The Mostowski collapse - a leftover from last time

Work in ZFC.

Exercise 1 *Define* $x\mathbf{R}y$ *iff* $x \in \mathrm{trcl}(y)$. *Show that* $\mathbf{R}$ *is well-founded* [1] *and set-like (on* $\mathbf{V}$*). Let* $\mathbf{G}$ *be the Mostowski collapsing function of* $(\mathbf{V}, \mathbf{R})$. *Show that* $\mathbf{G}(x) = \mathrm{rank}(x)$ *for each* x.[2]

Relativization, Absoluteness, ZFC

Exercise 2 *Which axioms of* ZFC *are true in* $\mathbf{ON}$*?*

Exercise 3 *Assume* γ *is a limit ordinal and show that all axioms of* ZFC *except maybe the replacement scheme are true in* $R(\gamma)$.

Exercise 4 *Find an example of a singular limit ordinal* γ *and an instance of the replacement scheme which is not true in* $R(\gamma)$.

Definition 1 *A formula in the language of set theory is called* Σ_1 *iff it is of the form* $\exists\, y_1, \ldots, y_n\ \psi$ *where* ψ *is* Δ_0, $n \in \omega$. *It is called* Π_1 *iff it is of the form* $\forall\, y_1, \ldots, y_n\ \psi$ *where* ψ *is* Δ_0, $n \in \omega$.

Exercise 5 *Show that, using the axioms of* ZFC, *the statement "*κ *is a cardinal" is equivalent to a formula* $\varphi(\kappa)$ *which is* Π_1.

Exercise 6 *If* $\mathbf{M} \subseteq \mathbf{N}$ *are both transitive models of* ZFC, κ *is a cardinal in* $\mathbf{N}$ *and* $\kappa \in M$, *then* κ *is a cardinal in* $\mathbf{M}$.

Exercise 7 *More generally, assume* $\mathbf{M} \subseteq \mathbf{N}$ *are both transitive and show that for all* $x_1, \ldots, x_n \in \mathbf{M}$,

- *If* $\varphi(x_1, \ldots, x_n)$ *is* Σ_1, *then* $\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \to \varphi^{\mathbf{N}}(x_1, \ldots, x_n)$.
- *If* $\varphi(x_1, \ldots, x_n)$ *is* Π_1, *then* $\varphi^{\mathbf{N}}(x_1, \ldots, x_n) \to \varphi^{\mathbf{M}}(x_1, \ldots, x_n)$.

We abbreviate the above by saying that Σ_1 *formulas are* upwards absolute, Π_1 *formulas are* downwards absolute.

[1] Hint: Show (and use) that $x\mathbf{R}y$ implies $\mathrm{rank}(x) < \mathrm{rank}(y)$.
[2] Hint: Use induction on $\mathrm{rank}(x)$. Show first that $\mathbf{G}(x)$ is an ordinal for each x.

Leftovers

Exercise 8 *Assume $\lambda \leq \kappa$ and show that there are κ^λ-many injective functions from λ to κ (i.e. $|\{f \in {}^\lambda\kappa \colon f \text{ injective }\}| = \kappa^\lambda$).*

Hint: Given $f \in {}^\lambda\kappa$, find a way to construct an injective $g_f \in {}^\lambda\kappa$ in an injective way, i.e. it should be the case that whenever $f_0 \neq f_1 \in {}^\lambda\kappa$, $g_{f_0} \neq g_{f_1}$.

Additional hint:
Use a fixed partition of κ into λ-many disjoint pieces, each of size κ. Define g_f so that $g_f(i)$ is the $f(i)^{\text{th}}$ element of the i^{th} disjoint piece for each $i < \lambda$. Check that this works!

Exercise 9 *Show (without assuming* GCH*) using the definition*

$$\kappa^{<\lambda} := \sup\{\kappa^\delta \colon \delta < \lambda, \delta \in \text{Card}\} :$$

- *If κ is strongly inaccessible, then $\kappa^{<\kappa} = 2^{<\kappa} = \kappa$.*
- *If κ is weakly inaccessible, then $\kappa^{<\kappa} = 2^{<\kappa}$.*

Hint for the 2nd item: If κ happens to be strongly inaccessible, we are done using the first item. Thus we may assume (and use) that there is $\delta < \kappa$ such that $2^\delta \geq \kappa$.