Well-founded sets

Exercise 1 Assume $f$ is a function from $A$ to $B$ with $\text{rank}(A) = \alpha$ and $\text{rank}(B) = \beta$. What can you say about $\text{rank}(f)$? Argue that we can calculate $\text{rank}(f)$ in the case when $f$ is onto $B$ (surjective).

Exercise 2 For which $\alpha$ is $|R(\alpha)| = \beth_\alpha$?

Exercise 3 If $\kappa$ is strongly inaccessible, then $|R_\kappa| = \kappa$. Are there cardinals $\kappa$ with this property which are not inaccessible?

Exercise 4 Find a set $x$ so that $|x| < |\text{trcl}(x)|$.

Exercise 5 Work in ZFC without the Axiom of Foundation. Show that every proper class $A$ for which $x \subseteq A \rightarrow x \in A$ holds is a superclass of $\text{WF}$ (in the obvious sense that $x \in \text{WF}$ implies $x \in A$).

Hint: $\emptyset \subseteq A$.

Definition 1 A set $x$ is called hereditarily finite (in german “hereditär endlich”) iff $|\text{trcl}(x)| < \omega$. $H_\omega$ denotes the set of all hereditarily finite sets.

Exercise 6 Show that $H_\omega = R(\omega)$.

A simplified approach to exercise 3 from last time

Exercise 7 Let $\lambda$ be any infinite cardinal and show that

$|\{X \subseteq \lambda: |X| = \lambda\}| = 2^\lambda$.

Hint: Assume not and use that whenever $X \subseteq \lambda$ has size less than $\lambda$, its complement (within $\lambda$) has size $\lambda$.

Exercise 8 Assume $\lambda \leq \kappa$ and show that there are $\kappa^\lambda$-many injective functions from $\lambda$ to $\kappa$ (i.e. $|\{f \in \lambda^\kappa: f \text{ injective}\}| = \kappa^\lambda$).

Hint: Given $f \in \lambda^\kappa$, find a way to construct an injective $g_f \in \lambda^\kappa$ in an injective way, i.e. it should be the case that whenever $f_0 \neq f_1 \in \lambda^\kappa$, $g_{f_0} \neq g_{f_1}$.

Note: Almost the same proof shows that whenever $\lambda$ is an infinite cardinal, there are $2^\lambda$ many bijections from $\lambda$ to $\lambda$. We will need (and use) this in the following but I suggest to omit the exact proof.

\[1\text{this is the special case where } \kappa = \lambda \text{ of exercise } 3 \text{ from last time}\]
Exercise 9 Show that whenever $\lambda \leq \kappa$ are infinite ordinals,\(^2\)

$$|\{X \subseteq \kappa : |X| = \lambda\}| = \kappa^\lambda.$$ 

Hint: Let $A := \{X \subseteq \kappa : |X| = \lambda\}$. Argue that $|A| \leq \kappa^\lambda$ is pretty obvious. We have to show that $\kappa^\lambda \leq |A|$. We want to find an injection from $\lambda \kappa$ to $A$. By exercise 8, it suffices to find an injection from the injective functions from $\lambda$ to $\kappa$ to $A$. Note that every injective $f : \lambda \rightarrow \kappa$ is naturally connected to an element of $A$, namely to $\text{range}(f)$. As there are $2^\lambda$ many bijections from $\lambda$ to $\lambda$ (see note above), $2^\lambda$-many functions functions $f$ are connected to the same element of $A$ for each element of $A$. This gives rise to the equation $|A| \otimes 2^\lambda = \kappa^\lambda$. Use this to obtain the desired result distinguishing the cases $2^\lambda < \kappa^\lambda$ and $2^\lambda = \kappa^\lambda$.\(^3\)

\(^2\)this is exactly exercise 3 from last time

\(^3\)I apologize for posing this problem in the last exercise without any hints or initial steps, it seems too hard - but maybe there’s an easier solution?