| < \kappa$ and $A \in M$, then there is a uniform $M$-ultrafilter $U$ on $\kappa$ with $A \in U$ and $\Psi_\text{id}(M, U)$.
(2) $\Gamma^\kappa_\text{id}$ is the ideal of bounded subsets of $\kappa$.

**Proof.** (1) Assume that $A \subseteq \kappa$ has size $\kappa$. Then, $\{A\}$ is a uniform, $<\kappa$-complete filter on $\kappa$, of size less than $\kappa$. If $\theta > \kappa$ is a regular cardinal and $M < \mathcal{H}(\theta)$ of size less than $\kappa$ with $\kappa \in M$ and $A \in M$, we can apply Lemma 4.3 to extend $\{A\}$ to a uniform, $<\kappa$-complete filter $U$ on $\kappa$ that measures $\mathcal{P}(\kappa) \cap M$. As mentioned above, this exactly means for $U$ to be a uniform, $<\kappa$-complete $M$-ultrafilter on $\kappa$. Let $j_U : M \rightarrow \langle \text{Ult}(M, U), \epsilon_U \rangle$ be the induced ultrapower embedding. By Proposition 3.4(3), we know that $\text{crit}(j) = \kappa$. Moreover, Lemma 3.6 shows that there is $\gamma \in N$ with $\gamma \epsilon_U j_U(A)$ that witnesses that $j_U$ jumps at $\kappa$. Since $\kappa$ is inaccessible, we can apply Lemma 4.1 to see that $j_U$ is $<\kappa$-powerset preserving. In this situation, we can use Proposition 3.4(4) and Lemma 3.6 to conclude that $U'_\gamma$ is a uniform $M$-ultrafilter on $\kappa$ that is $<\kappa$-complete and $<\kappa$-amenable for $M$ and contains the subset $A$.

(2) If $\theta > \kappa$ is a regular cardinal, $M < \mathcal{H}(\theta)$ with $|M| < \kappa$ and $U$ is a uniform $M$-ultrafilter on $\kappa$, then the uniformity of $U$ implies that every element of $U$ has size $\kappa$. By (1), this yields the desired equality. \qed

The following lemma provides us with the reverse direction for our desired characterization.

**Lemma 4.5.** Assume that for some $M < \mathcal{H}(\theta)$, there exists an elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ with $\text{crit}(j) = \kappa$. Then, $\kappa$ is regular and, if $j$ is also $<\kappa$-powerset preserving, then $\kappa$ is inaccessible.

**Proof.** Assume that $\kappa$ is singular and fix a cofinal function $c : \alpha \rightarrow \kappa$ with $\alpha < \kappa$ in $M$. Let $\gamma \in N$ witness that $j$ jumps at $\kappa$. By elementarity, there is $\delta \in N$ with $\epsilon_N j(\alpha)$ and $\epsilon_N j(c)(\delta))^N = j(c(\delta)) \epsilon_N \gamma$, a contradiction.

Now, assume that $j$ is also $<\kappa$-powerset preserving and that there is $\alpha \in M \cap \kappa$ with $2^\alpha > \kappa$. Pick a surjection $s : \mathcal{P}(\alpha) \rightarrow \kappa$ in $M$ and let $\gamma \in N$ witness that $j$ jumps at $\kappa$. By elementarity, there is $y \in N$ with $\langle N, \epsilon_N \rangle \models "y \subseteq j(\alpha) \land j(s)(y) = \gamma"$. In this situation, our assumption yields $x \in M$ with $j(x) = y$. Then $x \subseteq \alpha$ and $(j(s)(x))^N = j(s(x))^N = j(s(x))^N = j(s(x))^N$. \qed

**Corollary 4.6.** If $\kappa$ is an uncountable cardinal such that for many transitive weak $\kappa$-models $M$ there exists a $<\kappa$-powerset preserving elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ with $\text{crit}(j) = \kappa$, then $\kappa$ is inaccessible.

**Proof.** Fix a bijection $b$ between $\kappa$ and some $M_0 < \mathcal{H}(\kappa^+)$ and define $A = \{\langle \alpha, \beta \rangle | b(\alpha) \in b(\beta)\} \subseteq \kappa$, where $\langle \cdot, \cdot \rangle : \text{Ord} \times \text{Ord} \rightarrow \text{Ord}$ denotes the G"odel pairing function. Pick a transitive weak $\kappa$-models $M$ with $A \in M$ and a $<\kappa$-powerset preserving elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ with $\text{crit}(j) = \kappa$. Then $M_0 \in M$ and elementarity implies that $j \mid M_0$ is a $<\kappa$-powerset preserving elementary embedding with $\text{crit}(j) = \kappa$. Hence Lemma 4.5 implies that $\kappa$ is inaccessible. \qed

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14 That is, if $M$ is a model of ZFC$^-$ with $\kappa \in M$, an $M$-ultrafilter $U$ on $\kappa$ is uniform (respectively, uniform and $<\kappa$-complete for $M$) just in case $U$ is uniform (respectively, uniform and $<\kappa$-complete) in the sense of Definition 4.2.
The next result is now an immediate consequence of what has been shown above, and in particular implies Theorem 1.2 (4(b)).

**Theorem 4.7.** Let $\kappa$ be an uncountable cardinal, let $\delta < \kappa$, let $\varphi(v_0, v_1)$ be a first order $\epsilon$-formula and let $\theta > \kappa$ be a regular cardinal such that the statement $\varphi(\alpha, \delta)$ is absolute between $H(\theta)$ and $V$ for all $\alpha < \kappa$.

Then, the following statements are equivalent for every regular cardinal $\lambda < \kappa$:

1. The cardinal $\kappa$ is an inaccessible limit of ordinals $\alpha$ with the property $\varphi(\alpha, \delta)$ holds.
2. For any (equivalently, for some) $M \prec H(\theta)$ of size less than $\kappa$ with $\kappa \in M$, there exists a uniform $M$-ultrafilter $U$ on $\kappa$ with $\Psi_\alpha(M, U)$ and $\{ \alpha < \kappa \mid \varphi(\alpha, \delta) \} \subseteq U$.
3. For any (equivalently, for some) $M \prec H(\theta)$ of size less than $\kappa$ with $\kappa \in M$, there exists a $\kappa$-powerset preserving elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ with $crit(j) = \kappa$ such that $\langle N, \epsilon_N \rangle \models \varphi(\gamma, j(\delta))$ for some $\gamma \in N$ witnessing that $j$ jumps at $\kappa$.

**Proof.** The implication from (1) to (2) is immediate from Proposition 4.4, Proposition 3.1 (4) and Lemma 3.6 show that the universal and the existential statement in (2) are equivalent to the respective statements in (3). In addition, Finally, Lemma 4.5 shows that the existential statement in (3) implies that $\kappa$ is inaccessible and hence the existential statement in (2) allows us to use Proposition 4.4 to derive (1). \(\square\)

We will next obtain a further characterization of inaccessible cardinals, in which we may require stronger properties of the ultrafilters and elementary embeddings used. For this, we need two standard lemmas.

**Lemma 4.8.** If $\kappa < \theta$ are uncountable regular cardinals, $A \subseteq \kappa$ is stationary and $x \in H(\theta)$, then there is a transitive set $X$ of cardinality less than $\kappa$ and an elementary embedding $j : X \rightarrow H(\theta)$ with $crit(j) \in A$, $j(crit(j)) = \kappa$ and $x \in ran(j)$.

**Proof.** Let $(M_\alpha : \alpha < \kappa)$ be a continuous and increasing sequence of elementary substructures of $H(\theta)$ of cardinality less than $\kappa$ with $x \in M_\alpha$ and $\alpha \subseteq M_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. Since the set $\{M_\alpha \cap \kappa : \alpha < \kappa\}$ is club in $\kappa$, there is $\alpha < \kappa$ with $M_\alpha \cap \kappa \in S$. Set $\pi : M_\alpha \rightarrow X$ denote the corresponding collapse. Then $\pi^{-1} : X \rightarrow H(\theta)$ is an elementary embedding satisfying $crit(\pi^{-1}) = M_\alpha \cap \kappa \in S$, $\pi^{-1}(crit(\pi^{-1})) = \pi^{-1}(M_\alpha \cap \kappa) = \kappa$ and $x \in M_\alpha = ran(\pi^{-1})$. \(\square\)

**Lemma 4.9.** If $\kappa < \theta$ are regular uncountable cardinals, $A$ is a stationary subset of $\kappa$ and $M \prec H(\theta)$ with $|M| < \kappa$ and $A \in M$, then there exists a normal $M$-ultrafilter $U$ on $\kappa$ with $A \in U$.

**Proof.** By Lemma 4.8, there exists a transitive set $X$ of cardinality less than $\kappa$ and an elementary embedding $j : X \rightarrow H(\theta)$ with $j(crit(j)) = \kappa$, $crit(j) \in A$ and $M \in ran(j)$. Then $j[Y] \prec H(\theta)$ and $M \subseteq j[X]$, because $M$ has size less than $\kappa$. Set $U = \{ A \in j[X] \cap \mathcal{P}(\kappa) \mid crit(j) \in A \}$. Then $U$ is a $j[X]$-ultrafilter on $\kappa$ and, since $j[X] \cap \kappa = crit(j)$ and $j[X] \prec H(\theta)$, $U$ is $j[X]$-normal, $c, k$-complete in $V$. All elements of $U$ are stationary subsets of $\kappa$. Since $|M| < \kappa$, it follows that $U \cap M$ is a normal $M$-ultrafilter on $\kappa$ and $crit(j) \in A \in M$ implies that $A \in U$. \(\square\)

The above lemmas yield the following results, which in particular implies Theorem 1.2 (4(b)).

**Theorem 4.10.** The following statements are equivalent for every uncountable cardinal $\kappa$ and all regular cardinals $\theta > \kappa$:

1. The cardinal $\kappa$ is inaccessible.
2. For any (equivalently, for some) $M \prec H(\theta)$ of size less than $\kappa$ with $\kappa \in M$, there exists a $\kappa$-amenable, normal $M$-ultrafilter $U$ on $\kappa$.
3. For any (equivalently, for some) $M \prec H(\theta)$ of size less than $\kappa$ with $\kappa \in M$, there exists a $\kappa$-powerset preserving elementary embedding $j : M \rightarrow \langle N, \epsilon_N \rangle$ with $crit(j) = \kappa$, such that $j$ induces a normal $M$-ultrafilter on $\kappa$.

**Proof.** If $\kappa$ is inaccessible and $M \prec H(\theta)$ with $|M| < \kappa$ and $\kappa \in M$, then Lemma 4.9 yields a normal $M$-ultrafilter $U$ on $\kappa$. By Proposition 3.1 (3), Proposition 3.2, Lemma 3.8 and Lemma 3.9 for the statement $\varphi(M, U) \equiv \langle U \text{ is normal} \rangle$, if $M \prec H(\theta)$ with $|M| < \kappa$ and $\kappa \in M$, then every $\kappa$-amenable, normal $M$-ultrafilter $U$ on $\kappa$ yields an ultrapower embedding $j_U$ with $crit(j_U) = \kappa$ that induces a normal $M$-ultrafilter. Moreover, Lemma 4.11 shows that $j_U$ is $\kappa$-powerset preserving. This shows that both the universal and the existential statement in (2) imply the respective statements in (3). The equivalence between the corresponding statements in (2) and (3) then follows from Proposition 3.1 (4), Proposition 3.2 and Lemma 3.9 using the same formula $\varphi$ as above. Finally, Theorem 4.7 directly shows that the existential statements in (2) and (3) both imply (1). \(\square\)

\(^{15}\)i.e., arbitrary intersections of less than $\kappa$-many elements of $U$ in $V$ are nonempty.
5. Regular stationary limits and the non-stationary ideal

In this section, we characterize Mahlo-like cardinals, that is regular stationary limits of certain ordinals through the existence of \( M \)-normal filters for small models \( M \). We then use these characterizations to define the corresponding ideals, which turn out to be the non-stationary ideal below the considered set of ordinals. We start by proving the corresponding statement of Theorem 1.4 with the help of Lemma 4.9\(^{17}\).

**Proof of Theorem 1.4.** Fix a regular and uncountable cardinal \( \kappa \). First, let \( A \) be a stationary subset of \( \kappa \), let \( \theta > \kappa \) be a regular cardinal and let \( \mathcal{M} \prec H(\theta) \) with \( |\mathcal{M}| < \kappa \) and \( A \in \mathcal{M} \). In this situation, we can use Lemma 4.9 to find a normal \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \). Then \( U \) is uniform with \( \Psi_4(\mathcal{M}, U) \) and hence \( U \) witnesses that \( \kappa \notin I^\kappa_\omega \). In the other direction, let \( A \) be a non-stationary subset of \( \kappa \), let \( \mathcal{M} \prec H(\theta) \) with \( |\mathcal{M}| < \kappa \) and \( A \in \mathcal{M} \), and let \( U \) be a uniform, \( M \)-normal \( M \)-ultrafilter on \( \kappa \). By elementarity, we find a club subset \( C \) of \( \kappa \) that is disjoint from \( A \). By the \( M \)-normality and uniformity of \( U \), every club subset of \( \kappa \) in \( \mathcal{M} \) is contained in \( U \) and this shows that \( A \notin U \). We can conclude that \( A \in I^\kappa_\omega \). \( \square \)

The next result is an immediate consequence of Theorem 1.4, and in particular implies Theorem 1.2.(4(a)).

**Theorem 5.1.** Let \( \kappa \) be an uncountable cardinal, let \( \delta < \kappa \), let \( \varphi(v_0, v_1) \) be a first order \( \epsilon \)-formula and let \( \theta > \kappa \) be a regular cardinal such that the statement \( \varphi(\alpha, \delta) \) is absolute between \( H(\theta) \) and \( V \) for all \( \alpha < \kappa \). Then, the following statements are equivalent:

1. \( \kappa \) is a regular stationary limit of ordinals \( \alpha \) satisfying \( \varphi(\alpha, \delta) \).
2. For any (equivalently, for some) \( M \prec H(\theta) \) of size less than \( \kappa \) with \( \kappa \in M \), there exists a uniform, \( M \)-normal ultrafilter \( U \) on \( \kappa \) with \( \{ \alpha < \kappa \mid \varphi(\alpha, \delta) \} \in U \).
3. Same as (2), but we also require \( U \) to be normal.

4. For any (equivalently, for some) \( M \prec H(\theta) \) of size less than \( \kappa \) with \( \kappa \in M \), there exists a \( \kappa \)-embedding \( j : M \rightarrow \langle N, \epsilon_N \rangle \) with \( \text{crit}(j) = \kappa \) and \( \langle N, \epsilon_N \rangle \models \varphi(\kappa^N, j(\delta)) \).
5. Same as (4), but we also require \( \epsilon_N \) to be well-founded.
6. Same as (4), but we also require \( j \) to induce a normal ultrafilter.

**Proof.** The implication from (1) to (3) is given by Lemma 4.9. The combination of Proposition 3.1(3), Corollary 3.7, Lemma 3.8 and Lemma 3.9 shows that both the universal and the existential statement in (2) are equivalent to the corresponding statement in (4). The same is true for the corresponding statements in (3) and (6). Since Lemma 4.5 shows that the existential statement in (4) implies that \( \kappa \) is regular, we can use Theorem 1.4(2) and the implications derived above to conclude that the existential statement in (2) implies (1). All remaining implications are immediate. \( \square \)

6. Weakly compact cardinals and \( \kappa \)-amenability

In this section, we extend Kunen’s results from \( ^{23} \) and we characterize weakly compact cardinals \( \kappa \) through the existence of \( \kappa \)-amenable ultrafilters for models of size at most \( \kappa \). For this, we need a classical result on weakly compact cardinals, which we present using the notions introduced in Definition 1.2.

**Lemma 6.1.** ([11 Corollary 1.1.4], see also [15 Proposition 2.9]). An uncountable cardinal \( \kappa \) is weakly compact if and only if it has the filter extension property, i.e. whenever \( F \) is a uniform \( <\kappa \)-complete filter on \( \kappa \) of size at most \( \kappa \), and \( X \) is a collection of subsets of \( \kappa \) with \( X \) of size at most \( \kappa \), then there exists a uniform \( <\kappa \)-complete filter \( F' \supseteq F \) that measures \( X \).

**Corollary 6.2.** If \( \kappa \) is weakly compact, then \( I^\kappa_{\alpha} = I^{<\kappa}_{<\alpha} = I^\kappa_{<\alpha} \) is the bounded ideal on \( \kappa \).

**Proof.** Let \( A \) be unbounded in \( \kappa \) and let \( M \) be a weak \( \kappa \)-model with \( A \in M \). Then \( \mathcal{F} = \{ A \cap [\alpha, \kappa] \mid \alpha < \kappa \} \) is a uniform \( <\kappa \)-complete filter on \( \kappa \) of size \( \kappa \). Using Lemma 6.1, we find a uniform \( <\kappa \)-complete filter \( \mathcal{F} \supseteq F \) that measures \( P(\kappa) \cap M \). Then Proposition 3.1 and Lemma 4.1 show that \( \mathcal{U} = \Psi_{\alpha}(M, U) \). By uniformity, these computations show that both \( I^\kappa_{\alpha} \) and \( I^{<\kappa}_{<\alpha} \) are the bounded ideal on \( \kappa \) and, by Theorem 1.4(1), this also shows that they are equal to \( I^\kappa_{<\alpha} \). \( \square \)

The following lemma will allow us to characterize weak compactness through the existence of \( \kappa \)-amenable, \( <\kappa \)-complete ultrafilters.

**Lemma 6.3.** If \( \kappa \) is a weakly compact cardinal, then \( \lambda \leq \kappa \) is a cardinal, \( \theta > \kappa \) is a regular cardinal, \( A \) is an unbounded subset of \( \kappa \) and \( x \in H(\theta) \), then there is \( (\kappa, \lambda) \)-model \( M \prec H(\theta) \) with \( A \in H(\theta) \) and a uniform \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \) and \( \Psi_{\omega}(M, U) \).

\(^{16}\) Note that in particular, regular, inaccessible and Mahlo cardinals are Mahlo-like.

