Models of Set Theory II – Winter 2019/20

Peter Holy – Problem Sheet 1

Problem 1: Inspect any proof of the forcing theorem, and in particular of the definability of the forcing relation, for example from Philipp Lücke's lecture course or from Kunen's set theory book, and try to find a step in the proof which might not necessarily work when P is allowed to be class sized. Try to isolate the potential problem.

Definition: Let P be a notion of (class) forcing over V. We say that $D \subseteq P$ is a predense subclass of P if for every $p \in P$ there is $d \in D$ such that d is compatible with p.

Problem 2:

1. Show that there is a (finite) forcing notion P and a predense subset D of P that contains no maximal antichain $A \subseteq D$ of P.

2. Which implications provably hold between the following conditions, for notions of class forcing P:

   (a) $G \subseteq P$ is P-generic (i.e. G intersects any dense subclass of P).

   (b) G intersects any open dense subclass of P.

   (c) G intersects any predense subclass of P.

   (d) G intersects any maximal antichain of P.

3. Show that all of the above conditions are equivalent if we assume that there is a class that is a well-order of all sets (we will denote this statement as the *existence of a global well-order*, or also the *axiom of global choice*).

Problem 3:

1. Under the assumptions of Theorem 3.2, making use of the notation introduced there, show that the following hold:

   - If $\Delta(\sigma) \leq \alpha$, then $\sigma^G = \sigma^{G_\alpha}$.
   - $V[G] = \bigcup_{\alpha \in \mathrm{Ord}} V[G_\alpha]$.
   - $G_\alpha \in V[G]$ for every $\alpha \in \mathrm{Ord}$.
   - Every maximal antichain of P is set-sized (we say that P satisfies the Ord-*chain condition*).

2. Verify that $\Vdash = \Vdash_P$ for atomic formulas, without making use of generic filters, solely arguing in the ground model V: Show that for atomic formulas, $\Vdash$ obeys the same recursive conditions by which $\Vdash_P^*$ was defined. Try to argue why this may be a *better* proof than the one presented in the lecture.

Definition: We say that a partial order P satisfies *completeness of names* if whenever $p \Vdash_P \exists x \varphi(x)$, then there is a P-name $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$.

Problem 4: Assuming Global Choice, show that every partial order with the Ord-chain condition that also satisfies the forcing theorem satisfies completeness of names.

Note: Every partial order with the Ord-chain condition in fact satisfies the forcing theorem, but this is beyond the scope of the lecture (or exercises).