Well-Orders, Ordinals and Ordinal Arithmetic

Exercise 1
If $(A,<)$ is a well-order and $B \subseteq A$, then $(B, < \restriction B)$ is a well-order.

Notation: $< \restriction B$ denotes the restriction of $<$ to B, i.e.

$$< \restriction B = \{(x,y) \in <: x \in B \wedge y \in B\}.$$

Exercise 2 *If $(A,<)$ is a well-order and $f\colon A \to A$ is an isomorphism respecting $<$ (i.e. f is a bijection and $a_0 < a_1 \iff f(a_0) < f(a_1)$), then f is the identity on A.*

Exercise 3

1. *Show that $\emptyset$ is transitive,*
2. *find an example of a non-transitive set,*
3. *show that if I is any index set and for every $i \in I$, A_i is transitive, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i \in I} A_i$ are transitive,*
4. *find out exactly when $\{x\}$ and $\{x,y\}$ are transitive.*

Exercise 4 *If $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example why $\leq$ cannot be replaced by $<$.*

Note: We will always use α, β, γ and δ to denote ordinals without further mention. In that context, $+$ and $\cdot$ will, as above, always refer to ordinal addition and multiplication.

Exercise 5 *Assume $\alpha < \beta$ and show that there is a unique $\delta \leq \beta$ such that $\alpha + \delta = \beta$.*

Exercise 6 *If β is a limit ordinal, then $\alpha + \beta$, $\alpha \cdot \beta$ and $\beta \cdot \alpha$ are limit ordinals, for any α. If β is a successor ordinal, then $\alpha + \beta$ is a successor ordinal for any α. If α and β are both successor ordinals, then $\alpha \cdot \beta$ is a successor ordinal.*

Exercise 7 *Find a sequence $\langle A_i \colon i < \omega \rangle$ such that $A_i \subseteq A_j \subseteq \mathrm{ON}$ whenever $i < j < \omega$ which violates the following statement:*

$$\bigcup_{i \in \omega} \mathrm{type}(A_i, \in) = \mathrm{type}(\bigcup_{i\in\omega} A_i, \in).$$

Hint: A sequence with each A_i finite and $\bigcup_{i\in I} A_i = \omega+\omega$ will do the job; there are of course many other possibilities.

Definition 1 *If $\Gamma = \langle \gamma_\alpha \colon \alpha \in \mathrm{ON}\rangle$ is a (class-sized) sequence of ordinals, we say that Γ is normal iff it is strictly monotonously increasing and continuous, i.e. $\alpha_0 < \alpha_1 \to \gamma_{\alpha_0} < \gamma_{\alpha_1}$ and $\bigcup_{\alpha<\beta} \gamma_\alpha = \gamma_\beta$ whenever β is a limit ordinal. We say that α is a fixed point of Γ iff $\gamma_\alpha = \alpha$.*

Exercise 8 *Every (class-sized) normal sequence Γ of ordinals has a fixed point. Moreover, it has arbitrarily large fixed points.*

Hint: Try to build a strictly increasing sequence of ordinals of length ω with it's supremum a fixed point of Γ.

Exercise 9 *Show, without using the Axiom of Choice, that for every set X, the following are equivalent:*

- *X can be wellordered, i.e. there exists R such that (X, R) is a well-order.*
- *There exists a function $f\colon (\mathcal{P}(X) \setminus \{\emptyset\}) \to X$ such that for every $Y \in \mathcal{P}(X) \setminus \{\emptyset\}$, $f(Y) \in Y$.*

Argue that this implies that the Axiom of Choice as stated in Kunen's book (every set can be well-ordered) is equivalent to the Axiom of Choice as usually defined in mathematics (for every family A of nonempty sets, there exists a choice function for A, i.e. a function f with domain A such that for every $a \in A$, $f(a) \in a$).

Note: $\mathcal{P}(X)$ denotes the power set of X, i.e.

$$\mathcal{P}(X) = \{Y \colon Y \subseteq X\}.$$

The existence of $\mathcal{P}(X)$ is postulated by the power set axiom.