\(^{17}\) Note that the forward direction of this proof is quite different to that of the seemingly similar Proposition 4.4.
Proof. We recursively construct and $\omega$-sequences $\langle M_n \mid n < \omega \rangle$ of weak $\kappa$-models $M_n \prec H(\theta)$, and $\langle U_n \mid n < \omega \rangle$ of $M_n$-ultrafilters on $\kappa$. Pick a weak $\kappa$-model $M_0 \prec H(\theta)$ with $x \in M_0$, and let $U_0$ be the cobounded filter on $\kappa$. Then $U_0$ is $<\kappa$-complete and uniform. Now, assume that $M_n$ and $U_n$ are already constructed and let $M_{n+1} \prec H(\theta)$ be a weak $\kappa$-model with $M_n, U_n \in M_{n+1}$, and, using Lemma 6.1, let $U_{n+1}$ be a uniform $<\kappa$-complete $M_{n+1}$-ultrafilter extending $U_n$. Set $M = \bigcup_{n<\omega} M_n$ and let $U = \bigcup_{n<\omega} U_n$. Then, $U$ is a uniform $M$-ultrafilter that is $<\kappa$-complete for $M < H(\theta)$. If $\bar{x} = \langle x_\alpha \mid \alpha < \kappa \rangle \in M$ is a sequence of subsets of $\kappa$, then $\bar{x} \in M_n$ for some $n < \omega$. Hence, each $x_\alpha$ is measured by $U_n \subseteq U$, and thus, by our choice of $M_{n+1}$, we know that $\langle \alpha < \kappa \mid x_\alpha \in U \rangle \in M_{n+1} \subseteq M$, showing that $U$ is $\kappa$-amenable for $M$ and therefore proving the lemma for $\lambda = \kappa$. Given $\lambda < \kappa$, we simply take a $(\lambda, \kappa)$-model $\langle M, U \rangle \prec \langle M, U \rangle$ with $x \in M$. Then, by elementarity, $\langle M, U \rangle$ has the desired properties.

Corollary 6.4. If $\kappa$ is weakly compact, then $I^*_\kappa = I^*_\kappa = I^*_\kappa$ is the bounded ideal on $\kappa$.

Proof. By uniformity, Lemma 6.3 implies that both $I^*_\kappa$ and $I^*_\kappa$ are the bounded ideal on $\kappa$. Moreover, by choosing $\kappa = \lambda$ and $\theta = \kappa^+$ in Lemma 6.3 we can conclude that $I^*_\kappa$ is also equal to this ideal.

Corollary 6.5 suggests that one should rather consider $I^*_\kappa = I^*_\kappa$ to be the ideal canonically connected to weak compactness. We will study this ideal in Section 7. The next result directly implies Theorem 1.2.1.

Theorem 6.5. The following statements are equivalent for every uncountable cardinal $\kappa$, every cardinal $\lambda \leq \kappa$ and every regular cardinal $\theta > \kappa$:

1. $\kappa$ is weakly compact.
2. For many $(\lambda, \kappa)$-models (equivalently, for some $(\lambda, \kappa)$-model) $M \prec H(\theta)$, there exists a uniform $M$-ultrafilter $U$ on $\kappa$ with $\Psi_{\omega}(M, U)$.
3. For many $(\lambda, \kappa)$-models (equivalently, for some $(\lambda, \kappa)$-model) $M \prec H(\theta)$, there exists a $\kappa$-powerset preserving elementary embedding $j : M \rightarrow (N, \epsilon_N)$ with $\text{crit}(j) = \kappa$.
4. For many transitive weak $\kappa$-models, there exists a uniform $M$-ultrafilter $U$ on $\kappa$ with $\Psi_{\omega}(M, U)$.
5. For many transitive weak $\kappa$-models, there exists a $\kappa$-powerset preserving elementary embedding $j : M \rightarrow (N, \epsilon_N)$ with $\text{crit}(j) = \kappa$.

Proof. The implication from (1) to (2) and (4) follows from Lemma 6.3 Proposition 3.1(5) shows that both statements in (2) are equivalent to the respective statements in (3) and both statements in (4) are equivalent to the respective statements in (5). Now, assume that (1) fails and $j : M \rightarrow (N, \epsilon_N)$ witnesses the existential statement in (3) holds. By picking a suitable elementary substructure, we may assume that $\lambda < \kappa$. Then Theorem 4.10 implies that $\kappa$ is inaccessible and our assumptions implies that there exists a $\kappa$-Aronszajn tree. By elementarity, there is such a tree $T$ in $M$ with underlying set $\kappa$. Then, $j(T)$ is a $j(\kappa)$-Aronszajn tree with underlying set $j(\kappa)$ in $N$. Pick $\gamma \in N$ witnessing that $\text{crit}(j(\kappa)) = \kappa$, let $\delta$ be a node of $j(T)$ on level $\gamma$ in $N$ and let $y \in N$ be the set of predecessors of $\delta$ in $j(T)$ in $N$. By $\kappa$-powerset preservation, there is $x \in M$ with $M \cap x = \{ \beta \in M \cap \kappa \mid j(\beta) \epsilon_N y \}$.

In the above result, instead of using all $M \prec H(\theta)$, as in Kunen’s result for countable models, and as in our earlier sections, we pass to a characterization using only many models $\lambda \prec H(\theta)$. The results of our later sections will show that this is in fact necessary, for if $M \prec H(\theta)$ were closed under countable sequences and satisfies (2) in Theorem 6.5, then we would obtain that $U$ induces a well-founded ultrapower of $M$, which would imply that $\kappa$ is completely ineffable by Theorem 11.4.

7. Weakly compact cardinals without $\kappa$-amenability

We start this section by recalling the well-known arguments for Theorem 12.1(5a). First, the statement of Item 15(a) is a well-known consequence of the fact that weak compactness is equivalent to the filter property (see [1 Theorem 1.1.3]). Next, in order to verify Item 5(a)(iii), assume that $\kappa$ is weakly compact, and that $\theta > \kappa$ is a regular cardinal. Note that for any $x \in H(\theta)$, there are many weak $\kappa$-models $\lambda \prec H(\theta)$ which are closed under countable sequences. For each such $M$, we can use the standard embedding characterization of weakly compact cardinal to find an elementary embedding $j : M \rightarrow N$ with $\text{crit}(j) = \kappa$. Then $j$ induces a uniform $M$-normal ultrafilter $U$. Since $M$ is closed under countable sequences, it follows that $U$ is stationary-complete, as desired.
Let us next recall the definition of the weakly compact ideal, which is due to Lévy.

**Definition 7.1.** Let \( \kappa \) be a weakly compact cardinal. The weakly compact ideal on \( \kappa \) consists of all \( A \subseteq \kappa \) for which there exists a \( \Pi^1_1 \)-formula \( \varphi(v^1) \) and \( Q \subseteq V_\kappa \) with \( V_\kappa \models \varphi(Q) \) and \( V_\alpha \models \neg \varphi(Q \cap V_\alpha) \) for all \( \alpha \in A \).

It is well-known that the weakly compact ideal is strictly larger than the non-stationary ideal whenever \( \kappa \) is a weakly compact cardinal: the former contains the stationary set of non-inaccessible cardinals [see 3 Theorem 2.8]. By a classical result of Levy (see 21 Proposition 6.11]), the weakly compact ideal is a normal ideal. We now provide a characterization of the weakly compact ideal which resembles our earlier characterizations, and which in particular shows that \( I_0^\kappa \) is the weakly compact ideal on \( \kappa \), yielding Theorem 1.4 (3). This result is a variant of results of Baumgartner in [3 Section 2]. In the proof of the first item, we proceed somewhat similar to the argument for [13 Theorem 1.3].

In the following, whenever \( M \) is a \( \Sigma_0 \)-correct ZFC-model, \( \kappa \) is a cardinal of \( M \) and \( U \) is an \( M \)-ultrafilter on \( \kappa \) that is \( M \)-normal with respect to \( \subseteq \) decreasing sequences and contains all final segments of \( \kappa \) in \( M \), then we write \( \kappa^U \) instead of \( \kappa^{\text{Ult}(M,U)} \) (see Proposition 5.2).

**Theorem 7.2.** Let \( \kappa \) be a weakly compact cardinal.

(1) If \( A \subseteq \kappa \) is not contained in the weakly compact ideal, \( \theta > \kappa \) is a regular cardinal and \( M \prec H(\theta) \) is a weak \( \kappa \)-model with \( A \in M \), then there is a uniform \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \) and \( \Psi_\delta(M,U) \).

(2) \( I_0^\kappa = I_{\omega_0}^\kappa \) is the weakly compact ideal on \( \kappa \).

**Proof.** (1) Assume that there is no uniform, \( M \)-normal ultrafilter \( U \) on \( \kappa \) with \( A \in U \). Let \( \pi : M \rightarrow X \) denote the transitive collapse of \( \kappa \), pick a bijection \( b : \kappa \rightarrow X \) with and define \( E = \{ \langle \alpha, \beta \rangle \in \kappa \times \kappa \mid b(\alpha) \in b(\beta) \} \). Let \( T \) be the elementary diagram of \( (\kappa, E) \), coded as a subset of \( V_\kappa \) in a canonical way. Now, let \( \varphi(E,T) \) be a \( \Pi^1_1 \)-statement expressing the conjunction of the following two statements over \( V_\kappa \):

(i) There is no \( U \subseteq \kappa \) such that \( (\kappa, E,U) \) thinks that \( U \) is a uniform, normal ultrafilter on \( b^{-1}(\kappa) \) that contains \( b^{-1}(A) \).

(ii) \( T \) is the elementary diagram of \( (\kappa, E) \).

Then \( V_\kappa \models \varphi(E,T) \) and, since \( A \) is not contained in the weakly compact ideal on \( \kappa \), we can find an inaccessible \( \alpha \in A \) with \( \alpha > b^{-1}(A) = (b^{-1} \circ \pi)(A) \) and \( V_\alpha \models \varphi(E \cap V_\alpha, T \cap V_\alpha) \). Since (ii) is reflected to \( \alpha \), the structure \( (\alpha, E \cap V_\alpha) \) is an elementary substructure of \( (\kappa, E) \). Set \( M = (\pi^{-1} \circ b)[\alpha] \). Then \( M \prec H(\theta) \) with \( |M| < \kappa \) and \( A \in M \). Since \( A \) is stationary in \( \kappa \), Theorem 1.4 (2) yields a uniform, \( M \)-normal ultrafilter \( U_0 \) on \( \kappa \) with \( A \in U_0 \). Set \( U = (b^{-1} \circ \pi)(U_0) \subseteq \kappa \). Then, \( (\kappa, E \cap V_\alpha, U) \) thinks that \( U \) is a uniform, normal ultrafilter on \( b^{-1}(\kappa) \) that contains \( b^{-1}(A) \), contradicting the fact that (i) reflects to \( \alpha \).

(2) By (1), we know that \( I_{\omega_0}^\kappa \) is contained in the weakly compact ideal on \( \kappa \). Now, pick \( A \in P(\kappa) \setminus I_{\omega_0} \). Then there is a regular cardinal \( \theta > \kappa \), a weak \( \kappa \)-model \( M \prec H(\theta) \), and a uniform, \( M \)-normal ultrafilter \( U \) on \( \kappa \) with \( A \in U \). By Proposition 3.2, the induced ultrapower map \( j_U : M \rightarrow \{ \text{Ult}(M,U), \epsilon_U \} \) is a \( \kappa \)-embedding. Then Lemma 3.6 implies that \( j_U^\kappa \epsilon_U j_U^\kappa \) is a \( \Pi^1_1 \)-formula and assume that there is \( Q \subseteq V_\kappa \) with \( V_\kappa \models j_U^\kappa \epsilon_U j_U^\kappa \). Now, let \( \varphi(v_0^U, v_1^U) \) be a \( \Pi^1_1 \)-formula and assume that there is \( Q \subseteq V_\kappa \) with \( V_\kappa \models j_U^\kappa \epsilon_U j_U^\kappa \). Since \( M \prec H(\theta) \), we may assume that \( Q \in M \) and the above statements hold in \( M \). In this situation, the elementarity of \( j \) implies that there is an isomorphism between \( (V_\kappa, \epsilon_U) \) and \( (\{ \text{Ult}(M,U), \epsilon_U \}, j_U^\kappa \) ). Set \( R = \{ x \in V_\kappa \mid j_U(x) \in S \} \). Then we have

\[
\text{dom}(j_U^\kappa) = \{ y \in \text{Ult}(M,U) \mid y \epsilon_U (j_U^\kappa \cap V_\kappa \epsilon_U \text{Ult}(M,U)) \}
\]

and \( j[R] = \{ y \in \text{Ult}(M,U) \mid y \epsilon_U S \} \). This allows us to conclude that \( V_\kappa \models \neg \varphi(Q, R) \), a contradiction. This shows that \( A \) is not contained in the weakly compact ideal on \( \kappa \).

The above computations show that \( I_{\omega_0}^\kappa \) is the weakly compact ideal on \( \kappa \). Finally, by choosing \( \theta = \kappa^+ \) in (1), we know that \( I_0^\kappa \) is contained in this ideal and we can show that these ideals are equal by proceeding as in the above argument, however picking a weak \( \kappa \)-model \( M \) containing the set \( Q \) as an element.

\[ \square \]

Note that, analogous to Theorems 1.4 and 5.1, Theorem 7.2 can be used to obtain an explicit characterization of weakly compact cardinals with the property that certain definable subsets of these cardinals do not lie in the corresponding weakly compact ideal. Similar generalizations can be proven for all stronger large cardinal notions discussed below.

We end this section by proving the remaining statement of Theorem 1.5 (2).

**Lemma 7.3.** If \( \kappa \) is weakly compact, then \( N_{\text{wc}}^\kappa \not\in I_0^\kappa \).

**Proof.** Assume that \( N_{\text{wc}}^\kappa \in I_0^\kappa \). Then Theorem 7.2 implies that \( A \) is an element of the weakly compact ideal. Let \( \kappa \) be the least weakly compact cardinal with this property. Then [29 Lemma 1.15] shows that the set \( A = \{ \alpha < \kappa \mid \alpha \text{ is weakly compact} \} \) contains a 1-club, i.e. there is a stationary subset of \( A \) that contains...
all of its reflection points. Since weak compactness implies stationary reflection, there is an \( \alpha < \kappa \) with the property that \( A \cap \alpha \) is stationary in \( \alpha \). Then \( \alpha \in A \), \( \alpha \) is weakly compact and \( A \cap \alpha \) is a 1-club in \( \alpha \). Since the results of \([29]\) show that a subset of a weakly compact cardinal is an element of the weakly compact ideal if and only if its complement contains a 1-club, we can conclude that \( N^\kappa_{wc} \in \Gamma^c_{wc} \), a contradiction. \( \square \)

8. Weakly ineffable and ineffable cardinals

Remember that, given a set \( A \), an \( A \)-list is a sequence \( \langle d_\alpha \mid \alpha \in A \rangle \) with \( d_\alpha \subseteq A \) for all \( \alpha \in A \). Given an uncountable regular cardinal \( \kappa \), a set \( A \subseteq \kappa \) is then called ineffable (respectively, weakly ineffable) if for every \( A \)-list \( \langle d_\alpha \mid \alpha \in A \rangle \), there is \( D \subseteq \kappa \) such that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is stationary (respectively, unbounded) in \( \kappa \). The ineffable (respectively, weakly ineffable) ideal on \( \kappa \) is the collection of all subsets of \( \kappa \) that are not ineffable (respectively, weakly ineffable). These ideals were introduced by Baumgartner, and he has shown them to be normal ideals on \( \kappa \) whenever \( \kappa \) is (weakly) ineffable (see \([2]\)). The key proposition is now an adaptation of \([1]\) Theorem 1.2.1, which shows that the ideals \( \Gamma^c_{wie} \) and \( \Gamma^c_{ie} \) agree with the weakly ineffable and with the ineffable ideal, yielding Theorem 1.4.4 and 1.4.6.

**Theorem 8.1.** (1) If \( \kappa \) is an uncountable cardinal, \( \vec{d} = \langle d_\alpha \mid \alpha \in A \rangle \) is an \( A \)-list with \( A \subseteq \kappa \), \( M \) is a weak \( \kappa \)-model with \( \vec{d} \in M \) and \( U \) is an \( M \)-ultrafilter with \( \Psi^A_{wie}(M,U) \) and with \( A \in U \), then there is \( D \subseteq \kappa \) such that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is unbounded in \( \kappa \).

(2) Let \( \kappa \) be a weakly ineffable cardinal.
(a) If \( A \subseteq \kappa \) is not contained in the weakly ineffable ideal, \( \theta > \kappa \) is a regular cardinal and \( M < H(\theta) \) is a weak \( \kappa \)-model with \( \vec{d} \in M \) and \( U \) is an \( M \)-ultrafilter on \( \kappa \) with \( A \in U \) and \( \Psi^A_{wie}(M,U) \), then there is \( D \subseteq \kappa \) such that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is stationary in \( \kappa \).

(b) \( \Gamma^\kappa_{wie} = \Gamma^\kappa_{wie} \) is the weakly ineffable ideal on \( \kappa \).

(3) If \( \kappa \) is an uncountable cardinal, \( \vec{d} = \langle d_\alpha \mid \alpha \in A \rangle \) is an \( A \)-list with \( A \subseteq \kappa \), \( M \) is a weak \( \kappa \)-model with \( \vec{d} \in M \) and \( U \) is an \( M \)-ultrafilter with \( \Psi^A_{wie}(M,U) \) and with \( A \in U \), then there is \( D \subseteq \kappa \) such that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is stationary in \( \kappa \).

(4) Let \( \kappa \) be an ineffable cardinal.
(a) If \( A \subseteq \kappa \) is not contained in the ineffable ideal, \( \theta > \kappa \) is a regular cardinal and \( M < H(\theta) \) is a weak \( \kappa \)-model with \( \vec{d} \in M \) and \( U \) is an \( M \)-ultrafilter on \( \kappa \) with \( A \in U \) and \( \Psi^A_{ie}(M,U) \), then there is \( D \subseteq \kappa \) such that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is stationary in \( \kappa \).

(b) \( \Gamma^\kappa_{wie} = \Gamma^\kappa_{wie} \) is the ineffable ideal on \( \kappa \).

**Proof.** We only prove (3) and (4), since the proof of the case of weakly ineffable cardinals proceeds in complete analogy (replacing stationary by unbounded, and replacing normal by genuine throughout).

