Problem 1: Show that there is no $\omega$-Aronszajn tree.

Problem 2:

1. Prove Lemma 11.4, stating that in case $T$ is a normal Suslin tree, and $\langle P, \leq \rangle = \langle T, \geq \rangle$, then $P \times P$ does not satisfy the countable chain condition.

2. Provide an example of a maximal antichain $A$ in a normal tree $T$ for which there exists an extension $T'$ of $T$ such that $A$ is not maximal in $T'$.

Problem 3: Show that for every infinite set $X$, there exists a uniform ultrafilter on $X$. 
Definition:

- Let $\kappa$ be a regular cardinal. An ultrafilter $U$ is $<\kappa$-complete in case whenever $\lambda < \kappa$ and $\langle X_i \mid i < \lambda \rangle$ is a sequence of elements of $U$, then $\bigcap_{i<\lambda} X_i \in U$.
- An ultrafilter $U$ on $\kappa$ is nonprincipal in case $\{\alpha\} \notin U$ for every $\alpha < \kappa$.
- An uncountable cardinal $\kappa$ is measurable if there exists a $<\kappa$-complete nonprincipal ultrafilter on $\kappa$.

Problem 4: Verify the following:

1. Use the theory of normal measures developed in Peter Koepke’s set theory lecture to show that if $\kappa$ is measurable, then $\{\alpha < \kappa \mid \alpha \text{ is inaccessible}\}$ is a stationary subset of $\kappa$.

Hint: Note that $\kappa$ is inaccessible, as was shown in Peter Koepke’s lecture. Let $U$ be a normal, nonprincipal, $<\kappa$-complete ultrafilter on $\kappa$, as one obtains from measurability (again, this was shown in Peter Koepke’s lecture). Now assume that $N = \{\alpha < \kappa \mid \alpha \text{ is not inaccessible}\} \in U$. Define a regressive function $f : N \to \kappa$ by mapping each $\alpha$ in $N$ to the least $\xi < \alpha$ such that either $\text{cof}(\alpha) = \xi$ or $2^\xi \geq \alpha$. By the normality of $U$, $f$ has to be constant on a set in $U$. Use this to derive a contradiction. Finally, use another result about normal ultrafilters from Peter Koepke’s lecture to easily derive that every set in $U$ is a stationary subset of $\kappa$.

2. Every supercompact cardinal is measurable and $\{\alpha < \kappa \mid \alpha \text{ is measurable}\}$ is a stationary subset of $\kappa$.

Hint: Given $j : H(\nu) \to H(\theta)$ with $j(\text{crit}(j)) = \kappa$, for $X \subseteq \text{crit}(j)$ in $H(\nu)$, define an ultrafilter $U$ on the subsets $X$ of $\text{crit}(j)$ in $H(\nu)$ by setting $X \in U$ iff $\text{crit}(j) \subseteq j(X)$.