Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 9

Problem 1: Let P be a notion of forcing that is σ-closed. Verify the following:

1. Forcing with P does not add new countable sequences of elements of V.
2. If $T \in V$ is a tree, then σ-closed forcing does not add new branches of cofinality ω to T.

Problem 2: Let κ be a regular and uncountable cardinal. Show that if there exists a κ-Aronszajn tree, then there exists a normal κ-Aronszajn tree.

Problem 3: Show that if κ is an infinite cardinal, then $\mathrm{Add}(\kappa^+, 1, 2)$, the forcing that adds a single new Cohen subset of κ^+, yields a forcing extension in which $2^\kappa = \kappa^+$ holds, because it collapses 2^κ to become of size κ^+ while not adding new subsets of κ.

Definition: We define the *minimal counterexample iteration P_κ for* PFA *of length κ with collapses* as a countable support iteration $\{P_\alpha, \dot{Q}_\alpha \mid \alpha < \kappa\}$, where we (inductively) define $\dot{Q}_\alpha$ as for the usual minimal counterexample iteration for PFA of length κ from the lecture in case α is an even ordinal, but we let $\dot{Q}_\alpha$ be such that $\Vdash_\alpha \dot{Q}_\alpha = \mathrm{Fn}(\omega_1, 1, \omega_2)$ when α is an odd ordinal, so we simply *demand* that at every odd stage in our iteration, the ω_2 of our intermediate model is collapsed to become of size ω_1 by the above σ-closed forcing.

Problem 4: Show that if κ is supercompact, then P_κ as defined above satisfies the following:

1. P_κ forces the PFA (by the very same argument as for the iteration used to force PFA in the lecture).
2. P_κ is κ-cc and hence preserves κ (by the same argument that I tried to give for Lemma 13.2 – the part of the argument that actually worked showed that the iteration used to force PFA in the lecture satisfies the κ-cc).
3. P_κ forces that $\check{\kappa} = \omega_2$, because it collapses all cardinals of the ground model strictly between ω_1 and κ.
4. P_κ forces that $2^{\aleph_0} = \aleph_2$, using nice names.

Remark: Hence, the above shows that starting from a supercompact cardinal, PFA is consistent with $2^{\aleph_0} = \aleph_2$. As I already remarked, PFA in fact implies $2^{\aleph_0} = \aleph_2$.

Remark 2: The argument that I wanted to do in the lecture in fact cannot work, for example if starting with a supercompact cardinal κ, however also assuming that PFA already held in our ground model, then there wouldn't be any counterexamples to PFA and the minimal counterexample iteration P_κ for PFA of length κ that we used in the lecture would just be the trivial forcing, so it would certainly not force that κ becomes ω_2.