(3) For every \( \xi < \kappa \), let \( x_\xi = \{ \alpha \in A \mid \xi \in d_\alpha \} \). Then \( \{ x_\xi \mid \xi < \kappa \} \subseteq M \). Now, given \( \xi < \kappa \), set \( u_\xi = x_\xi \) if \( x_\xi \in U \), and set \( u_\xi = A \setminus x_\xi \) otherwise. By our assumptions on \( U \), we have \( u_\xi \in U \) for all \( \xi < \kappa \) and hence \( H = \Delta_{\xi<\kappa} u_\xi \) is a stationary subset of \( \kappa \). Now, fix \( \alpha, \beta \in H \) with \( \alpha < \beta \) and \( \xi < \kappa \). Then \( \alpha, \beta \in u_\xi \). If \( x_\xi \in U \), then \( \alpha, \beta \in x_\xi \), and hence \( \xi \in d_\alpha \cap d_\beta \). In the other case, if \( x_\xi \notin U \), then \( \alpha, \beta \notin A \setminus x_\xi \) and hence \( \xi \notin d_\alpha \cup d_\beta \). In combination, this shows that \( d_\alpha = d_\beta \cap \alpha \) holds for all \( \alpha, \beta \) \( \in H \) with \( \alpha < \beta \). Define \( D = \bigcup_{\alpha, \beta} \{ d_\alpha \mid \alpha \in H \} \). Then our arguments show that the set \( \{ \alpha \in A \mid D \cap \alpha = d_\alpha \} \) is stationary in \( \kappa \).

(4)(a) Let \( \kappa \) be an ineffable cardinal and let \( A \subseteq \kappa \) be ineffable, \( \theta > \kappa \) be regular and \( M < H(\theta) \) be a weak \( \kappa \)-model. Pick an enumeration \( \{ x_\xi \mid \xi < \kappa \} \) of all subsets of \( \kappa \) in \( M \), and, for every \( \xi < \kappa \), set \( d_\alpha = \{ \xi < \kappa \mid \xi \in d_\alpha \} \). Then there is \( H \subseteq A \) stationary in \( \kappa \), and \( D \subseteq \kappa \) with \( D \cap \alpha = d_\alpha \) for all \( \alpha \in H \). Given \( \xi < \kappa \), set \( u_\xi = x_\xi \) if \( x_\xi \in D \), and set \( u_\xi = A \setminus x_\xi \) otherwise. Define \( U = \{ u_\xi \mid \xi < \kappa \} \). The next claim provides the desired conclusion.

**Claim.** \( U \) is a normal \( M \)-ultrafilter with \( A \in U \).

**Proof of the Claim.** \( H \setminus (\xi + 1) \subseteq x_\xi \) for all \( \xi \in D \) and \( D \setminus x_\xi \subseteq \xi + 1 \) for all \( \xi < \kappa \setminus D \). Hence, we have \( H \setminus (\xi + 1) \subseteq u_\xi \) for all \( \xi < \kappa \) and this directly implies that \( U \) is a \( M \)-ultrafilter. Moreover, it shows that \( H \subseteq \Delta_{\xi<\kappa} u_\xi \), and hence \( U \) is normal. \( \square \)

(4)(b) Let \( \kappa \) be ineffable and assume that \( A \subseteq \kappa \) is not an element of \( \Gamma^\kappa_{wie} \cap \Gamma^\kappa_{wie} \). Then every \( A \)-list is contained in a weak \( \kappa \)-model \( M \) with the property that there is a normal \( M \)-ultrafilter \( U \) on \( \kappa \) with \( A \in U \). By (3), this shows that \( A \) is ineffable. This argument shows that the ineffable ideal is contained in both \( \Gamma^\kappa_{wie} \) and \( \Gamma^\kappa_{wie} \). In the other direction, (4)(a) directly shows that \( \Gamma^\kappa_{wie} \) is contained in the ineffable ideal. Moreover, by choosing \( \theta = \kappa^+ \) in (4)(a), the same conclusion can be established for \( \Gamma^\kappa_{wie} \).

The above theorem immediately yields Theorem 1.2.5b and 1.2.5c.

We close this section by verifying Theorem 1.5.3 and 1.5.4.

**Lemma 8.2.** If \( \kappa \) is weakly ineffable, then \( N^\kappa_{wc} \subseteq \Gamma^c_{wie} \) and \( N^\kappa_{wc} \notin \Gamma^c_{wie} \).
Definition 9.2. Verify analogous results for this larger class of large cardinal notions. We start by introducing a number of information between two players, the sequences $\theta > \kappa$.

For every $\alpha \in A$, define

$$d_{\alpha} = \langle \varphi_0[\alpha] \rangle \cup \{<1, b(x)> | x \in X_\alpha \} \subseteq \alpha,$$

where $X_\alpha \subseteq V_\alpha$ and $[\varphi_0] \in V_\omega$ is the Gödel number of a $\Pi^1_1$-formula $\varphi_0(\nu^1)$ such that $V_\alpha \models \varphi_0(X_\alpha)$ and $V_\beta \models \neg \varphi_0(X_\alpha \cap V_\beta)$ for all $\beta \leq \alpha$. Assume, for a contradiction, that $A$ is weakly ineffable. Then, $(d_{\alpha} \in A)$ is an $A$-list, and, by the weak ineffability of $A$, we find $D \subseteq \kappa$ such that $U = \{\alpha \in A | D \cap \alpha = d_{\alpha}\}$ is an unbounded subset of $\kappa$. Pick $\alpha, \beta \in U$ with $\alpha < \beta$. Then $\varphi_0 \equiv \varphi_0$ and $X_\alpha = X_\beta \cap V_\alpha$. Hence, $V_\alpha \models \varphi_0(X_\beta \cap V_\alpha)$, a contradiction. Since $N^{\kappa}_{\text{tree}} \subseteq \prod^{\kappa}_{\text{tree}}$, the above arguments show that $N^{\kappa}_{\text{tree}} \in \Gamma^{\kappa}_{\text{tree}}$.

For the second statement, assume for a contradiction that $\kappa$ is the least weakly ineffable cardinal for which $N^{\kappa}_{\text{tree}} \in \Gamma^{\kappa}_{\text{tree}}$. Let $d = (d_{\alpha} \in \alpha \in N^{\kappa}_{\text{tree}})$ be an $N^{\kappa}_{\text{tree}}$-list, and set $A = \kappa \cap \{\alpha < \kappa | \alpha \text{ is weakly ineffable}\} \notin \Gamma^{\kappa}_{\text{tree}}$. For every $\alpha \in A$, the fact $d \upharpoonright \alpha$ is an $N^{\kappa}_{\text{tree}}$-list implies that there is $D_{\alpha} \subseteq \alpha$ such that $\{\xi \in N^{\kappa}_{\text{tree}} \upharpoonright \kappa \subseteq \alpha \} = \text{d}_{\alpha} \in \alpha$ is an unbounded subset of $\kappa$. But then, $(D_{\alpha} \in A)$ is an $A$-list, and hence there is $D \subseteq \kappa$ such that $\{\alpha \in A | D \cap \alpha = d_{\alpha}\}$ is an unbounded subset of $\kappa$. In this case, the set $\{\alpha \in N^{\kappa}_{\text{tree}} | D \cap \alpha = d_{\alpha}\}$ is also unbounded in $\kappa$. These computations show that the subset $N^{\kappa}_{\text{tree}}$ of $\kappa$ is weakly ineffable, contradicting our initial assumption.

Lemma 8.3. If $\kappa$ is ineffable, then $N^{\kappa}_{\text{tree}} \in \Gamma^{\kappa}_{\text{tree}}$ and $N^{\kappa}_{\text{tree}} \notin \Gamma^{\kappa}_{\text{tree}}$.

Proof. First, let $A$ denote the set of all inaccessibles $\alpha < \kappa$ which are not weakly ineffable. For every $\alpha \in A$, pick an $\alpha$-list $(d^\alpha_{\xi} \in \xi < \alpha)$ witnessing that $\alpha$ is not weakly ineffable. Given $\alpha \in A$, define $D_{\alpha} = \{\xi < \alpha \in d^\alpha_{\xi}, \xi < \alpha \in d^\alpha_{\xi} \subseteq \alpha\}$. Assume, for a contradiction, that $\alpha$ is ineffable and pick $D \subseteq \kappa$ such that the set $S = \{\alpha \in A \mid D \cap \alpha = D_{\alpha}\}$ is stationary in $A$. Then there is a unique $\kappa$-list $(d^\alpha_{\xi} \in \kappa < \xi \in \alpha)$ with $d^\alpha_{\xi} = d^\alpha_{\xi}$ for all $\xi \in S$ and $\kappa < \xi$. Since $\kappa$ is ineffable, there $E \subseteq \kappa$ with the property that the set $T = \{\xi < \kappa \mid E \cap \xi = d^\alpha_{\xi}\}$ is stationary in $\kappa$. Pick $\xi \in S \cap \text{Lim}(T) \subseteq \alpha$. If $\xi \in T \cap \alpha$, then $d^\alpha_{\xi} = d^\alpha_{\xi} E \cap \xi$. Since $T \cap \alpha$ is unbounded in $\alpha$, this shows that the set $\{\xi < \alpha \mid E \cap \xi = d^\alpha_{\xi}\}$ is unbounded in $\alpha$, contradicting the fact that $(d^\alpha_{\xi} \in \xi < \alpha)$ witnesses that $\alpha$ is not weakly ineffable.

For the second statement, assume for a contradiction that $\kappa$ is the least ineffable cardinal for which $N^{\kappa}_{\text{tree}} \in \Gamma^{\kappa}_{\text{tree}}$. Let $d_{\alpha} = \alpha \in N^{\kappa}_{\text{tree}}$ be an $N^{\kappa}_{\text{tree}}$-list, and let $A = \{\alpha < \kappa | \alpha \text{ is ineffable}\} \notin \Gamma^{\kappa}_{\text{tree}}$. For every $\alpha \in A$, we find a set $D_{\alpha} \subseteq \alpha$ such that the set $\{\xi \in N^{\kappa}_{\text{tree}} \mid D_{\alpha} \cap \xi = d_{\alpha}\}$ is stationary in $\alpha$. Then, $(D_{\alpha} \in A)$ is an $A$-list, and hence there is $D \subseteq \kappa$ such that $S = \{\alpha \in A \mid D \cap \alpha = D_{\alpha}\}$ is a stationary subset of $\kappa$. Let $C$ be a club subset of $\kappa$, and pick $\alpha \in \text{Lim}(C) \cap S \subseteq \alpha$. Then $\alpha$ is regular, $C \cap \alpha$ is a club in $\alpha$, and there is $\xi \in C \cap N^{\kappa}_{\text{tree}}$ with $d_{\alpha} = D_{\alpha} \cap \xi = D \cap \xi$. This allows us to conclude that the set $\{\xi \in N^{\kappa}_{\text{tree}} \mid D \cap \xi = d_{\alpha}\}$ is stationary in $\kappa$. These arguments show that $N^{\kappa}_{\text{tree}}$ is ineffable, contradicting our assumption.

9. A Formal Notion of Ramsey-like Cardinals

In this section, we generalize the $\alpha$-Ramsey cardinals from [15] to the class of $\Psi$-$\alpha$-Ramsey cardinals, and verify analogous results for this larger class of large cardinal notions. We start by introducing a number of generalizations of notions from [15]. In the later sections of this paper, we will consider a number of special cases of these fairly general concepts. Our generalizations will be based on games that are similar to those from [15], which however allow for quite general extra winning conditions $\Psi$. We will usually only require them to satisfy the property introduced in the next definition.

Definition 9.1. We say that a first order $\epsilon$-formula $\Psi(v_0, v_1)$ remains true under restrictions if $\Psi(X, F \cap X)$ holds whenever $\theta \neq X \subseteq M$ and $\Psi(M, F)$ holds.

Definition 9.2. Given uncountable regular cardinals $\kappa < \theta$ with $\kappa = \kappa^{<\kappa}$, a limit ordinal $\gamma \leq \kappa^+$, an unbounded subset $A$ of $\kappa$, and a first order formula $\Psi(v_0, v_1)$, we let $G\Psi(A)$ denote the game of perfect information between two players, the Challenger and the Judge, who take turns to produce $\subseteq$-increasing sequences $(M_\alpha | \alpha < \gamma)$ and $(F_\alpha | \alpha < \gamma)$, such that the following holds for every $\alpha < \gamma$:

1. At any stage $\alpha < \gamma$, the Challenger plays a $\kappa$-model $M_\alpha \prec H(\theta)$ such that the sequences $(M_\alpha | \alpha < \gamma)$ and $(F_\alpha | \alpha < \gamma)$ are contained in $M_\alpha$, and then the Judge plays an $M_\alpha$-ultrafilter $F_\alpha$ on $\kappa$.
2. $A \in F_\alpha \cap M_\alpha$.

In the end, we let $M_\gamma = \bigcup_{\alpha \leq \gamma} M_\alpha$ and $F_\gamma = \bigcup_{\alpha \leq \gamma} F_\alpha$. The Judge wins the run of the game if $F_\gamma$ is an $M_\gamma$-normal filter such that $\Psi(M_\gamma, F_\gamma)$ holds. Otherwise, the Challenger wins.

Note that if the Judge ever plays a filter $F_\alpha$ that is not normal, then the Challenger wins, for if $\Delta F_\alpha$ is non-stationary, then $F_\gamma$ cannot be $M_\gamma$-normal. On the other hand, if every $F_\alpha$ is $M_\alpha$-normal, then clearly also $F_\gamma$ is $M_\gamma$-normal.

Definition 9.3. Let $\kappa$ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, let $A$ be an unbounded subset of $\kappa$, let $\theta > \kappa$ be a regular cardinal, let $\gamma \leq \kappa^+$ be a limit ordinal, and let $\Psi(v_0, v_1)$ be a first order $\epsilon$-formula.
Let \( \alpha, \kappa \) will show that Scheme A holds true as well for \( \Psi \). 

**Lemma 9.5.** Let \( \gamma \) be regular, and let \( \Psi(\alpha, v_1) \) be a first order \( \epsilon \)-formula. 

The above definition of Ramsey-like cardinals fits well with the main topics of this paper: Given an uncountable regular cardinals, let \( \alpha \leq \kappa \) be a first order \( \epsilon \)-formula.

In Section 11, we will show that a cardinal \( \kappa \) is \( \Psi \)-Ramsey if and only if \( \kappa \) is \( \Psi \)-Ramsey as a subset of itself.

The above definition covers many instances of specific Ramsey-like cardinals that have already been defined in the set-theoretic literature. 

- In Section 11, we will show that a cardinal \( \kappa \) is completely ineffable if and only if it is \( T^{\omega}_\kappa \)-Ramsey, where \( T(M, U) \) denotes the (trivial) property that \( U = U \).
- \[ \text{Definition 1.2} \] A cardinal \( \kappa \) is weakly Ramsey if it is \( \text{wf} M, U \)-Ramsey, where \( \text{wf} M, U \) denotes the property that the ultrapower \( U \) is well-founded.
- \[ \text{Definition 2.11} \] Given an ordinal \( \beta \leq \omega_1 \), a cardinal \( \kappa \) is \( \beta \)-iterable if it is \( \text{wf} \beta^\kappa \)-Ramsey, where \( \text{wf} \beta^\kappa \) denotes the property that \( \beta \) produces not only a well-founded ultrapower, but also \( \beta \)-many well-founded iterates of \( \beta \).
- \[ \text{Definition 4.5} \] A cardinal \( \kappa \) is super weakly Ramsey if it is \( \text{wf}^{+} \)-Ramsey.
- \[ \text{Definition 5.1} \] A cardinal \( \kappa \) is \( \beta \)-Ramsey if it is \( \text{wf}^\beta \gamma \)-Ramsey.
- \[ \text{Definition 4.11} \] Given an ordinal \( \beta \leq \omega_1 \), a cardinal \( \kappa \) is \( (\omega, \beta) \)-Ramsey if it is \( \text{wf} \beta^\kappa \)-Ramsey.
- \[ \text{Theorem 1.3} \] A cardinal \( \kappa \) is Ramsey if and only if it is \( \text{cc}^\kappa \)-Ramsey, where \( \text{cc} \) denotes the property that \( U \) is countably complete.
- \[ \text{Proof of Theorem 3.19} \] A cardinal \( \kappa \) is weakly super Ramsey if it is \( \text{cc}^\kappa \)-Ramsey.
- Ineffably Ramsey cardinals were introduced by Baumgartner in [6]. Adapting the above result on Ramsey cardinals, it will follow in Section 17 that a cardinal \( \kappa \) is ineffably Ramsey if and only if it is \( \text{sc}^\kappa \)-Ramsey, where \( \text{sc} \) denotes the property that \( U \) is stationary-complete.
- In [7] Definition 3.2, Feng introduced a hierarchy of Ramsey-like cardinals denoted as \( \Pi_\beta \)-Ramsey cards, for \( \beta \in \text{Ord} \), with the notion of \( \text{completely Ramsey} \) cardinals at its top, with \( \Pi_\beta \)-Ramsey cardinals being exactly Ramsey cardinals, and with \( \Pi_1 \)-Ramsey cardinals being exactly ineffably Ramsey cardinals. All of these cardinal can be seen to fit into our hierarchy of Ramsey-like cardinals.
- \[ \text{Definition 5.1} \] Given an uncountable regular cardinal \( \alpha \), a cardinal \( \kappa \geq \alpha \) is \( \alpha \)-Ramsey if it is \( T^{\omega}_\alpha \)-Ramsey (or, equivalently, \( \text{cc}^\alpha \)-Ramsey). \( \omega_1 \)-Ramsey cardinals were also called \( \omega \)-closed Ramsey in [7] Definition 2.6.
- \[ \text{Definition 1.4} \] A cardinal \( \kappa \) is strongly Ramsey if it is \( T^{\omega}_\kappa \)-Ramsey (or, equivalently, \( \text{cc}^\kappa \)-Ramsey).
- \[ \text{Definition 1.5} \] A cardinal \( \kappa \) is super Ramsey if it is \( T^{+}_\kappa \)-Ramsey (or, equivalently, \( \text{cc}^+ \)-Ramsey).
- \[ \text{Definition 2.7} \] A cardinal \( \kappa \) is a genuine \( \alpha \)-Ramsey cardinal if it is \( \text{cc}^\kappa \)-Ramsey, where \( \text{cc} \) denotes the property that \( \Delta_{\alpha, \gamma} = \kappa \) for every enumeration \( \langle X_\alpha \mid \alpha < \kappa \rangle \) of \( U \).
- \[ \text{Definition 2.7} \] A cardinal \( \kappa \) is a normal \( \alpha \)-Ramsey cardinal if it is \( \Delta^{\omega}_\kappa \)-Ramsey, where \( \Delta \) denotes the property that \( \Delta U \) is stationary.
- A cardinal \( \kappa \) is locally measurable if and only if it is \( \Psi_{\text{ms}}^{\kappa} \)-Ramsey.
- A cardinal \( \kappa \) is measurable if and only if it is \( \Psi_{\text{ms}}^{\kappa} \)-Ramsey for some (equivalently, for all) regular \( \kappa \).

