Predicate Calculus pt. 2

Exercises 7-10 from last time.

Exercise 1 *A set of propositional formulas $\mathcal{T}$ is called* satisfiable *iff there is an assignment of the occuring variables which makes all formulas in $\mathcal{T}$ true. The* compactness theorem of propositional logic *says:*

$\mathcal{T}$ is satisfiable iff every finite subset of $\mathcal{T}$ is satisfiable.

Proof the compactness theorem of propositional logic by assuming that every finite subset of $\mathcal{T}$ is satisfiable and enlarging $\mathcal{T}$ to a maximal set of propositional formulas $\mathcal{T}^*$ (in the same variables) so that every finite subset of $\mathcal{T}^*$ is satisfiable and let

$$\mu(p) = \mathcal{W} \iff p \in \mathcal{T}^*.$$

Show that μ makes all formulas in $\mathcal{T}$ true.

Exercise 2 *A (symmetric, irreflexive) graph $G = (V, E)$ consists of a set of vertices V and a binary, symmetric, irreflexive relation E on V, the edge relation of the graph. If xEy, we say that the vertices x and y are connected (by an edge). An N-coloring of G assigns to each vertex one of the colors $1, \ldots, N$ so that connected vertices are assigned different colors. Use the compactness theorem of propositional logic to show that G is N-colorable iff every finite subgraph of G is N-colorable.*

Hint: Induce a propositional variable $p_{e,n}$ for every vertex e and color n.

Exercise 3 *Let $\mathcal{A}$ be an L-structure. A substructure $\mathcal{C}$ is called* elementary substructure *iff*

$$\mathcal{A} \models \phi[c_1, \ldots, c_n] \iff \mathcal{C} \models \phi[c_1, \ldots, c_n]$$

for all L-formulas $\phi(x_1, \ldots, x_n)$ and $c_1, \ldots, c_n \in C$. Show that the so-called Tarski-criterion *holds: C is the universe of an elementary substructure of $\mathcal{A}$ iff for all $\phi(x, y_1, \ldots, y_n)$ and all $d_1, \ldots, d_n \in C$, the following holds: If there is $a \in A$ so that $\mathcal{A} \models \phi[a, d_1, \ldots, d_n]$, then there is $c \in C$ so that $\mathcal{A} \models \phi[c, d_1, \ldots, d_n]$.*

Hint: One direction is easy; the other proceeds by induction on formula complexity.

Exercise 4

1. *If L is a countable language, then there are only countably many L-terms and L-formulas.*

2. *If L is a countable language, every L-structure $\mathcal{A}$ has a countable elementary substructure.*

Hint for 2: use 1 and Exercise 3.

Exercise 5 *We call a class of L-structures* elementary *iff it is the class of all models of a theory T. Show:*

1. *The class of all infinite L-structures is elementary.*

2. *The class of all finite L-structures is not elementary.*