Set Theory pt. 4

More details on exercise 8 from one before last time.

**Exercise 1** Show that the Axiom of Choice follows from Zorn’s Lemma.

**Hint:** Let \( x \) be a collection of non-empty sets and consider a maximal partial choice function.

**Definition 1** If \( A \) is a set of ordinals, we let \( \sup A \) denote the supremum of \( A \), i.e. the least ordinal \( \alpha \) so that for all \( \gamma \in A \), \( \gamma \leq \alpha \). We define addition, multiplication and exponentiation of ordinals as follows (\( \gamma \) always denotes a limit ordinal):

\[
\begin{align*}
\alpha + 0 &= \alpha \\
\alpha + (\beta + 1) &= (\alpha + \beta) + 1 \\
\alpha + \gamma &= \sup\{\alpha + \beta : \beta < \gamma\} \\
\alpha \cdot 0 &= 0 \\
\alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha \\
\alpha \cdot \gamma &= \sup\{\alpha \cdot \beta : \beta < \gamma\} \\
\alpha^0 &= 1 \\
\alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \\
\alpha^\gamma &= \sup_{\beta < \gamma} \alpha^\beta
\end{align*}
\]

**Exercise 2** Show that

\[1\]

1. If \( A \) is a set of ordinals, \( \sup A = \bigcup A \); therefore \( \sup A \) always exists,
2. \( 1 + \omega = \omega \),
3. \( \omega + \omega \cdot \omega = \omega \cdot \omega \),
4. \( \omega \cdot \omega^\omega = \omega^\omega \).

**Note:** In the following, \( \alpha, \beta, \gamma, \delta \) always denote ordinals without further mention.

**Exercise 3 (transfinite induction)** Show that the principle of transfinite induction, which will be necessary for some of the subsequent exercises, is a theorem of ZFC:

For any formula \( \varphi \) (which may also use parameters),

\[
(\forall \gamma ((\forall \beta < \gamma \varphi(\beta)) \to \varphi(\gamma))) \to \forall \gamma \varphi(\gamma).
\]

\[1\] like usually in mathematics, exponentiation binds stronger than multiplication, which in turn binds stronger than addition, i.e. if we write \( 8 + 2^5 \cdot 7 \), this is supposed to mean \( 8 + ((2^5) \cdot 7) \)
**Hint:** Assume for a contradiction that the left-hand side of the implication holds but \( \gamma \) is least such that \( \lnot \varphi(\gamma) \).

**Exercise 4** If \( \alpha < \beta \), then \( \gamma + \alpha < \gamma + \beta \) and \( \alpha + \gamma \leq \beta + \gamma \). Give an example why \( \leq \) cannot be replaced by \(<\).

**Exercise 5** Assume \( \alpha < \beta \) and show that there is a unique \( \delta \leq \beta \) such that \( \alpha + \delta = \beta \).

**Exercise 6** If \( \beta \) is a limit ordinal, then \( \alpha + \beta \), \( \alpha \cdot \beta \) and \( \beta \cdot \alpha \) are limit ordinals, for any \( \alpha \). If \( \beta \) is a successor ordinal, then \( \alpha + \beta \) is a successor ordinal for any \( \alpha \). If \( \alpha \) and \( \beta \) are both successor ordinals, then \( \alpha \cdot \beta \) is a successor ordinal. What about ordinal exponentiation?

**Exercise 7** Does \( \alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma \) hold?

**Exercise 8**

- Let \( \alpha \) be any ordinal and show that there exists a largest \( \delta \) such that \( \omega^\delta \leq \alpha \).
- Let \( \alpha \) be any ordinal and let \( \delta \) be maximal such that \( \omega^\delta \leq \alpha \). Then there exists a largest \( n < \omega \) such that \( \omega^\delta \cdot n \leq \alpha \).

**Exercise 9 (Cantor’s Normal Form Theorem)** Each ordinal \( \alpha \neq 0 \) can be written in the following form

\[
\alpha = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n,
\]

where \( 1 \leq n < \omega \), \( \alpha \geq \beta_1 > \ldots > \beta_n \geq 0 \) and \( 1 \leq k_i < \omega \) for each \( i = 1, \ldots, n \).

**Exercise 10** Show that there is a least ordinal number \( \varepsilon_0 \) so that \( \omega^{\varepsilon_0} = \varepsilon_0 \). Show (using induction and Cantor’s Normal Form Theorem) that every ordinal number below \( \varepsilon_0 \) can be written in a form which only uses the constant 0 and the functions \( x + y \), \( x \cdot y \) and \( \omega^x \).