The following lemma is a straightforward generalization of [8] Theorem 5.6.

**Lemma 9.5.** Let \( \kappa < \theta \) be uncountable regular cardinals with \( \kappa = \kappa^{<\kappa} \), let \( A \) be an unbounded subset of \( \kappa \), let \( \gamma \leq \kappa \) be regular, and let \( \Psi(v_0, v_1) \) be a first order \( \epsilon \)-formula. Then, the following statements hold:

1. If \( A \) has the \( \Psi_{\text{ms}}^{\kappa} \)-filter property, then \( A \) is \( \Psi_{\text{ms}}^{\kappa} \)-Ramsey.
Proof. First, assume that \( A \) has the \( \Psi_\gamma^\omega \)-filter property. We proceed similar to the proof of Theorem 6.5. Given \( x \in H(\theta) \), let \( \sigma \) be any strategy for the \textit{Challenger} in the game \( G\Psi_\gamma^\omega(A) \) that ensures that \( x \in M_\alpha \). Since, by our assumption, \( \sigma \) cannot be a winning strategy for the \textit{Challenger}, it follows that there is a run \( \langle \langle M_\alpha \mid \alpha < \gamma \rangle, \langle F_\alpha \mid \alpha < \gamma \rangle \rangle \) of this game which the \textit{Judge} wins. Then, \( M = \bigcup_{\alpha<\gamma} M_\alpha \) is closed under \(<\gamma\)-sequences, and \( F_\gamma = \bigcup_{\alpha<\gamma} F_\alpha \) is a uniform \( M \)-normal ultrafilter such that \( A \in F_\gamma \), and \( \Psi(M,F_\gamma) \) holds. By the same argument as in the proof of Lemma 6.3, the filter \( F_\gamma \) is \( \kappa \)-amenable for \( M \), as desired.

Now, assume that \( \theta > 2^{\theta^\theta} \) is regular, and that \( A \) is \( \Psi_\gamma^\omega \)-Ramsey, as witnessed by \( M \prec H(\theta) \) and \( U \).

Proof. Let \( \theta_0 \) and \( \theta_1 \) both be regular cardinals greater than \( \kappa \), and assume that the \textit{Challenger} has a winning strategy \( \sigma_0 \) in the game \( G\Psi_\gamma^{\theta_0}(A) \). We construct a winning strategy \( \sigma_1 \) for the \textit{Challenger} in the game \( G\Psi_\gamma^{\theta_1}(A) \). Whenever the \textit{Challenger} would play \( M_\alpha \) in a run of the game \( G\Psi_\gamma^{\theta_1}(A) \) where he is following his winning strategy \( \sigma_0 \), then \( \sigma_1 \) shall tell him to play some \( M_\alpha' \) which is a valid move in the game \( G\Psi_\gamma^{\theta_1}(A) \) such that \( M_\alpha' \supseteq \mathcal{P}(\kappa) \cap M_\alpha \). Every possible response \( F_\alpha^* \) of the \textit{Judge} in the game \( G\Psi_\gamma^{\theta_1}(A) \) induces a response \( F_\alpha = F_\alpha^* \cap M_\alpha \) in the game \( G\Psi_\gamma^{\theta_0}(A) \). We use this induced response together with the strategy \( \sigma_0 \) to obtain the next move of the \textit{Challenger} in the game \( G\Psi_\gamma^{\theta_0}(A) \), and continue playing these two games in this way for \( \gamma \)-many steps. As the \textit{Challenger} is following a winning strategy in the game \( G\Psi_\gamma^{\theta_0}(A) \), it follows that \( F_\alpha \) is either not \( M_\alpha \)-normal or \( \Psi(M_\alpha,F_\alpha) \) fails. But, using our assumptions, the same is the case for \( M_\alpha' = \bigcup_{\alpha<\gamma} M_\alpha' \) and \( F_\alpha^* = \bigcup_{\alpha<\gamma} F_\alpha^* \), showing that \( \sigma_1 \) is indeed a winning strategy.

The following is now immediate from Lemma 6.5 and Lemma 9.7.

Corollary 9.8. Let \( \kappa \) be an uncountable cardinal, let \( \gamma \leq \kappa^+ \), and let \( \Psi(M,U) \) be a property that remains true under \( \kappa \)-restrictions. Then, an un bounded subset \( A \) of \( \kappa \) has the \( \Psi_\gamma^\omega \)-filter property if and only if it has the \( \Psi_\gamma^\omega \)-filter property.

The next result immediately yields Theorem 1.2 and 2.1, and also justifies the entries for \( T_\omega^\omega \)-Ramsey, \( \infty^\omega \)-Ramsey, \( \Delta^\omega_\gamma \)-Ramsey, \( \delta^\omega_\gamma \)-Ramsey, and the final entry for \( \Delta^\gamma_\gamma \)-Ramsey cardinals in Table 2. We will show in Section 13 that the notions of \( \Delta^\gamma_\gamma \)-Ramseyness, \( \infty^\gamma_\gamma \)-Ramseyness and \( \delta^\gamma_\gamma \)-Ramseyness coincide.

Theorem 9.9. Let \( \kappa \) be an uncountable cardinal, let \( A \) be an unbounded subset of \( \kappa \), let \( \gamma \leq \kappa \) be a regular cardinal, and let \( \Psi(v_0,v_1) \) be a first-order \( \epsilon \)-formula that remains true under restrictions. Then, the following statements are equivalent for all \( \gamma \leq \lambda \leq \kappa \) with \( \lambda^{<\gamma} = \lambda \):

1. \( A \) is \( \Psi_\gamma^\omega \)-Ramsey.
2. For any regular cardinal \( \theta > \kappa \), and any \( (\lambda,\kappa) \)-models \( M \prec H(\theta) \) closed under \(<\gamma\)-sequences, there exists a uniform, \( \kappa \)-amenable, \( M \)-normal ultrafilter \( U \) on \( \kappa \) such that \( A \in U \) and \( \Psi(M,U) \) holds.
3. For any regular cardinal \( \theta > \kappa \), and any \( (\lambda,\kappa) \)-models \( M \prec H(\theta) \) closed under \( <\gamma\)-sequences, there exists a \( \kappa \)-powerset preserving \( \kappa \)-embedding \( j : M \to \langle N, \epsilon_N \rangle \) such that \( \kappa^N \in j(A) \) and \( \Psi(M,U) \) holds.

Proof. The implication from (1) to (2) is trivial in case \( \lambda = \kappa \). If \( \lambda < \kappa \), we take a \( (\lambda,\kappa) \)-model \( \langle M, U \rangle \prec \langle M, \bar{U} \rangle \) closed under \( <\gamma\)-sequences and containing \( x \) and \( A \) as elements. Then, by elementarity, and since \( \Psi \) remains true under restrictions, \( \langle M, U \rangle \) is as desired. The equivalence between (2) and (3) follows from

\[ (2) \text{ If } \Psi \text{ remains true under restrictions and } \vartheta > 2^{<\theta^\theta} \text{ is a regular cardinal, and } A \text{ is } \Psi_\gamma^\omega \text{-Ramsey, then } A \text{ has the } \Psi_\gamma^\omega \text{-filter property.} \]
Proposition 3.1 (5), Corollary 3.7, and Lemma 3.9. Next, note that Lemma 4.5 shows that (3) implies that \( \kappa \) is inaccessible. The implication from (2) to (1) is again trivial in case \( \lambda = \kappa \). For smaller \( \lambda \), note that the size of \( M \) did not matter in the proof of Lemma 9.5 (2), as long as \( \gamma + 1 \subseteq M \). This shows that (2) implies \( A \) to have the \( \mathcal{M}_\gamma \)-filter property. Applying Lemma 9.5 (1) then yields \( A \) to be \( \mathcal{P}_\gamma \)-Ramsey. \( \square \)

Let us now introduce ideals that are canonically induced by our Ramsey-like cardinals.

**Definition 9.10.** Let \( \Psi(v_0, v_1) \) be a first order \( \varepsilon \)-formula and let \( \kappa \) be a \( \Psi_\alpha \)-Ramsey cardinal with \( \theta \geq \kappa \) regular and \( \alpha \leq \kappa \) regular and infinite. We define the \( \Psi_\alpha \)-Ramsey ideal on \( \kappa \) to be the set

\[
I_{\Psi_\alpha}(\kappa) = \{A \subseteq \kappa \mid A \text{ is not } \Psi_\alpha \text{-Ramsey}\}.
\]

If \( \theta = \kappa \), the above ideals are particular instances of the ideals defined in Definition 1.3: Given an ordinal \( \alpha \) and a property \( \Psi(M, U) \) of models \( M \) and \( M \)-ultrafilters \( U \), if \( \Phi_\alpha(M, U) \) is the induced property defined in the discussion following Definition 9.4, then \( I_{\Phi_\alpha} = I_{\Psi_\alpha}(\kappa) \) holds for every \( \Psi_\alpha \)-Ramsey cardinal \( \kappa \). In particular, the discussion following Definition 1.3 and Lemma 2.3 show that these ideals are proper and normal. Similarly, if we further strengthen the property \( \Phi_\alpha(M, U) \) to obtain a property \( \Phi_\alpha(M, U) \) that also demands the model \( M \) to be an elementary submodel of \( H(\kappa^+) \), then \( I_{\Phi_\alpha} = I_{\Psi_\alpha}(\kappa) \) holds for every \( \Psi_\alpha \)-Ramsey cardinal \( \kappa \), and hence these ideals are also proper and normal.

**Proposition 9.11.** Let \( \Psi(v_0, v_1) \) and \( \Omega(v_0, v_1) \) be first order \( \varepsilon \)-formulas that remain true under restrictions, such that \( \Omega \) implies \( \Psi \), let \( \alpha \leq \beta \leq \kappa \) be regular infinite cardinals, let \( \theta \geq \beta \geq \kappa \) be regular cardinals, and let \( \kappa \) be an \( \Omega_\beta \)-Ramsey cardinal. Then, \( I_{\Omega_\beta}(\kappa) \subseteq I_{\Omega_\alpha}(\kappa) \).

**Proof.** Assume that \( A \notin I_{\Omega_\beta}(\kappa) \). Then, for any \( x \in H(\theta) \cup P(\kappa) \), there is a weak \( \kappa \)-model \( M \), elementary in \( H(\theta) \) in case \( \theta > \kappa \), and transitive in case \( \theta = \kappa \), that is closed under \( \prec \)-sequences, with \( x \in M \), with \( \theta \in M \) in case \( \theta > \kappa \), and with a uniform, \( \alpha \)-amenable, \( M \)-normal ultrafilter \( U \) on \( \kappa \) with \( A \in U \), such that \( \Omega(M, U) \) holds. But then, using that \( \Omega \) implies \( \Psi \), which remains true under restrictions, either \( M \cap H(\kappa) \) (in case \( \theta > \kappa \)) or \( M \cap H(\kappa^+) \) (in case \( \theta = \kappa \)) witnesses, together with \( U \), that \( A \notin I_{\Omega_\beta}(\kappa) \). \( \square \)

If \( \kappa \) is a \( \Psi_\alpha \)-Ramsey cardinal, it follows by a trivial cardinality argument that the ideals \( I_{\Psi_\alpha}(\kappa) \) stabilize for sufficiently large \( \theta \).\(^{18}\) We can thus make the following definition, that corresponds to Definition 1.3.

**Definition 9.12.** Let \( \Psi(v_0, v_1) \) be a first order \( \varepsilon \)-formula that remains true under restrictions and let \( \kappa \) be a \( \Psi_\alpha \)-Ramsey cardinal with \( \alpha \leq \kappa \) regular and infinite. We define the \( \Psi_\alpha \)-Ramsey ideal on \( \kappa \) to be the set

\[
I_{\Psi_\alpha}(\kappa) = \bigcup \{I_{\Psi_\alpha}(\kappa) \mid \theta > \kappa \text{ regular}\}.
\]

Given an ordinal \( \alpha \) and a property \( \Psi(M, U) \) of models \( M \) and \( M \)-ultrafilters \( U \), if \( \Phi_\alpha(M, U) \) is the induced property defined in the discussion following Definition 9.4, then the above remarks directly show that \( I_{\Phi_\alpha} = I_{\Psi_\alpha}(\kappa) \) holds for all \( \Psi_\alpha \)-Ramsey cardinals \( \kappa \). In particular, these ideals are normal and proper. In addition, Proposition 9.11 shows that, for properties \( \Psi \) and \( \Omega \) that remain true under restrictions such that \( \Omega \) implies \( \Psi \), and for regular infinite cardinals \( \alpha \leq \beta \leq \kappa \), if \( \kappa \) is an \( \Omega_\beta \)-Ramsey, then \( I_{\Phi_\alpha}(\kappa) \subseteq I_{\Phi_\beta}(\kappa) \).

In the remainder of this section, we prove results concerning the relations of the ideals produced by Definition 9.10 and Definition 9.12. The following sets will be central for this analysis. Given regular cardinals \( \alpha < \kappa \) and a first order \( \varepsilon \)-formula \( \Psi(v_0, v_1) \), we make the following definitions:

- \( N_{\Psi_\alpha}(\kappa) = \{\gamma \in (\alpha, \kappa) \mid \gamma \text{ is not a } \Psi_\alpha \text{-Ramsey cardinal}\} \).
- \( N_{\Psi_\alpha}(\kappa) = \{\gamma < \kappa \mid \gamma \text{ is not a } \Psi_\alpha \text{-Ramsey cardinal}\} \).
- \( N_{\Psi_\alpha}(\kappa) = \{\gamma \in (\alpha, \kappa) \mid \gamma \text{ is not a } \Psi_\alpha \text{-Ramsey cardinal}\} \).
- \( N_{\Psi_\alpha}(\kappa) = \{\gamma < \kappa \mid \gamma \text{ is not a } \Psi_\alpha \text{-Ramsey cardinal}\} \).

The following lemmas now show that under mild assumptions on the formula \( \Psi \), the \( \Psi_\alpha \)-Ramsey, \( \Psi_\alpha \)-Ramsey and \( \Psi_\alpha \)-Ramsey cardinals are strictly increasing in terms of consistency strength, thus strengthening and generalizing [13] Proposition 5.2 and Proposition 5.3] and [9] Theorem 3.14. They also show that if \( \kappa \) is a \( \Psi_\alpha \)-Ramsey cardinal, then \( I_{\Psi_\alpha}(\kappa) \supseteq I_{\Psi_\alpha}(\kappa) \).

**Lemma 9.13.** Let \( \Psi(v_0, v_1) \) be a first order \( \varepsilon \)-formula that remains true under restrictions, let \( \alpha \) be a regular cardinal, and let \( \kappa \geq \alpha \) be a \( \Psi_\alpha \)-Ramsey cardinal such that \( \Psi \) is absolute between \( V \) and \( H(\kappa^+) \). Then, the following statements hold true.

1. \( N_{\Psi_\alpha}(\kappa) \in I_{\Psi_\alpha}(\kappa) \).
2. \( N_{\Psi_\alpha}(\kappa) \notin I_{\Psi_\alpha}(\kappa) \).

\(^{18}\)In general, we do not know of any way to find a non-trivial bound on what a sufficiently large \( \theta \) would be relative to \( \kappa \).
Proof. (1) Assume that $A = N\Psi^*_\alpha(\kappa) \notin I\Psi^*_\alpha(\kappa)$. Then, there is a weak $\kappa$-model $M \prec H(\kappa^+)$ and a $\kappa$-powerset preserving $\kappa$-embedding $j : M \rightarrow (N, \varepsilon_N)$ such that $A \in M$ and $\kappa^N \varepsilon_N j(A)$. First assume that $\alpha < \kappa$. Then our assumptions on $\Psi$ imply that the set $A$ consists of all $\gamma$ in $(\alpha, \kappa)$ that are not $\Psi^*_\alpha$-Ramsey cardinals in $M$. Therefore, $\kappa^N$ is not a $\Psi^*_\alpha$-Ramsey cardinal in $(N, \varepsilon_N)$. However, since $j$ is $\kappa$-powerset preserving and $M$ is an elementary submodel of $H(\kappa^+)$, we can use the isomorphism provided by Lemma 3.5 (2) to conclude that $\kappa^N$ is $\Psi^*_\alpha$-Ramsey in $(N, \varepsilon_N)$, a contradiction. In the other case, if $\alpha = \kappa$, then our assumptions ensure that $A$ consists of all $\gamma < \kappa$ that are not $\Psi^*_\alpha$-Ramsey cardinals in $M$ and hence $\kappa^N$ is not a $\Psi^*_\alpha$-Ramsey cardinal in $(N, \varepsilon_N)$. As above, we can use Lemma 3.5 (2) to derive a contradiction.

(2) First, assume that $\kappa > \alpha$ and $\kappa$ is the last $\Psi^*_\alpha$-Ramsey cardinal $\gamma > \alpha$ with the property that $N\Psi^*_\alpha(\gamma) \in I\Psi^*_\alpha(\kappa)$. By Definition 3.10 Proposition 3.1 (5) and Corollary 3.7 there is a weak $\kappa$-model $M \prec H(\kappa^+)$, and a $\kappa$-powerset preserving $\kappa$-embedding $j : M \rightarrow (N, \varepsilon_N)$ with $\kappa^N \notin j(N\Psi^*_\alpha(\kappa))$. Therefore, $\kappa^N$ is a $\Psi^*_\alpha$-Ramsey cardinal below $j(\kappa)$ in $(N, \varepsilon_N)$, and hence by minimality, $N\Psi^*_\alpha(\kappa^N) \notin I\Psi^*_\alpha(\kappa^N)$ holds in this model. By our assumptions on $\Psi$, $M$ computes both $N\Psi^*_\alpha(\kappa)$ and $\Psi^*_\alpha(\kappa)$ correctly. In this situation, Lemma 3.5 (2) shows that $N\Psi^*_\alpha(\kappa^N) \notin I\Psi^*_\alpha(\kappa)$ holds in $(N, \varepsilon_N)$, a contradiction. □

The next result shows that, in many important cases, filters of the form $\Psi^*_\alpha(\kappa)$ are proper subsets of the corresponding filters $\Psi^*_\alpha(\kappa)$.

