# Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 2

**Problem 1:** Verify the following, for given regular infinite cardinals $\kappa$, $\lambda$ and $\nu$:

1. $\mathrm{Add}(\kappa, \lambda, \nu)$ is isomorphic to a suitable product in which all factors are the forcing $\mathrm{Add}(\kappa, 1, \nu)$.
2. $\mathrm{Add}(\kappa, \lambda, \nu)$ is a complete subforcing of $\mathrm{Add}(\kappa, \mathrm{Ord}, \nu)$.
3. $\mathrm{Add}(\kappa, \mathrm{Ord}, \nu)$ satisfies the forcing theorem.
4. ZFC does not hold after forcing with $\mathrm{Add}(\kappa, \mathrm{Ord}, \nu)$.

**Problem 2:** Verify the following, assuming Global Choice in case $X$ is a proper class.

1. If $P$ is the $I$-supported product of $\{P_x \mid x \in X\}$, each $P_x$ is $<\lambda$-closed, and $I$ is closed under unions of size less than $\lambda$, then $P$ is $<\lambda$-closed.
2. The second part of Lemma 4.4, which is the following statement: If $\lambda$ is inaccessible, $\kappa < \lambda$ is regular, $|P_x| < \lambda$ for each $x \in X$, $I$ is an ideal on $X$, each element of $I$ has size less than $\kappa$, and $P$ is the $I$-supported product of $\{P_x \mid x \in X\}$, then $P$ satisfies the $\lambda$-chain condition.

**Problem 3:** Verify the following:

1. If $X \subseteq Y$, $I$ is an ideal consisting of subsets of $Y$, and $\{P_y \mid y \in Y\}$ is a collection of (set) forcing notions, then the $I$-supported product of $\{P_x \mid x \in X\}$ is a complete subforcing of the $I$-supported product of $\{P_y \mid y \in Y\}$. In particular, each $P_y$ is a complete subforcing of the $I$-supported product of $\{P_y \mid y \in Y\}$.

2. Let $\alpha$ be a limit ordinal, and let $P$ be the finite support product of nontrivial forcing notions $\{P_\beta \mid \beta < \alpha\}$. Let $G$ be $P$-generic, and let $G_\beta$ denote the induced $P_\beta$-generic for $\beta < \alpha$. Then, $V[G] \supsetneq \bigcup_{\beta<\alpha} V[G_\beta]$.

3. Let $P$ be an $I$-supported product of $\{P_n \mid n \in \omega\}$ for an arbitrary ideal $I$ on $\omega$, and assume that for every $n < \omega$, $P_n$ preserves all cofinalities, and $P_n \Vdash 2^{\aleph_0} = \aleph_{n+1}$. Let $G$ be $P$-generic, and let $G_n$ denote the induced $P_n$-generic for $n < \omega$. Then, $V[G] \supsetneq \bigcup_{n<\omega} V[G_n]$.

**Problem 4:** Finish the argument for the proof of Theorem 5.1 by showing that for every infinite regular cardinal $\lambda$,

$$(2^\lambda)^{V[G]} = F(\lambda),$$

which is done similar to the argument from Philipp's lecture that after forcing with $\mathrm{Add}(\lambda, \theta, 2)$, for $\theta$ with cofinality greater than $\lambda$ over a model of the GCH, $2^\lambda = \theta$ holds.