

# Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 10

**Problem 1:** Finish the proof of Theorem 14.1, by arguing similarly as for Easton's theorem, that the forcing defined in the proof of Theorem 14.1 satisfies the forcing theorem, that its extensions satisfy ZFC, and that they satisfy the GCH. (In fact, the argument is somewhat easier than the argument for Easton's theorem – Lemma 5.2 becomes trivial in our present context.)

*Hint:* It is probably best to factor  $P$  as  $P = P^{<\lambda} * P^{\geq\lambda}$  for regular and uncountable cardinals  $\lambda$  which are preserved by forcing with  $P$ , then showing that for such  $\lambda$ ,  $P^{<\lambda}$  is  $\lambda$ -cc and  $P^{\geq\lambda}$  is  $<\lambda^+$ -closed.

**Problem 2:** Prove Theorem 14.2. That is, assume the GCH, and let  $P$  be the iteration of length  $\text{Ord}$  with Easton support, where at each stage  $\alpha$  of our iteration, we force with the trivial forcing, unless  $\alpha$  is an infinite successor cardinal, in which case we force with the forcing  $\text{Add}(\alpha, \alpha^{++}, 2)$ . Show that any  $P$ -generic extension satisfies  $V = \text{HOD}$ . Proceed roughly as follows:

1. Similar to the proof for Easton's theorem (again, the argument is somewhat easier), show that  $P$  satisfies the forcing theorem, forcing with  $P$  preserves all cardinals and preserves ZFC.
2. Show that, for any  $p \in P$ , and any infinite successor cardinal  $\alpha$ , it holds true that  $p \Vdash 2^\alpha = \alpha^+$  if and only if  $p(\alpha)$  decides to force with the trivial forcing at stage  $\alpha$ , i.e.  $p(\alpha) = \dot{1}_{\text{triv}}$ , where  $\dot{1}_{\text{triv}}$  denotes the canonical  $P_\alpha$ -name for the weakest (and only) condition of the trivial forcing (which however is strictly stronger than  $\dot{1}_\alpha$ ).
3. Argue that if  $G$  is  $P$ -generic, then any  $x \in V[G]$  has a  $P_\alpha$ -name for some ordinal  $\alpha$ .
4. Show that for any  $p \in P$  and any  $P$ -name  $\dot{x}$  for a set of ordinals, there is  $q \leq p$  and  $\gamma \in \text{Ord}$ ,  $\xi \in \text{Ord}$  such that  $q \Vdash \dot{x} \subseteq \xi$  and

$$q \Vdash \dot{x} = \{\eta < \xi \mid 2^{\aleph_{\eta+1}} = \aleph_{\eta+2}\}.$$

5. Argue that by the above,  $V[G] = \text{HOD}^{V[G]}$ , i.e. every set in  $V[G]$  is ordinal-definable in  $V[G]$ .