Lemma 9.14. Let $\Psi(v_0, v_1)$ be a first order $\epsilon$-formula that remains true under restrictions and is absolute between $\mathcal{V}$ and $H(\theta)$ for sufficiently large regular cardinals $\theta$. If $\kappa$ is a $\Psi^*_\alpha$-Ramsey cardinal for some regular $\alpha < \kappa$, then $N\Psi^*_\alpha(\kappa) \notin I\Psi^*_\alpha(\kappa)$.

Proof. Assume that $B = N\Psi^*_\alpha(\kappa) \notin I\Psi^*_\alpha(\kappa)$. Let $\theta > (2^\epsilon)^+$ be a sufficiently large regular cardinal. Then, there is a weak $\kappa$-model $M \prec H(\theta)$ with a $\kappa$-powerset preserving $\kappa$-embedding $j : M \rightarrow (N, \varepsilon_N)$ such that $\kappa^N \varepsilon_N j(B)$. Since our assumption on $\Psi$ imply that $M$ computes $N\Psi^*_\alpha(\kappa)$ correctly, the model $(N, \varepsilon_N)$ thinks that $\kappa^N$ is not $\Psi^*_\alpha(\kappa)^+$-Ramsey. However, by $\kappa$-powerset preservation and by Lemma 3.5 (2), $(N, \varepsilon_N)$ also thinks that $\kappa^N$ is $\Psi^*_\alpha(\kappa)^+$-Ramsey, a contradiction. □

Lemma 9.15. Let $\Psi(M, U)$ be a first order property such that $\Psi(M_1, U_1)$ holds whenever $\kappa$ is an infinite cardinal and $M_0, M_1, U_0$ and $U_1$ satisfy the following properties:

- $M_i$ is a transitive weak $\kappa$-model for all $i < 2$.
- $U_i$ is a uniform, $\kappa$-amenable and $M_i$-normal $M_i$-ultrafilter on $\kappa$ for all $i < 2$.
- $\Psi(M_0, U_0)$ holds and $M_1, U_1 \in H(\kappa^+)M_0$.
- Some surjection $s : \kappa \rightarrow V_\kappa$ is an element of $M_0$.
- $\Psi(j_{M_0}(M_1), j_{U_0}(U_1))$ holds in $(\text{Ult}(M_0, U_0), \varepsilon_{U_0})$, where $j_{U_0}$ is the $\epsilon$-isomorphism induced by the ultra-power embedding $j_{U_0} : M_0 \rightarrow \text{Ult}(M_0, U_0)$ and by $s$, as in Lemma 3.5 (2).

Then, if $\kappa$ is a $\Psi^*_\alpha$-Ramsey cardinal with $\alpha \leq \kappa$ and $\theta \in \{\kappa, \kappa^+\}$, then $N\Psi^*_\alpha(\kappa) \notin I\Psi^*_\alpha(\kappa)$.

Proof. First, assume that there is an ordinal $\alpha$ and a $\Psi^*_\alpha$-Ramsey cardinal $\kappa \geq \alpha$ with $N\Psi^*_\alpha(\kappa) \in I\Psi^*_\alpha(\kappa)$. Let $\kappa$ be minimal with this property and pick $x \subseteq \kappa$ witnessing that $N\Psi^*_\alpha(\kappa)$ is not $\Psi^*_\alpha$-Ramsey. Pick a surjection $s : \kappa \rightarrow V_\kappa$. Since $\kappa$ is $\Psi^*_\alpha$-Ramsey, there is a weak $\kappa$-model $M_0$ closed under $\kappa$-sequences and a uniform, $\kappa$-amenable $M_0$-normal $M_0$-ultrafilter $U_0$ such that $x, s \in M_0$ and $\Psi(M_0, U_0)$ holds. If $\alpha < \kappa$, then we set $\beta = j_{U_0}(\kappa)$. In the other case, if $\alpha = \kappa$, then we set $\beta = j_{U_0}(\kappa)$. Then $U_0$ is a $\Psi^*_\beta$-Ramsey cardinal with $N\Psi^*_\beta(\kappa) \notin I\Psi^*_\beta(\kappa)$ in $\text{Ult}(M_0, U_0)$. Hence, in $\text{Ult}(M_0, U_0)$, there is a weak $\kappa^U_0$-model $M$ closed under $\kappa^U_0$-sequences and a uniform, $\kappa^U_0$-amenable $M$-normal $M$-ultrafilter $U$ such that $j_{U_0}(x) \in M$, $j_{U_0}(\kappa^U_0) \in U$ and $\Psi(M, U)$ holds. Pick $M_1, U_1 \in H(\kappa^+)M_0$ with $j_{U_0}(M_1) = M$ and $j_{U_0}(U_1) = U$. Then $M_1$ is a weak $\kappa$-model closed under $\kappa$-sequences, $U_1$ is a uniform, $\kappa$-amenable and $M_1$-normal $M_1$-ultrafilter on $\kappa$ and our assumptions on $\Psi$ imply that $\Psi(M_1, U_1)$ holds. Moreover, we have $x \in M_1$ and, since $j_{U_0}(N\Psi^*_\alpha(\kappa)) = N\Psi^*_\alpha(\kappa^U_0)N^U_0$, we know that $N\Psi^*_\alpha(\kappa) \subseteq U_1$. But this shows that $M_1$ and $U_1$ witness that $N\Psi^*_\alpha(\kappa) \notin I\Psi^*_\alpha(\kappa)$, a contradiction.

The case $\kappa = \kappa^+$ works analogously, using the observation that, if $M_0 \prec H(\kappa^+)$ is a weak $\kappa$-model, $U_0$ is a uniform, $\kappa$-amenable and $M_0$-normal $M_0$-ultrafilter on $\kappa$, $M \prec H(\kappa^U_0)$ is a weak $\kappa^U_0$-model in $\text{Ult}(M_0, U_0)$ and $M_1 \in H(\kappa^+M_0$ with $j_{U_0}(M_1) = M$, then $M_1$ is a weak $\kappa$-model with $M_1 \prec H(\kappa^+)$. □

The above lemma directly yields the related parts of Theorem 1.5 (4), 1.5 (9) and 1.5 (10). It also provides the corresponding statements for $\beta$-iterable, super weakly Ramsey and super Ramsey cardinals.

Corollary 9.16. Let $\alpha \leq \kappa \leq \theta$ be cardinals with $\theta \in \{\kappa, \kappa^+\}$. 
The bottom of the Ramsey-like hierarchy

10. THE BOTTOM OF THE RAMSEY-LIKE HIERARCHY

The weakest principles that can be extracted from the general definitions of the previous section are the $T^*_\omega$-Ramsey and the $T^+\omega$-Ramsey cardinals. It already follows from Theorem 6.5 that if $\kappa$ is $T^*_\omega$-Ramsey, then $\kappa$ is weakly compact. Moreover, it is trivial to check that whenever $\kappa$ is a $T^*_\omega$-Ramsey cardinal, then $IT^*_\omega(\kappa)$, the smallest of our Ramsey-like ideals, is a superset of the ideal $I^*_\omega$.

Lemma 10.1. If $\kappa$ is a $T^*_\omega$-Ramsey cardinal, then $I^*_\text{wec} \cup \{N^*_\text{wec}\} \subseteq IT^*_\omega(\kappa)$, $NT^*_\omega(\kappa) \not\subseteq IT^*_\omega(\kappa)$ and $I^*_\text{wec} \not\subseteq IT^*_\omega(\kappa)$.

Proof. First, let $A \subseteq \kappa$ be $T^*_\omega$-Ramsey and fix an $A$-list $\vec{d} = \langle d_\alpha \mid \alpha \in A \rangle$. Pick a weak $\kappa$-model $M$ with $\vec{d} \in M$ and a $\kappa$-amenable, $M$-normal ultrafilter $U$ on $\kappa$ with $A \in U$. Set $N = \text{Ult}(M, U)$. Since $j_U$ is $\kappa$-powerset preserving, the set $D = \{\alpha < \kappa \mid j_U(\alpha) \in N \subseteq (j_U(\vec{d}))^N\}$ is an element of $M$. Then $\{\alpha \in A \mid D \cap \alpha = d_\alpha\} \in U$ and, since $U$ is uniform, we can conclude that $A$ is weakly ineffable. These computations show that $\kappa$ is weakly ineffable with $I^*_\text{wec} \subseteq IT^*_\omega$. Moreover, Corollary 9.16 directly shows that $NT^*_\omega \not\subseteq IT^*_\omega$.

Next, assume that $N^*_\text{wec} \not\subseteq IT^*_\omega(\kappa)$. Then, there is a transitive weak $\kappa$-model $M$ and a $\kappa$-amenable, $M$-normal ultrafilter $U$ on $\kappa$ such that $N^*_\text{wec} \subseteq U$. Now, for every $\kappa$-size collection of subsets of $\kappa$ in $M$, we can use $U$ to find a normal ultrafilter on that collection in $M$. In particular, $\kappa$ is ineffable in $M$ (see [1] Corollary 1.3.1) or Theorem 5.1. By the $\kappa$-powerset preservation of the embedding $j_U$, the fact that the ineffability of $\kappa$ is a property of $V_{\kappa+1}$ implies that $\kappa^U$ is ineffable in $\text{Ult}(M(U), j_U(U))$. On the other hand, we have $j_U(U) \subseteq (\text{Ult}(M(U), j_U(U)))$, yielding that $\kappa^U$ is not ineffable in $\text{Ult}(M(U), j_U(U))$, a contradiction.

Finally, if $\kappa$ is not ineffable, then the remarks following Definition 1.3 show that $\kappa \in I^*_\text{wec} \setminus IT^*_\omega$. Hence, we may assume that $\kappa$ is ineffable. Since $T^*_\omega$-Ramseyness is a $\Pi^1_3$-property and the classical argument of Jensen and Kunen in [17] proving the $\Pi^1_3$-indescribability of ineffable cardinals shows that, given an a $\Pi^1_3$-statement $\Omega$ that holds in $V_\kappa$, the set of all non-reflection points of $\Omega$ in $V_\kappa$ is not ineffable, we can use Theorem 5.1 to conclude that $NT^*_\omega \in I^*_\text{wec} \setminus IT^*_\omega$. □

Proposition 10.2. If $\kappa$ is $T^+\omega$-Ramsey, then $I^*_\text{wec} \subseteq IT^+\omega(\kappa)$ and $NT^+\omega(\kappa) \not\subseteq IT^+\omega(\kappa)$.

Proof. The first statement is proven exactly as the related part of Lemma 10.1, additionally using that, by elementarity, every element of $U$ is stationary. The second statement follows from Corollary 9.16. □

11. COMPLETELY INEFFABLE CARDINALS

We start by recalling the definition of complete ineffability.

Definition 11.1. Let $\kappa$ be an uncountable regular cardinal.

1. A nonempty collection $S \subseteq \mathcal{P}(\kappa)$ is a stationary class if the following statements hold:
   (a) Every $A \in S$ is a stationary subset of $\kappa$.
   (b) If $A \in S$ and $A \subseteq B \subseteq \kappa$, then $B \in S$.
2. A subset $A$ of $\kappa$ is completely ineffable if there is a stationary class $S \subseteq \mathcal{P}(\kappa)$ with $A \in S$ and the property that for every $S \in S$ and every function $f : [S]^2 \to 2$, there is $H \in S$ that is homogeneous for $f$.
3. The cardinal $\kappa$ is completely ineffable if the set $S$ is completely ineffable in the above sense.

It is trivial to check that if there exists a stationary class $S \subseteq \mathcal{P}(\kappa)$ witnessing the complete ineffability of $\kappa$, then the union of all such stationary classes is again a stationary class witnessing the complete ineffability of $\kappa$, and it is therefore the unique maximal stationary class that does so. From [15] Corollary 3] and its proof, and from the definition of the completely ineffable ideal in [19], it is immediate that the completely ineffable ideal is the complement of this maximal stationary class. The following lemma is an easy adaptation of Kunen’s result that ineffability can be characterized either in terms of homogeneous sets for colourings or for lists (see [17] Theorem 4) [19]. It is probably a folklore result, and its substantial direction is implicit in the proof of [20] Theorem 3.12].

Lemma 11.2. A stationary class $S \subseteq \mathcal{P}(\kappa)$ with $0 \not\in \bigcup S^{[\kappa]}$ witnesses that $A \subseteq \kappa$ is completely ineffable if and only if $S$ witnesses $A$ to be completely ineffable with respect to lists, in the sense that $A \in S$ and for

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Note that this is a harmless extra assumption, for if some stationary class witnesses $A \subseteq \kappa$ to be completely ineffable, then there is such a stationary class $S$ which also satisfies $0 \not\in \bigcup S$. 

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[17] Theorem 4] also provides a characterization of ineffability in terms of regressive colourings. An analogous result would be possible for complete ineffability, however we do not need this in our paper, and hence omitted to present it.
every $S \in \mathcal{S}$, and every $S$-list $\vec{d} = (d_\alpha \mid \alpha \in S)$, there is $H \in \mathcal{S}$ with $d_\alpha = d_\beta \cap \alpha$ for all $\alpha, \beta \in H$ with $\alpha < \beta$.

Proof. First, assume that the stationary class $\mathcal{S}$ witnesses that $A \subseteq \kappa$ is completely ineffable. Pick $S \in \mathcal{S}$, and an $S$-list $\vec{d} = (d_\alpha \mid \alpha \in S)$. Order the bounded subsets of $\kappa$ by letting $a < b$ if there is an $\alpha < \kappa$ such that $a \cap \alpha = b \cap \alpha$ and $a \in b \setminus a$. Define a colouring $c : [\kappa]^2 \to 2$ by setting, for $\alpha < \beta$, $c((\alpha, \beta)) = 1$ in case $d_\alpha < d_\beta$ or $d_\alpha = d_\beta$, and setting $f((\alpha, \beta)) = 0$ otherwise. Let $H \in \mathcal{S}$ be homogeneous for $c$. Since we cannot have a descending $\kappa$-sequence in the ordering $\prec$, it follows that $c$ takes constant value $1$ on $H$. For every $\xi < \kappa$, consider the sequence $(d_\alpha \cap \xi \mid \alpha \in H, \alpha > \xi)$. Since this is a weakly $\kappa$-increasing $\kappa$-sequence of subsets of $\xi$, we can define a function $f : \kappa \to \kappa$ by letting $f(\xi)$ be the minimal $\xi \geq \xi$ such that $d_\alpha \cap \xi = d_\beta \cap \xi$ whenever $\alpha, \beta \in H$ with $\eta \leq \alpha < \beta$. Then, $f$ is a continuous, increasing function that maps $\kappa$ cofinally into $\kappa$. Let $C$ be the closed unbounded subset of $\kappa$ of fixed points of $f$. Since $H$ is stationary, we may pick some $\zeta \in C \cap H$, for which we thus have $d_\zeta = d_\eta \cap \zeta$ for every $\alpha > \zeta \in H$. We may now define a colouring $\vec{d} : [H]^2 \to 2$ by letting $d((\alpha, \beta)) = 1$ if and only if both $\alpha$ and $\beta$ are greater than $\zeta$. If $I \in \mathcal{S}$ is homogeneous for $g$, it follows that $d$ has to take value $1$ on $I$, because it only takes value $0$ on a bounded, and therefore non-stationary, subset of $\kappa$. Thus, if $\alpha < \beta$ are both elements of $I$, it follows that $d_\alpha = d_\beta \cap \alpha$, showing that $\mathcal{S}$ witnesses $A$ to be completely ineffable with respect to lists, as desired.

In the other direction, let $\mathcal{S}$ be the stationary class with $0 \notin \bigcup \mathcal{S}$ that witnesses that $A$ is completely ineffable with respect to lists. Pick $S \in \mathcal{S}$, and a colouring $c : [S]^2 \to 2$. Define an $S$-list $\vec{d} = (d_\alpha \mid \alpha \in S)$ by setting $d_\alpha = \{\beta < \alpha \mid c((\alpha, \beta)) = 1\}$. By our assumption, we find $H \in \mathcal{S}$ such that $d_\alpha = d_\beta \cap \alpha$ whenever $\alpha < \beta$ are both elements of $H$. Let $f : H \to 2$ be defined by setting $f(\alpha) = 1$ if and only if $c_\alpha \in d_\beta$ for some (equivalently, for all) $\beta \geq \alpha$ in $H$. Now, define an $H$-list $\vec{c} = (c_\alpha \mid \alpha \in H)$ by setting $c_\alpha = \alpha$ in case $f(\alpha) = 1$, and setting $c_\alpha = 0$ otherwise. By our assumption, we find $I \in \mathcal{S}$ such that $f$ is homogeneous on $I$. Assume that $f$ takes value $i \in \{0, 1\}$ on $I$. Then, if $\alpha < \beta$ are both elements of $I$, our definitions yield that $f((\alpha, \beta)) = i$, i.e. $I \in \mathcal{S}$ is homogeneous for $f$, as desired.

The following result is the crucial link connecting completely ineffable and Ramsey-like cardinals, and in particular implies Theorem 1.4 of [9], by the above and by Lemma 9.5. In particular, it shows that a cardinal is completely ineffable if and only if it is $\text{T}_\kappa^\kappa$-Ramsey. Its proof is a generalization, adaption and simplification of [24, Theorem 3.12].

Theorem 11.3. Given an uncountable regular cardinal $\kappa$, a subset $A$ of $\kappa$ is completely ineffable if and only if $\kappa = \kappa^{< \kappa}$ holds and $A$ has the $\text{T}_\kappa^\kappa$-filter property.

Proof. First, assume that $A$ has the $\text{T}_\kappa^\kappa$-filter property, and $\kappa = \kappa^{< \kappa}$ holds. Let $\theta > \kappa$ be regular. By Lemma 9.5, $A$ is the $\text{T}_\kappa^\kappa$-Ramsey. Let $\mathcal{S}$ denote the collection of all subsets of $\kappa$ which are $\text{T}_\kappa^\kappa$-Ramsey. Then, $A \in \mathcal{S}$ and $\mathcal{S}$ is a stationary class. Pick $X \in \mathcal{S}$ and $c : [X]^2 \to 2$. Pick $M \prec H(\theta)$ with $c \in M$, and an $M$-normal, $\kappa$-amenable $\kappa$-ultrafilter $U$ on $\kappa$ with $X \in U$. Let $\bar{Y} = (Y_\alpha \mid \alpha \in X)$ be defined by setting $Y_\alpha = \{\beta > \alpha \mid c((\alpha, \beta)) = 0\}$. Define $\bar{Z} = (Z_\alpha \mid \alpha \in X)$ by setting $Z_\alpha = Y_\alpha$ in case $Y_\alpha \in U$, and let $Z_\alpha = X \setminus Y_\alpha$ in $U$ otherwise. Then, $\bar{Z} \in U$ implies $\bar{Z} \in U$. Let $H \in U$ either be $\bar{Z} \cap \alpha \in X \setminus Y_\alpha \in \bar{U}$ or $\bar{Z} \cap \alpha \in X \setminus Y_\alpha \not\in \bar{U}$. Then, it is easy to check that $H \subseteq X$ is homogeneous for $c$. Moreover, $H \in U \subseteq \mathcal{S}$ thus shows that $\mathcal{S}$ is indeed a stationary class containing $A$, as desired.

For the reverse direction, assume that $A \subseteq \kappa$ is completely ineffable, as witnessed by the stationary class $\mathcal{S}$. Let $\theta > \kappa$ be a regular cardinal. We describe a strategy for the $\text{Judge}$ in the game $\text{GT}_\kappa^\kappa(A)$. As required by the rules of this game, the $\text{Challenger}$ and the $\text{Judge}$ take turns playing $\kappa$-models $M_n$ and $M_n$-ultrafilters $U_n$. We let the $\text{Judge}$ also pick, in each step $n < \omega$, an enumeration $X_n = (X_\xi \mid \xi < \kappa) \subseteq \mathcal{P}(\kappa) \cap M_n$ and a set $H_n \in \mathcal{S}$ such that the following hold (the first two items are required by the rules of the game $\text{GT}_\kappa^\kappa(A)$):

- $A \subseteq U_n$,
- If $n > 0$, then $U_n \supseteq U_{n-1}$ and $H_n \subseteq H_{n-1}$,
- $X_\xi \subseteq U_n$ if and only if $\gamma \in X_\xi$ for all $\xi < \gamma \in H_n$ if and only if $\gamma \in X_\xi$ for some $\xi < \gamma \in H_n$.

Assume that we have done this for $m < n$. We want to define the above objects at stage $n$. For the sake of a uniform argument when $n = 0$, let $H_{-1} = A$, and let $C_{-1} = \kappa$. For $\alpha \in H_{n-1}$, define $r_\alpha \subseteq \alpha$ by letting $\xi \in r_\alpha$ if and only if $\alpha \in X_\xi$. By Lemma 11.2, we find a stationary set $H_n \subseteq H_{n-1}$ in $\mathcal{S}$ on which the $\alpha$'s is cohere, that is, for every $\alpha < \beta$ in $H_n$, $r_\alpha = r_\beta \cap \alpha$. But this means that for $\alpha < \beta \in H_n$, and $\xi < \alpha$, $\alpha \in X_\xi$ if and only if $\beta \in X_\xi$. This shows that if we now define $U_n$ using $H_n$ as required above, it will satisfy the required equivalence. In particular, this implies that $H_n \subseteq \Delta U_n$, making $U_n$ a normal $M_n$-measure.

It remains to show that $U_{n-1} \subseteq U_n$ in case $n > 0$. Thus, let $X \in U_{n-1}$ be given, say $X = X_\xi \subseteq X_\xi^{-1}$. By the definition of $U_{n-1}$, every $\xi < \gamma \in H_{n-1}$ is an element of $X$. In particular, we find some such $\gamma > \xi$ in $H_n$, witnessing that $X \in U_n$, as desired.

\[\text{I.e., } \mathcal{S} \text{ is the complement of the completely ineffable ideal.}\]
We are now ready to generalize Kleinberg's result from [22]. Given the above, the following is now an easy consequence of Theorem 9.9 and in particular implies Theorem 1.2 (2a).

**Theorem 11.4.** Given an uncountable cardinal \( \kappa \), the following statements are equivalent for all regular \( \theta > 2^\kappa \) and all \( \lambda \leq \kappa \):

1. The cardinal \( \kappa \) is completely ineffable.
2. For many \((\lambda, \kappa)\)-models \( M < H(\theta) \), there exists a uniform, \( \kappa \)-amenable \( M \)-normal ultrafilter on \( \kappa \).
3. For many \((\lambda, \kappa)\)-models \( M < H(\theta) \), there exists a \( \kappa \)-powerset preserving \( \kappa \)-embedding \( j : M \rightarrow (N, \in_N) \).

**Proof.** That (1) implies (2) is immediate from Theorem 11.3 and Theorem 9.9. The equivalence between (2) and (3) follows from Corollary 3.3. Assuming that (2) holds, observe that the proof of the implication from (2) to (1) in Theorem 9.9 shows that \( \kappa \) has the \((\omega,\kappa^+)\)-filter property. By Lemma 9.7, it follows that \( \kappa \) is completely ineffable.

In the remainder of this section, we verify Theorems 1.5 (5).

**Lemma 11.5.** If \( \kappa \) is completely ineffable, then \( I_{\text{ie}}^c \cup \{N_{\text{ie}}^c \} \subseteq I_{\text{scie}}^c \) and \( N_{\text{icie}}^c \notin I_{\text{scie}}^c \).

**Proof.** The first statement is immediate from Lemma 10.1 and Proposition 10.2. Assume that \( \kappa \) is the least completely ineffable cardinal with this property. Let \( \theta \) be a sufficiently large regular cardinal, and let \( M < H(\theta) \) be a weak \( \kappa \)-model with a \( \kappa \)-amenable, \( M \)-normal ultrafilter \( U \). By our assumption, it follows that \( \kappa^U \) is completely ineffable in \( \text{Ult}(M,U) \), however \( N_{\text{icie}}^c \notin I_{\text{scie}}^c \) in \( \text{Ult}(M,U) \) by elementarity of \( j_U \) and by our minimality assumption on \( \kappa \). Let \( S \) be the maximal stationary class witnessing that \( \kappa^U \) is completely ineffable in \( \text{Ult}(M,U) \), that is, the complement of \( I_{\text{scie}}^c \) in \( \text{Ult}(M,U) \). The iterative construction of the maximal stationary class \( T \) witnessing that \( \kappa \) is completely ineffable in \( M \) (see [22]) together with the \( \kappa \)-powerset preservation of \( j_U \) easily yields that the \( j_U \)-preimage of the collection of elements of \( S \) in \( \text{Ult}(M,U) \) is contained in \( T \). However, this yields that \( N_{\text{icie}}^c \in T \), and since \( T \) is the complement of \( I_{\text{scie}}^c \) in \( M \), this is clearly a contradiction. \( \square \)

12. **Weakly Ramsey cardinals, Ramsey cardinals and ineffably Ramsey cardinals**

We start this section by proving several statements from Theorem 1.5 (6).

**Lemma 12.1.** If \( \kappa \) is a weakly Ramsey cardinal, then \( I_{\text{ie}}^c \cup \{N_{\text{ie}}^c \} \subseteq I_{\text{scie}}^c \) and \( N_{\text{icie}}^c \notin I_{\text{scie}}^c \).

**Proof.** First, note that, since the properties \( T \) and \( \text{wf} \) remain true under restrictions, we can combine Proposition 9.11 and Lemma 10.1 to conclude that \( I_{\text{ie}}^c \subseteq I_{\text{scie}}^c(\kappa) \leq \text{lwf}_{\omega}(\kappa) = I_{\text{scie}}^c \). Moreover, the proof of [19, Theorem 3.7] directly shows that \( N_{\text{icie}}^c \in I_{\text{scie}}^c \). In addition, Corollary 9.16 (2) directly implies that \( N_{\text{icie}}^c \notin \text{lwf}_{\omega}(\kappa) = I_{\text{scie}}^c \). Finally, since weak Ramseyness is a \( \Pi_2 \)-property, the argument used in the last part of the proof of Lemma 10.1 also shows that \( I_{\text{ie}}^c \notin I_{\text{scie}}^c \).

In [3], Theorem 1.3 and [28, Theorem 5.1], isolating from folklore results (see for example [25]), Gitman, Sharpe and Welch have shown that a cardinal \( \kappa \) is Ramsey if and only if (in our notation) it is \( \text{cc}_{\omega}^\kappa \)-Ramsey.

**Definition 12.2.** A cardinal \( \kappa \) is **ineffably Ramsey** if and only if every function \( c : [\kappa]^{<\omega} \rightarrow 2 \) has a homogeneous set that is stationary in \( \kappa \).

In [3], Baumgartner also introduced the **Ramsey ideal** and the **ineffably Ramsey ideal** at \( \kappa \), which can be described as follows (see also [8]). \( A \subseteq \kappa \) is Ramsey if every regressive function function \( c : [A]^{<\omega} \rightarrow \kappa \) has a homogeneous set of size \( \kappa \). The Ramsey ideal on \( \kappa \) is the collection of all subsets of \( \kappa \) that are not Ramsey. \( A \subseteq \kappa \) is ineffably Ramsey if every regressive function \( c : [A]^{<\omega} \rightarrow \kappa \) has a homogeneous set that is stationary in \( \kappa \) and the ineffably Ramsey ideal on \( \kappa \) is the collection of all subsets of \( \kappa \) that are not ineffably Ramsey.

The same argument as for [5, Theorem 2.10] yields Item (1) of the following. A completely analogous argument then verifies Item (2) below, showing in particular that \( \kappa \) is ineffably Ramsey if and only if it is \( \text{sc}_{\omega}^\kappa \)-Ramsey.

**Proposition 12.3.**

1. If \( \kappa \) is a Ramsey cardinal, then \( I_{\mathcal{R}}^c = \text{Icc}_{\omega}(\kappa) \) is the Ramsey ideal on \( \kappa \).
2. If \( \kappa \) is ineffably Ramsey, then \( I_{\text{ie}}^c = \text{Icc}_{\omega}(\kappa) \) is the ineffably Ramsey ideal on \( \kappa \).

Finally, using results from [8] and [22], we verify several statements from Theorem 1.5 (7) and (8).

**Lemma 12.4.** If \( \kappa \) is a Ramsey cardinal, then \( N_{\mathcal{R}}^c \notin I_{\mathcal{R}}^c \) and \( I_{\text{ie}}^c \notin I_{\mathcal{R}}^c \).

**Proof.** The first statement follows directly from [8, Theorem 4.5]. Since Ramseyness is a \( \Pi_2 \)-property, the argument used in the last part of the proof of Lemma 10.1 also shows that \( I_{\text{ie}}^c \notin I_{\mathcal{R}}^c \). \( \square \)
Lemma 12.5. If $\kappa$ is an ineffably Ramsey cardinal, then $N^\kappa_{\text{IR}} \not\subseteq I^\kappa_{\text{IR}}$ and $I^\kappa_{\text{scR}} \not\subseteq I^\kappa_{\text{R}}$.

Proof. The first statement again follows directly from [8, Theorem 4.5]. For the second statement, we may assume that $\kappa$ is completely ineffable, because otherwise the remarks following Definition 1.3 show that $\kappa \in I^\kappa_{\text{scR}} \setminus I^\kappa_{\text{IR}}$. Then the proof of [22, Theorem 4] shows that, given a $\Sigma^3_1$-statement $\Omega$ that holds in $V_\kappa$, the set of all non-reflection points of $\Omega$ in $\kappa$ is not completely ineffable. Since ineffableRamseyness is $\Pi^1_3$-definable, the results of Section 11 now show that $N^\kappa_{\text{IR}} \not\subseteq I^\kappa_{\text{scR}} \setminus I^\kappa_{\text{R}}$. □

13. $\Delta^\kappa_\omega$-Ramsey Cardinals

In this section we provide the short and easy proof that – perhaps somewhat surprisingly – the notions of $\text{sc}^\kappa_\omega$-Ramsey, $\infty^\omega_\omega$-Ramsey, and $\Delta^\kappa_\omega$-Ramsey cardinals are equivalent. [22] Together with Theorem 9.9 this result shows why $\Delta^\kappa_\omega$-Ramseyness appears three times in Table 2 yielding Theorem 1.2 (2c), and in particular completesthe tables presented in our introductory section.

Proposition 13.1. Let $\kappa$ be a cardinal. Then $\kappa$ is $\text{sc}^\kappa_\omega$-Ramsey if and only if $\kappa$ is $\infty^\omega_\omega$-Ramsey if and only if $\kappa$ is $\Delta^\kappa_\omega$-Ramsey.

Proof. Assume that $\kappa$ is $\text{sc}^\kappa_\omega$-Ramsey, and let $A \subseteq \kappa$. Pick a sufficiently large regular cardinal $\theta$, and let $M_0 \prec H(\theta)$ with $A \in M_0$ be a weak $\kappa$-model. Consider a run of the game $G_{\text{sc}^\theta_\omega}(\kappa)$, in which the Challenger starts by playing $M_0$. As the Challenger has no winning strategy in this game, there is a run of this game which is won by the Judge. Let $M = M_\omega$ and $F = F_\omega$ be the final model and filter produced by this run. This means that $M \prec H(\theta)$ is a weak $\kappa$-model with $A \in M$, and that $F$ is a $\kappa$-amenable, $\kappa$-normal and stationary-complete $M$-ultrafilter. But $\Delta F = \bigcap_{i \in \omega} \Delta F_i$ since each $\Delta F_i \in F$, it follows that $\Delta F$ is stationary-complete. □

We want to close this section by mentioning that $\omega_1$-Ramsey cardinals are limits of $\Delta^\kappa_\omega$-Ramsey cardinals. This (and the slightly stronger statement that we will actually mention below) is shown exactly as in [15, Theorem 5.10], using Corollary 9.8.

Proposition 13.2. $N^\kappa_{\text{IR}} \subseteq I^\kappa_{\text{IT}_\omega}$.

14. Strongly Ramsey and Super Ramsey Cardinals

In this section, we prove several statements about strong and super Ramsey cardinals contained in Theorem 1.5 and 1.9 and 10. We start by using ideals similar to the ones used in the proof of Lemma 9.13 to derive the following result.

Proposition 14.1. If $\kappa$ is a strongly Ramsey cardinal, then $N^\kappa_{\alpha^+}(\kappa) \subseteq I^\kappa_{\alpha^+}(\kappa)$ for all regular $\alpha < \kappa$.

Proof. Pick a $\kappa$-model $M$ and a uniform $M$-ultrafilter $U$ on $\kappa$ that is $M$-normal and $\kappa$-amenable for $M$. Then $\text{Ult}(M, U)$ is well-founded and $H(\kappa^+)^M = H(\kappa^+)^{\text{Ult}(M, U)} \in \text{Ult}(M, U)$. Fix $x \in P(\kappa)^M$. Using the closure properties of $M$ and the fact that $j_C$ is $\kappa$-powerset preserving, we can construct a continuous sequence $\langle X_i \in H(\kappa^+)^M \mid i < \alpha^+ \rangle$ of elementary submodels of $H(\kappa^+)^M$ with $x \in X_0$ and $X_i \subseteq M + \{X_i \cap U \in M_{i+1} \}$ for all $i < \alpha^+$. Set $M_0 = M_{\omega}^+$ and $U_* = U \cap M_\omega \in H(\kappa^+)^M \subseteq \text{Ult}(M, U)$. Our construction then ensures that, in $\text{Ult}(M, U)_*$, we have $x \in M_* < H(\kappa^+)$ is a weak $\kappa$-model closed under $\alpha$-sequences and $U_*$ is a uniform $M_\omega$-ultrafilter that is $M_\omega$-normal and $\kappa$-amenable for $M_*$. These computations show that $\kappa$ is a $T^\alpha_{\omega^+}$-Ramsey cardinal in $\text{Ult}(M, U)$. □

The next result yields several statements from Theorem 1.5 and 9.

Lemma 14.2. If $\kappa$ is a strongly Ramsey cardinal, then $I^\kappa_{\text{R}} \cup \{N^\kappa_{\text{IR}}\} \subseteq I^\kappa_{\text{stR}}$, $N^\kappa_{\text{stR}} \not\subseteq I^\kappa_{\text{stR}}$ and $I^\kappa_{\text{cR}} \not\subseteq I^\kappa_{\text{stR}}$.

Proof. By definition, we have $I^\kappa_{\text{R}} = \text{icc}_{\omega^+}(\kappa) \subseteq \text{icc}_{\omega^+}(\kappa) = I^\kappa_{\text{stR}}$. Corollary 9.16 (1) shows that $N^\kappa_{\text{stR}} = N^\kappa_{\text{IR}} \not\subseteq I^\kappa_{\text{cR}} = I^\kappa_{\text{stR}}$. Next, since $T^\alpha_{\omega^+}$-Ramsey cardinal $\kappa$ is ineffably Ramsey, Proposition 14.1 shows that $N^\kappa_{\text{IR}} \not\subseteq N^\kappa_{\omega^+}(\kappa) \subseteq I^\kappa_{\text{IR}}$. Finally, it is easy to see that a cardinal $\kappa$ is strongly Ramsey if and only if for all $x \subseteq \kappa$ and all $P \subseteq P(\kappa)$, either $P$ is not equal to the set of all bounded subsets of $\kappa$ or there exists a transitive weak $\kappa$-model $M$, a surjection $s : \kappa \to M$ and a uniform $M$-ultrafilter $U$ such that $s[P] \subseteq M$ and $U$ is $\kappa$-amenable for $M$ and $M$-normal. Since this equivalence shows that strong Ramseyess is a $\Pi^1_3$-property, the argument used in the last part of the proof of Lemma 10.1 can be modified to show that $I^\kappa_{\text{cR}} \not\subseteq I^\kappa_{\text{R}}$. □

We now study the ideal induced by super Ramseyess and its relation to the ineffably Ramsey ideal.

Proposition 14.3. If $\kappa$ is a $T^\alpha_{\omega^+}$-Ramsey cardinal, then $\text{isc}_{\alpha^+}(\kappa) \subseteq I^\kappa_{\alpha^+}(\kappa)$.

\footnote{In particular, this contrasts the hierarchy of large cardinals treated in [26, Section 3].}
Proof. Let \( M \prec H(\kappa^+) \) be a weak \( \kappa \)-model closed under countable sequences, let \( U \) be a uniform \( M \)-ultrafilter that is \( \kappa \)-amenable for \( M \) and \( M \)-normal, let \( \langle X_n \mid n < \omega \rangle \) be a sequence of elements of \( U \) and set \( X = \bigcap_{n<\omega} X_n \). Then \( X \in U \), \( X \) is stationary in \( M \), and elementarity implies that \( X \) is stationary in \( V \). This shows that that \( SC(M, U) \) holds.

The next lemma proves several statements from Theorem 1.5[10].

Lemma 14.4. If \( \kappa \) is a super Ramsey cardinal, then \( I^\kappa_{stR} \cup I^\kappa_{srR} \cup \{ N^\kappa_{stR} \} \subseteq I^\kappa_{srR} \) and \( N^\kappa_{srR} \not\subseteq I^\kappa_{srR} \).

Proof. By definition, we have \( I^\kappa_{srR} = \text{IT}^\kappa_0(\kappa) \subseteq \text{IT}^\kappa_1(\kappa) = I^\kappa_{srR} \). Next, Proposition 14.3 allows us to show that \( I^\kappa_{srR} = I_{sc}^\kappa(\kappa) \subseteq \text{IT}^\kappa_1(\kappa) \subseteq \text{IT}^\kappa_0(\kappa) = I^\kappa_{srR} \). Moreover, Lemma 9.13 directly shows that \( N^\kappa_{stR} = \text{NT}^\kappa_0(\kappa) \subseteq \text{IT}^\kappa_1(\kappa) = I^\kappa_{srR} \). Finally, Corollary 9.16[1] shows that \( N^\kappa_{srR} = \text{NT}^\kappa_1(\kappa) \not\subseteq \text{IT}^\kappa_0(\kappa) = I^\kappa_{srR} \).

15. Locally measurable cardinals

In this section, we prove a few results about locally measurable cardinals which allow us to compare these cardinals and their ideals to the ones studied above, yielding several statements from Theorem 1.5[1].

Proposition 15.1. If \( \kappa \) is locally measurable, then \( N^\kappa_{ms} \not\subseteq I^\kappa_{ms} \) and \( I^\kappa_{ic} \not\subseteq I^\kappa_{ms} \).

Proof. As noted in Section 9 if we set \( \Psi \equiv \Psi_{ms} \), then local measurability coincides with \( \Psi_{Rs} \)-Ramseyness and hence \( N^\kappa_{ms} = N\Psi_{Rs}(\kappa) \) as well as \( I^\kappa_{ms} = I\Psi_{Rs}(\kappa) \). Since \( \Psi \) satisfies the assumptions of Lemma 9.15, we can use the lemma to conclude that \( N\Psi_{Rs}(\kappa) \not\subseteq I\Psi_{Rs}(\kappa) \). Next, since local measurability is a \( \Pi^1_1 \)-property, we can modify the proof of Lemma 10.1 to show that \( I^\kappa_{ic} \not\subseteq I^\kappa_{ms} \).

Lemma 15.2. If \( \kappa \) is a locally measurable cardinal, then \( \kappa \) is strongly Ramsey and \( I^\kappa_{stR} \subseteq I^\kappa_{ms} \).

Proof. Pick \( A \in \mathcal{P}(\kappa) \setminus I_{ms}^\kappa \) and some \( x \subseteq \kappa \). Then, there is a weak \( \kappa \)-model \( M \) and an \( M \)-normal ultrafilter \( U \) such that \( x, H(\kappa), U \in M \) and \( A \in U \). But then, \( U \in M = ZFC^- \) implies that \( \mathcal{P}(\kappa)^M \subseteq M \), and hence \( H(\kappa)^M \subseteq M \). Then, \( M \) contains a continuous sequence \( \langle M_i \mid i < \kappa \rangle \) of elementary submodels of \( H(\kappa)^M \) with \( \kappa, x, A \in M_0 \) and \( (\text{\<\<}^\kappa M_i)^M \cup \{ M_i \cap U \} \subseteq M_{i+1} \) for all \( i < \kappa \). By construction, the model \( M_\kappa \) has cardinality \( \kappa \) in \( M \) and, since \( H(\kappa) \subseteq M \), we know that \( \text{\<\<}^\kappa M_\kappa \subseteq M \). But this implies that \( M_\kappa \) is a \( \kappa \)-model. Since our construction also ensures that \( U \) is \( M_\kappa \)-normal and \( \kappa \)-amenable for \( M_\kappa \), we can conclude that \( A \not\in I^\kappa_{stR} \).

The following result shows that locally measurable cardinals are consistency-wise strictly above all the large cardinals mentioned in Table 2 noting that \( \Delta^\kappa_{Rs} \)-Ramsey cardinals are implication-wise stronger than all these large cardinal notions.

Theorem 15.3. If \( \kappa \) is locally measurable, then \( N\Delta_\kappa^\kappa \in I^\kappa_{ms} \).

Proof. Assume that \( \kappa \) is locally measurable. Let \( M \) be a transitive weak \( \kappa \)-model with \( V_\kappa \in M \), let \( U \) be an \( M \)-normal \( M \)-ultrafilter with \( U \in M \), let \( \theta > \kappa \) be a regular cardinal in \( \text{Ult}(M, U) \) and let \( x \in H(\theta)^{\text{Ult}(M, U)} \). Using the fact that \( U \) is a normal ultrafilter on \( \kappa \) in \( M \) and \( \text{Ult}(M, U) \) can be identified with the ultrapower \( \text{Ult}(M, U)^M \) of \( M \) by \( U \) constructed in \( M \), we know that \( \text{Ult}(M, U) \) is closed under \( \kappa \)-sequences in \( M \), and we can find an increasing continuous sequence \( \langle M_i \mid i < \kappa \rangle \) of weak \( \kappa \)-models in \( \text{Ult}(M, U) \) with the properties that \( x \in M_0 \), that \( M_i \prec H(\theta)^{\text{Ult}(M, U)} \), and that \( M_{\kappa}^\kappa \cup \{ M_i \cap U \} \subseteq M_{i+1} \) for all \( i < \kappa \). Then, \( x \in M_\kappa \prec H(\theta)^{\text{Ult}(M, U)} \), and our construction ensures that \( U_\kappa = U \cap M_\kappa \in \text{Ult}(M, U) \) is a \( \kappa \)-amenable \( M_\kappa \)-ultrafilter in \( \text{Ult}(M, U) \). Moreover, since \( U \) is a normal ultrafilter in \( M \), the filter \( U_\kappa \) is normal in \( \text{Ult}(M, U) \). These computations show that \( \kappa \) is \( \Delta^\kappa_{Rs} \)-Ramsey in \( \text{Ult}(M, U) \), and therefore \( N\Delta_\kappa^\kappa \not\subseteq U \). This allows us to conclude that the set \( N\Delta_\kappa^\kappa \) is contained in \( I^\kappa_{ms} \).

Note that \( \Delta^\kappa_{Rs} \)-Ramsey cardinals are in particular super Ramsey, and therefore the above theorem provides a proof for the statement \( N_{srR} \in I^\kappa_{ms} \) from Theorem 1.5[1].

16. The measurable ideal

We close our paper with the investigation of the ideal induced by the property \( \Psi_{ms}(M, U) \) w.r.t. Scheme A and Scheme C and its relations with the ideals studied above. We start by verifying Theorem 1.5[10] and Theorem 1.5[12], and then make some further observations concerning this ideal and its induced partial order \( \mathcal{P}(\kappa)/I^\kappa_{ms} \).

Lemma 16.1. If \( \kappa \) is a measurable cardinal, then \( I^{\leq \kappa}_{ms} = I^{\kappa}_{ms} \) and this ideal is equal to the complement the union of all normal ultrafilters on \( \kappa \).
Proof. First, if \( A \subseteq \kappa \) with \( A \notin \Gamma_{\kappa}^V \cap \Gamma_{\kappa}^\kappa, \) then there is a regular cardinal \( \theta > (2^\kappa)^+ \), an infinite cardinal \( \lambda \leq \kappa \), a weak \( (\lambda, \kappa) \)-model \( M \prec H(\theta) \) with \( A \in M \) and an \( M \)-ultrafilter \( U \) on \( A \) with \( A \in U \) and \( \Psi_{\kappa}(M, U) \).

Then \( U \) is \( M \)-normal and \( U = F \cap M \) for some \( F \) in \( M \). Therefore elementarity implies that \( F \) is a normal ultrafilter on \( A \) with \( A \). In the other direction, assume that \( \lambda \leq \kappa \) is an infinite cardinal, \( F \) is a normal ultrafilter on \( \kappa \) and \( A \in F \). If \( \theta > (2^\kappa)^+ \) is regular and \( x \in H(\theta) \), then we can pick a weak \( (\lambda, \kappa) \)-model \( M \prec H(\theta) \) with \( x, A, F \in M \).

In this situation, it is easy to see that \( \Psi_{\kappa}(M, F \cap M) \) holds and hence \( A \notin \Gamma_{\kappa}^V \cap \Gamma_{\kappa}^{\kappa} \).

\[ \square \]

**Lemma 16.2.** If \( \kappa \) is measurable, then \( 1\Delta_\kappa^V(\kappa) \cap \Gamma_{\kappa}^\kappa \cup \{ \Gamma_{\kappa}^V \} \subseteq \Gamma_{\kappa}^\kappa \) and \( \Gamma_{\kappa}^\kappa \notin \Gamma_{\kappa}^{\kappa} \).

**Proof.** Using Lemma 16.1, it is easy to see that \( 1\Delta_\kappa^V(\kappa) \cup \Gamma_{\kappa}^\kappa \subseteq \Gamma_{\kappa}^\kappa \). Moreover, if \( U \) is a normal ultrafilter on \( \kappa \), then \( V \) and \( \text{Ult}(V, U) \) contain the same weak \( \kappa \)-models and hence \( \kappa \) is locally measurable in \( \text{Ult}(V, U) \).

By Lemma 16.1, this shows that \( \Gamma_{\kappa}^\kappa \subseteq \Gamma_{\kappa}^\kappa \). Finally, assume that there is a measurable cardinal \( \kappa \) with \( \Gamma_{\kappa}^\kappa \in \Gamma_{\kappa}^\kappa \), and let \( \kappa \) be minimal with this property. By Lemma 16.1, we can pick a normal ultrafilter \( U \) on \( \kappa \) with \( \kappa \notin \Gamma_{\kappa}^\kappa \). Set \( M = \text{Ult}(V, U) \). Then \( \kappa \) is measurable in \( M \). Moreover, since \( H(\kappa^+) \subseteq M \), we have \( \Gamma_{\kappa}^\kappa = (\Gamma_{\kappa}^\kappa)^M \), and therefore, the minimality of \( \kappa \) implies that \( \Gamma_{\kappa} \notin \Gamma_{\kappa}^{\kappa} \). Again, by Lemma 16.1, this yields a normal ultrafilter \( F \) on \( \kappa \) in \( M \) with \( \Gamma_{\kappa} \notin \Gamma_{\kappa}^{\kappa} \).

Therefore, elementarity implies that \( \kappa \) is atomic. In contrast, Theorem 1.4.1 directly implies that for every inaccessible cardinal \( \kappa \), the corresponding partial order \( P(\kappa)/\Gamma_{\kappa}^\kappa \) is not atomic. The following results will allow us to show that, for many of the large cardinal properties characterized in this paper that are weaker than measurability, their corresponding ideals do not induce atomic quotient partial orders.

**Lemma 16.3.** Let \( I \) be a normal ideal on an uncountable regular cardinal \( \kappa \).

1. If \( A_1 \) is an atom in the partial order \( P(\kappa)/I \), then \( U_A = \{ B \subseteq \kappa \mid A \setminus B \in I \} \) is a normal ultrafilter on \( \kappa \) containing \( A \), and \( \Gamma_I \cap \text{P}(A) \) is a normal ultrafilter on \( \kappa \) containing \( A \), with \( \Gamma^+ \cap \text{P}(A) = U_A \cap \text{P}(A) \).

2. If the partial order \( P(\kappa)/I \) is atomic, then \( \kappa \) is a measurable cardinal with \( \Gamma_{\kappa}^\kappa \subseteq I \), and the ideal \( I \) is precipitous.

**Proof.** Assume that there is \( B \in P(\kappa) \setminus U_A \) with \( \kappa \notin B \). Then \( A \cap B, A \setminus B \in I^+ \), and this implies that \( A_1 \cap B_1 \) and \( A_1 \setminus B_1 \) are incompatible conditions in \( P(\kappa)/I \) below \( A_1 \), a contradiction. Since \( I \) is a normal ideal on \( \kappa \), this shows that \( U_A \) is a normal ultrafilter on \( \kappa \). Moreover, if \( B \in U_A \), then \( A \setminus B \in I \) and \( A \in I^+ \) implies that \( B \in I^+ \). Finally, if \( B \in P(A) \setminus U_A \), then the above computations show that \( A \setminus B \in U_A \) and hence, \( B \in I \).

By Lemma 16.1, the existence of an atom in \( P(\kappa)/I \) implies the measurability of \( \kappa \). Next, if \( A \in I^+ \), then our assumption yields a \( B \in I^+ \cap P(A) \) with the property that \( B \) is an atom in \( P(\kappa)/I \) and, by Lemma 16.1, the filter \( U_B \) witnesses that \( A \) is an element of \( (\Gamma_{\kappa}^\kappa)^+ \). This shows that \( I^+ \subseteq \Gamma_{\kappa}^\kappa \) and therefore \( \Gamma_{\kappa}^\kappa \subseteq \Gamma_I \).

Finally, let \( \sigma \) be a strategy for the Player \( \text{Nonempty} \) in the precipitous game \( G_1 \) (see Lemma 22.21), with the property that whenever Player \( \text{Empty} \) plays \( A \in I^+ \) for their first move of the game, then \( \text{Nonempty} \) replies by playing \( B \in I^+ \cap P(A) \) so that \( B_1 \) is an atom in \( P(\kappa)/I \). Now, if \( A_n \mid n < \omega \) is a run of \( G_1 \) in which \( \text{Nonempty} \) played according to \( \sigma \), then the above arguments show that \( U = U_A \) is a normal ultrafilter on \( \kappa \) with \( A_n \notin \mathcal{U} \) for all \( n < \omega \). Hence, \( \emptyset \notin \bigcap_{n < \omega} A_n \in U \). This shows that \( \sigma \) is a winning strategy for \( \text{Nonempty} \) in \( G_1 \), and therefore \( I \) is precipitous.

Note that in combination with Lemma 16.3, the above lemma allows us to derive the third statements of all items in Theorem 1.4 from the previously proven parts of the theorem.

In the remainder of this section, we consider the question whether the partial order induced by the measurable ideal has to be atomic. The following lemma gives a useful criterion for the atomicity of these partial orders. Note that the assumption of the lemma is satisfied if the Mitchell order on the collection of normal ultrafilters on the given measurable cardinal is linear. As noted in [11], this statement holds in all known canonical inner models for large cardinal hypotheses, and is expected to also be true in potential canonical inner models for supercompact cardinals.

**Lemma 16.4.** If \( \kappa \) is a measurable cardinal with the property that any set of pairwise incomparable elements in the Mitchell ordering at \( \kappa \) has size at most \( \kappa \), then the partial order \( P(\kappa)/\Gamma_{\kappa}^\kappa \) is atomic.
Proof. Let $I = I_{<\kappa}^{\text{ms}}$, and fix $A \in \mathbb{I}^+$. Let $F$ denote the collection of all normal ultrafilters on $\kappa$ that contain $A$, and let $F_0$ denote the set of all elements of $F$ that are minimal in $F$ with respect to the Mitchell ordering. Note that any two elements of $F_0$ are incomparable, hence $F_0$ has size at most $\kappa$ by our assumption. Lemma 16.1 implies that $F_0 \neq \emptyset$. We may thus pick some $U \in F_0$.

Claim. There exists $B \in \mathcal{P}(A) \cap U$ with the property that $U$ is the unique element of $F_0$ that contains $B$.

Proof of the Claim. We may assume that $F_0 \neq \{U\}$. Let $\alpha : \kappa \rightarrow F_0 \setminus \{U\}$ be a surjection. Given $\alpha < \kappa$, fix $B_\alpha \in U \setminus \{U\}$. Define $B = A \setminus \Delta_{\alpha < \kappa} B_\alpha \in \mathcal{P}(A) \cap U$. Then, $B \notin U(\alpha)$ for any $\alpha < \kappa$, for otherwise we would have $B \cap (\alpha, \kappa) \in \mathcal{P}(A) \cap U(\alpha)$, and hence $B_\alpha \in U(\alpha)$ for some $\alpha < \kappa$, contradicting our assumption on $B_\alpha$. \hfill $\square$

Claim. There exists $C \in \mathcal{P}(B) \cap U$ with the property that $U$ is the unique normal ultrafilter on $\kappa$ that contains $C$.

Proof of the Claim. First, assume that $U$ has Mitchell rank 0. Then

$$C = \{\alpha \in B \mid \alpha \text{ is not measurable} \} \in \mathcal{P}(B) \cap U.$$ 

Let $U'$ be a normal ultrafilter on $\kappa$ that contains $C$. Then $A, B \in U'$ and $U'$ has Mitchell rank 0. This implies $U' \in F_0$ and, by the previous claim, we can conclude that $U = U'$.

Now, assume that $U$ has Mitchell rank greater than 0. Define

$$C = \{\alpha \in B \mid \alpha \text{ is measurable and } B \cap \alpha \notin F \text{ for every normal ultrafilter } F \text{ on } \alpha\}.$$ 

Then $U \in F_0$ implies that $C \in \mathcal{P}(B) \cap U$. Let $U'$ be a normal ultrafilter on $\kappa$ that contains $C$. Then $A, B \in U'$ and $U' \in F_0$. By the above claim, we know that $U = U'$. \hfill $\square$

Claim. The condition $[C]_I$ is an atom in the partial order $\mathcal{P}(\kappa)/I$.

Proof of the Claim. Pick $D, E \in \mathbb{I}^+$ with $[D|_I, E|_I] \leq \mathcal{P}(\kappa)/I [C]_I$. By Theorem 1.4(9), we can find normal ultrafilters $U_0$ and $U_1$ on $\kappa$ with $D \in U_0$ and $E \in U_1$. Then $C \in U_0 \cap U_1$, $U = U_0 = U_1$, $D \cap E \in U \subseteq \mathbb{I}^+$, and $[D \cap E|_I] \leq \mathcal{P}(\kappa)/I [D|_I, E|_I]$. \hfill $\square$

This completes the proof of the lemma.

In contrast to the situation studied in the above lemma, the next result shows that it is possible to combine ideas from a classical construction of Kunen and Paris from [24] with results of Hamkins from [12] to obtain a measurable cardinal $\kappa$ with the property that the ideal $I_{<\kappa}^{\text{ms}}$ induces an atomless partial order.

Theorem 16.5. Let $\kappa$ be a measurable cardinal. Then, in a generic extension of the ground model $V$, $\kappa$ is measurable and the partial order $\mathcal{P}(\kappa)/I_{<\kappa}^{\text{ms}}$ is atomless. Moreover, if the ideal $I_{<\kappa}^{\text{ms}}$ is precipitous in $V$, then the ideal $I_{<\kappa}^{\text{ms}}$ is precipitous in the given generic extension.

Proof. Let $F_0$ denote the collection of all normal ultrafilters on $\kappa$ in $V$, let $c$ be $\text{Add}(\omega, 1)$-generic over $V$, let $\mathcal{F}$ denote the collection of all normal ultrafilters on $\kappa$ in $V[c]$ and set $\lambda = \kappa^+ = (\kappa^+)^{V[c]}$. By the Lévy–Solovay Theorem (see [21] Proposition 10.13) and results of Hamkins (see [12] Corollary 8 and Lemma 13), there is a bijection $b : \mathcal{F} \rightarrow F_0$ with $b(U) = U \cap \mathcal{P}(\kappa)^{V[c]}$ and $U = \{A \in \mathcal{P}(\kappa)^{V[c]} \mid \exists B \in b(U) \ A \supseteq B\}$ for all $U \in \mathcal{F}$. Given $U \in F$, let $j_U : V[c] \rightarrow M_U = \text{Ult}(V[c], U)$ denote the corresponding ultrapower embedding. Work in $V[c]$, and let $\bar{\mathbb{F}}$ denote the Easton-support product of all partial orders of the form $\text{Add}(\mathfrak{r}^{++}, 1)$ for some infinite cardinal $\nu < \kappa$. Then $\bar{\mathbb{F}}$ has cardinality $\kappa$ and satisfies the $\kappa$-chain condition. Let $f$ be the function with domain $\kappa$ and $f(\alpha) = \bar{\mathbb{F}} \upharpoonright (\alpha, \kappa)$ for all $\alpha < \kappa$. Given $U \in F$, set $R_U = [f]_U$ and define $\bar{R}_U$ to be the $<\lambda$-support product of many copies of $R_U$. Then $R_U$ is a $<\lambda$-closed partial order in $M_U$ and there is a canonical isomorphism between $j_U(\bar{\mathbb{F}})$ and $\bar{\mathbb{F}} \times R_U$ in $M_U$. Since we have $\kappa^+ \leq M_U \subseteq M_U$, the partial orders $R_U$ and $\bar{R}_U$ are also $<\lambda$-closed in $V[c]$. Finally, let $\bar{\mathbb{S}}$ denote the $<\lambda$-support product of all partial orders of the form $\bar{\mathbb{F}} \times R_U$ with $U \in F$.

Let $G \times H$ be $(\bar{\mathbb{F}} \times \bar{\mathbb{S}})$-generic over $V[c]$. Given $U \in F$ and $\gamma < \kappa$, we let $H_{U, \gamma}$ denote the filter induced by $H$ on the $\gamma$-th factor of $\bar{\mathbb{S}}_{U, \gamma}$, and we let $\bar{j}_{U, \gamma} : V[c, G] \rightarrow M_U[G, H_{U, \gamma}]$ denote the corresponding canonical lifting of $j_U$ (see [4] Proposition 9.1). Finally, we set $1 = (I_{<\kappa}^{\text{ms}})^{V[c, G, H]}$.

Claim. $\mathcal{P}(\kappa)^{V[c, G, H]} \subseteq V[c, G]$.

Proof of the Claim. Fix $A \in \mathcal{P}(\kappa)^{V[c, G, H]}$. Then, there is a $\bar{\mathbb{F}}$-nice name $\dot{A}$ in $V[c, H]$ with $A = \dot{A}^G$. Since $\bar{\mathbb{S}}$ is $<\lambda$-closed in $V[c]$, the above remarks imply that $\bar{\mathbb{F}}$ satisfies the $\kappa$-chain condition and has cardinality $\kappa$ in $V[c, H]$. This shows that $\dot{A}$ is an element of $V[c]$ and we can conclude that $A = \dot{A}^G \in V[c, G]$. \hfill $\square$
The above claim directly implies that if $U \in \mathcal{F}$ and $\gamma < \lambda$, then
\[ U_\gamma = \{ A \in \mathcal{P}(\kappa)^{V[c,G]} \mid \kappa \in j_{U_\gamma}(A) \} \]
is a normal ultrafilter on $\kappa$ in $V[c,G,H]$. Moreover, it is easy to see that, given a $\bar{P}$-name $\dot{A} \in V[c]$ for a subset of $\kappa$, $\dot{A}^G$ is an element of $U_\gamma$ if and only if there is a condition $\bar{p} \in G$, an element $E$ of $U$ and a function $g \in V[c]$ with domain $\kappa$ and $[g]_V \in H_{U,\gamma}$ with the property that $\text{supp}(\bar{p}) \subseteq \alpha$, $g(\alpha) \in \bar{P}$ \((\alpha, \kappa)\) and $\bar{p} \cup g(\alpha) \upharpoonright \bar{p} \vdash \dot{\alpha} \in \dot{A}^\kappa$ for all $\alpha \in E$.

Claim. If $W$ is a normal ultrafilter on $\kappa$ in $V[c,G,H]$, then $W \cap V[c] \in \mathcal{F}$.\hfill \(\square\)

Proof of the Claim. Since the partial order $\bar{P} \times \bar{S}$ is $\sigma$-closed in $V[c]$, the results of [12] mentioned above yield an ultrafilter $U \in \mathcal{F}$ with $W \cap V = b(U) = U \cap V$. Since $U = \{ A \in \mathcal{P}(\kappa)^{V[c]} \mid \exists B \in W \cap V \ A \subseteq B \} \in \mathcal{F}$ is an ultrafilter in $V[c]$, we can conclude that $U = W \cap V[c]$.

Claim. $I = \mathcal{P}(\kappa)^{V[c,G]} \setminus \bigcup \{ U_\gamma \mid U \in \mathcal{F}, \gamma < \lambda \} = \{ A \in \mathcal{P}(\kappa)^{V[c,G]} \mid \exists B \in (I_{ms}^c)^V \ A \subseteq B \}$.

Proof of the Claim. Fix a $\bar{P}$-name $\dot{A}$ for a subset of $\kappa$ in $V[c]$, and let $O$ denote the set of all $\bar{p} \in \bar{P}$ with
\[ D_{\bar{p}} = \{ \alpha < \kappa \mid \bar{p} \upharpoonright \alpha \vdash \dot{\alpha} \notin \dot{A}^\kappa \} \subseteq \bigcap \mathcal{F}. \]

First, assume that there is a $\bar{p} \in G \cap O$. Then $\dot{A}^G \cap D_{\bar{p}} = \emptyset$ and, if $W$ is a normal ultrafilter on $\kappa$ in $V[c,G,H]$, then $D_{\bar{p}} \in W \cap V[c] \in \mathcal{F}$ and hence $\dot{A}^G \notin W$. By Theorem 1.4 [9], this shows that $\dot{A}^G \notin I$. In particular, we have $\dot{A}^G \notin U_\gamma$ for all $U \in \mathcal{F}$ and $\gamma < \lambda$. Finally, these arguments also directly show that $\dot{A}^G \subseteq \kappa \setminus D_{\bar{p}} \in (I_{ms}^c)^V$.

Now, assume that $G \cap O = \emptyset$. Since $O$ is an open subset of $\bar{P}$ in $V[c]$, there is $\bar{p}_0 \in G$ with the property that no condition below $\bar{p}_0$ is an element of $O$. Fix some condition $(\bar{p}_1, \bar{s}_1)$ below $(\bar{p}_0, \bar{s}_0)$ in $\bar{P} \times \bar{S}$. Then, there is $U \in \mathcal{F}$ with $E = \{ \alpha < \kappa \mid \bar{p}_1 \upharpoonright \alpha \vdash \dot{\alpha} \notin \dot{A}^\kappa \} \subseteq U$. This allows us to find a sequence $(\bar{q}_0 \in \bar{P} \mid \alpha < \kappa)$ in $V[c]$ with $\bar{q}_0 \upharpoonright \bar{p}_1 \bar{p}_1 \upharpoonright \alpha \vdash \dot{\alpha} \notin \dot{A}^\kappa$. Finally, if $B \in (I_{ms}^c)^V$, then $\kappa \setminus B \in U \subseteq U_\gamma$ and hence $\dot{A}^G \notin B$.\hfill \(\square\)

Claim. In $V[c,G,H]$, the partial order $\mathcal{P}(\kappa)/I$ is atomless.

Proof of the Claim. Pick a $\bar{P}$-name $\dot{A}$ in $V[c]$ with $\dot{A}^G \notin I$. By the previous claim, there is $U \in \mathcal{F}$ and $\gamma < \lambda$ with $A^G \subseteq U_\gamma$. In this situation, earlier remarks show that we can find $(\bar{p}_0, \bar{s}_0) \in G \times H$, $E \in U$ and a function $g \in V[c]$ with domain $\kappa$ and $[g]_V = \bar{s}_0(U)(\gamma)$ such that $\text{supp}(\bar{p}_0) \subseteq \alpha$, $g(\alpha) \in \bar{P} \mid (\alpha, \kappa)$ and $\bar{p}_1 \cup g(\alpha) \upharpoonright \bar{p}_1 \vdash \dot{\alpha} \in \dot{A}^\kappa$ for all $\alpha \in E$. Fix a condition $(\bar{p}_1, \bar{s}_1)$ below $(\bar{p}_0, \bar{s}_0)$ in $\bar{P} \times \bar{S}$ and $\delta \in \lambda^U \supseteq \bar{s}_0(U)$. Then, we can find $F \in U$ and a function $h \in V[c]$ with domain $\kappa$ and $[h]_V = \bar{s}_1(U)(\gamma)$ such that $\text{supp}(\bar{p}_1) \subseteq \alpha$ and $g(\alpha) \in \bar{P} \mid (\alpha, \kappa)$ holds for all $\alpha \in F$. Since partial orders of the form $\bar{P} \mid (\alpha, \kappa)$ with $\alpha < \kappa$ are atomless, we can find functions $h_\gamma, h_\delta \in V[c]$ with domain $\kappa$ with the property that $h_\gamma(\alpha)$ and $h_\delta(\alpha)$ are incompatible conditions below $h(\alpha)$ in $\bar{P} \mid (\alpha, \kappa)$ for all $\alpha \in F$. Then, there is a $\bar{P}$-name $\dot{B} \in V[c]$ with the property that whenever $K$ is $\bar{P}$-generic over $\dot{A}^c$, then $\dot{B}^K = \{ \alpha \in A \mid h_\gamma(\alpha) \in K \}$. Then $\bar{p}_1 \cup h_\gamma(\alpha) \upharpoonright \bar{p}_1 \vdash \dot{\alpha} \in \dot{A}^\kappa$ and $\bar{p}_1 \cup h_\delta(\alpha) \upharpoonright \bar{p}_1 \vdash \dot{\alpha} \notin \dot{A}^\kappa$ for all $\alpha \in E \cap F$. Moreover, there is a condition $(\bar{p}, \bar{s})$ below $(\bar{p}_1, \bar{s}_1)$ in $\bar{P} \times \bar{S}$ with $\bar{s}(U)(\gamma) \subseteq [h]_V$ and $\bar{s}(U)(\delta) \subseteq [h]_V$. A genericity argument now shows that there is $B \in U_\gamma \cap \mathcal{P}(\dot{A}^G)$ with the property that $\dot{A}^G \cap B \in U_\delta$ for some $\delta < \lambda$. In particular, the condition $[\dot{A}^G]_I$ is not an atom in the partial order $\mathcal{P}(\kappa)/I$ in $V[c,G,H]$.\hfill \(\square\)

Claim. If the ideal $I_{ms}^c$ is precipitous in $V$, then the ideal $I$ is precipitous in $V[c,G,H]$.

Proof of the Claim. A result of Kakuda (see [20] Theorem 1) shows that the set
\[ \{ A \in \mathcal{P}(\kappa)^{V[c]} \mid \exists B \in (I_{ms}^c)^V \ A \subseteq B \} \]
is a precipitous ideal on $\kappa$ in $V[c]$. As observed above, this ideal is equal to $(I_{ms}^c)^V$ for $c$ in $V[c,H]$. Since the partial order $\bar{S}$ is $\sigma$-closed in $V[c]$, this shows that $(I_{ms}^c)^V$ is also a precipitous ideal on $\kappa$ in $V[c,H]$. Since the
For many large cardinal properties corresponding to some property of models and filters, we either were able to show or it was known that the collection of all smaller cardinals without the given property is not contained in the induced ideal. Since each of these arguments has its individual proof that relies on the specific large cardinal property, it is natural to ask whether this conclusion holds true in general.

**Question 17.1.** Assume that Scheme \( [2] \) (respectively, Scheme \( [3] \) or Scheme \( [4] \)) holds true for some large cardinal property \( \Phi(\kappa) \) and some property \( \Psi(M,U) \) of models and filters. Does \( N^\kappa_\Phi \not\in I^\kappa_\Psi \) (respectively, \( N^\kappa_\Phi \not\in I^\kappa_\Psi \)) hold for every cardinal \( \kappa \) with \( \Phi(\kappa) \)?

For some large cardinal properties that can be characterized through Scheme \( [4] \), we were not able to show that \( N^\kappa_\Phi \not\in I^\kappa_\Psi \) always holds. The difficulties in these arguments are mostly caused by the fact that elementary submodels of some large \( H(\theta)^{3^\theta} \) cannot be transferred between some weak \( \kappa \)-model \( M \) and its ultrapowers. In particular, the following statements are left open:

**Question 17.2.**
(1) Does \( N^\omega \not\in I^{\omega R} \) hold for every \( \omega \)-Ramsey cardinal \( \kappa \)?
(2) Does \( N^\kappa_\omega \not\in I^{\omega R} \) hold for every \( \Delta^\omega_0 \)-Ramsey cardinal \( \kappa \)?

For Ramsey-like cardinals characterized through the validity of Scheme \( [2] \), Lemma 9.13 is our main tool to show that \( N^\kappa_\Psi \not\in I^\kappa_\Psi \). Since we considered several properties of models and ultrafilters that are not absolute between \( V \) and the corresponding ultrapowers, we naturally arrive at the following question:

**Question 17.3.** Let \( \kappa \) be a \( \Psi^\Phi_\omega \)-Ramsey cardinal with \( \alpha \leq \kappa, \theta \in \{ \kappa, \kappa^+ \} \) and \( \Psi \in \{ \omega, \infty, \Delta \} \). Is it true that \( N\Psi^\Phi_\omega(\kappa) \not\in I\Psi^\Phi_\omega(\kappa) \)?

The individual results of our paper strongly support the idea that most natural large cardinal notions below measurability canonically induce large cardinal ideals in a way that the relationship between these ideals reflects the relationship between the corresponding large cardinal notions, as is exemplified by the results listed in Theorem 15. Therefore, it is natural to ask whether this can be done more generally:

**Question 17.4.** Given large cardinal properties \( \Phi_0 \) and \( \Phi_1 \) and properties \( \Psi_0 \) and \( \Psi_1 \) of models and filters that are each connected through one of our characterization schemes in a canonical way, . . .

- . . . is it true that \( \Phi_0(\kappa) \) provably implies \( \Phi_1(\kappa) \) for every cardinal \( \kappa \) if and only if it can be proven that for every cardinal \( \kappa \) satisfying \( \Phi_0(\kappa) \), the ideal on \( \kappa \) induced by \( \Phi_1 \) and \( \Psi_1 \) is contained in the ideal on \( \kappa \) induced by \( \Phi_0 \) and \( \Psi_0 \)?
- . . . is it true that \( \Phi_0 \) has strictly larger consistency strength than \( \Phi_1 \) if and only if it can be proven that for every cardinal \( \kappa \) satisfying \( \Phi_0(\kappa) \), the set \( N_{\Phi_1} \) is an element of the ideal on \( \kappa \) induced by \( \Phi_0 \) and \( \Psi_0 \)?

In this paper, we introduced several new large cardinal concepts, whose relationships to each other are only partially established. For example, every \( \Delta^\omega_0 \)-Ramsey cardinal is trivially \( cc^\omega_\Delta \)-Ramsey. However, we do not know if this implication can be reversed, and, in the light of Proposition 14.1, we ask the following:

**Question 17.5.** Are \( cc^\omega_\Delta \)-Ramseyness and \( \Delta^\omega_0 \)-Ramseyness distinct large cardinal notions? Are their consistency strengths distinct?

The following related question is also open:

**Question 17.6.** Are ineffable Ramseyness, \( \infty^\omega \)-Ramseyness, and \( \Delta^\omega_0 \)-Ramseyness distinct large cardinal notions? Are their consistency strengths distinct? What is their relationship with \( cc^\omega_\Delta \)-Ramsey cardinals?

The only open question concerning possible direct implications between large cardinal concepts from the set-theoretic literature that appear in our paper is the following, which we already mention in the introduction to our paper. The corresponding question about ideals is open as well.

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23Note that, in the case of weak compactness, the results of Sections 6 and 7 already show that these questions can have negative answers by showing that the various characterizations of weak compactness induce both the bounded and the weakly compact ideal. Therefore, it does not make sense to consider these questions for arbitrary instances of our characterization schemes.
Question 17.7. Are super Ramsey cardinals completely ineffable? Correspondingly, if \( \kappa \) is a super Ramsey and completely ineffable cardinal, is the completely ineffable ideal always contained in the super Ramsey ideal?

The results of Section \[16\] show that the atomicity of partial orders of the form \( \mathcal{P}(\kappa)/I^{\mathfrak{m}}_{\text{ms}} \) depends on the ambient model of set theory. In all models constructed in this section however, the ideal \( I^{\mathfrak{m}}_{\text{ms}} \) is precipitous, which motivates the following question:

Question 17.8. If \( \kappa \) is a measurable cardinal, is the ideal \( I^{\mathfrak{m}}_{\text{ms}} \) precipitous?

References


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24 Let us mention the following two related results: Hellsten has shown ([14]) that for a weakly compact cardinal, the weakly compact ideal can consistently be precipitous. Johnson ([19]) has shown that for a completely ineffable cardinal, the completely ineffable ideal is never precipitous.