Condensation and Large Cardinals - a simplified version of my dissertation

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Abstract

We give a corrected and simplified, self-contained account of the proof of the main theorem of the author's dissertation ([4]): We show that over any model of set theory we may perform a cofinality-preserving forcing to obtain a model of set theory which satisfies Local Club Condensation while preserving an ω-superstrong cardinal. To simplify reference, chapter numbers in this note correspond with chapter numbers in [4].

1 Canonical Functions

Lemma 1.1 *Assume β has regular cardinality κ and for every $\gamma \leq \beta$, f_γ is a bijection from* $\operatorname{card}\gamma$ *to γ. Then there is a club of $\delta < \kappa$ such that*

$$f_\alpha[\delta] = f_\beta[\delta] \cap \alpha \textit{ for all } \alpha \in f_\beta[\delta] \setminus \kappa.$$

Proof: See [2] or [4].

2 Large Cardinal Basics

Definition 2.1 *κ is ω-superstrong if there is an elementary embedding $j\colon \mathbf{V} \to \mathbf{M}$ with critical point κ such that $V_{j^\omega(\kappa)} \subseteq M$.*[1]

3 Forcing Basics

Definition 3.1 *If P is a notion of forcing and η is a cardinal, we say that P is η^+-strategically closed iff Player I has a winning strategy in the following two player game of perfect information: Player I and Player II alternately make moves where in each move, each player plays a condition of P. Player I has to start and play $\mathbf{1}_P$ in the first move. Player II is allowed to play any condition stronger than the condition just played by Player I in each of his moves. Player I has to play a condition stronger than all previously played conditions in each move, Player I has to make a move at every limit step of the game. We say that Player I wins if he can find conditions to play in any such game of length η^+ (arriving at η^+, the game ends, no condition has to be played at stage η^+).*

[1] Such an embedding with $V_{j^\omega(\kappa)+1} \subseteq M$ is known to be inconsistent by Kunen's Theorem.

4 Local Club Condensation

The definition of Local Club Condensation applies to models $\mathbf{M}$ of set theory with a hierarchy of levels of the form $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ with the properties that $\mathbf{M} = \bigcup_{\alpha\in\mathrm{Ord}} M_\alpha$, each M_α is transitive, $\mathrm{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \in M_\beta$ and if γ is a limit ordinal, $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We will also let M_α denote the structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$, where context will usually clarify the intended meaning.

Local Club Condensation is the statement that if α has uncountable cardinality κ and $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \leq \gamma < \kappa\rangle$ of substructures of $\mathcal{A}_\alpha$ whose domains have union M_α, where each $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle, \ldots)$ is s.t. $|B_\gamma| = |\gamma|$, $\gamma \subseteq B_\gamma$ and each $(B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$.

We will usually be in the situation that $\mathbf{M} = (\mathbf{L}[A], A)$ for some $A \subseteq \mathrm{Ord}$ and $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle = \langle L_\alpha[A] \colon \alpha \in \mathrm{Ord}\rangle$. We say that $\mathbf{M}$ is of the form $\mathbf{L}[A]$ in that case. The following will be useful in Section 8:

Lemma 4.1 *Local Club Condensation is equivalent to the following, seemingly weaker statement: If α has uncountable cardinality κ, then the structure $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, F)$ has a continuous chain $\langle \mathcal{B}_\gamma \colon \gamma \in C\rangle$ of substructures $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle, F)$ of $\mathcal{A}_\alpha$ with $\bigcup_{\gamma\in C} B_\gamma = M_\alpha$, $C \subseteq \kappa$ is club, C consists only of cardinals if κ is a limit cardinal, each B_γ has cardinality $\mathrm{card}\,\gamma$, contains γ as a subset and each $(B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle)$ is isomorphic to some $(M_{\bar\alpha}, \in, \langle M_\beta \colon \beta < \bar\alpha\rangle)$, where F denotes the function $(f, x) \mapsto f(x)$ whenever $f \in M_\alpha$ is a function with $x \in \mathrm{dom}(f)$.*

Proof: See [4] or [2].

5 History, Motivation

See [4].

6 Forcing Acceptability

The corresponding chapter of [4] contains a number of serious mistakes and is somewhat misleading as well. A corrected account of all the material of that chapter (and more) has appeared in [3, Section 1].

7 A small history of fragments of Condensation

See [4].

8 Forcing Local Club Condensation

In this section we will show how to extend (by cofinality-preserving forcing) a given model $\mathbf{V}$ of set theory to a model of Local Club Condensation while preserving large cardinals. This is the central result of the thesis. We assume that the starting universe $\mathbf{V}$ satisfies GCH. We will define a *reverse Easton-like* class sized forcing P and show that there are P-generic extensions of the universe as desired. We will define P inductively. P_ω, the forcing up to ω is trivial. Assume P_α, the forcing P up to α is defined. Let S_α denote the lottery sum of all elements of the form $(0, f_\alpha)$ and $(1, f_\alpha)$ where f_α is a bijection from $\operatorname{card}\alpha$ to α in $\mathbf{V}$. Let $\check{\mathbf{1}}$ denote the standard name for the weakest condition $\mathbf{1}$ of a forcing. We define $P^\oplus_\alpha$ to be a subset of $P_\alpha * S_\alpha$ which is not dense in $P_\alpha * S_\alpha$. Namely, let $P^\oplus_\alpha = \{(t, p(\alpha)(0)) \in P_\alpha * S_\alpha \colon t \in P_\alpha \wedge p(\alpha)(0) = \check{\mathbf{1}}$ or $\exists f_\alpha \colon \operatorname{card}\alpha \to \alpha \ \exists p_\alpha \ \mathbf{1}_{P_\alpha} \Vdash p_\alpha \in \{0,1\} \wedge p(\alpha)(0) = (p_\alpha, \check{f}_\alpha)\}$. A $P^\oplus_\alpha$-generic $G^\oplus_\alpha$ thus either decides for $p_\alpha = 0$ or $p_\alpha = 1$ at stage α and chooses a ground model bijection $f_\alpha^{G^\oplus_\alpha}$ from $\operatorname{card}\alpha$ to α. We usually denote this bijection by f_α without making actual reference to the generic (or condition) that chose it as this should always be clear from context. For two compatible conditions s_0 and s_1 in S_α, let $s_0 \cup s_1$ denote the stronger of both. If $G^\oplus_\alpha$ is $P^\oplus_\alpha$-generic, it specifies a predicate $g_{\alpha+1} \subseteq \alpha + 1$ (which we shall identify with a function $g_{\alpha+1} \colon \alpha + 1 \to 2$) by

$$g_{\alpha+1}(\beta) = 1 \leftrightarrow G^\oplus_\alpha \text{ decides } p_\beta = 1.$$

If $\operatorname{card}\alpha = \omega$ or $\operatorname{card}\alpha$ is singular, we let $P_{\alpha+1} = P^\oplus_\alpha$. Whenever $\operatorname{card}\alpha > \omega$ is regular and $G^\oplus_\alpha$ is $P^\oplus_\alpha$-generic with corresponding predicate $g = g_{\alpha+1}$, let $C(G^\oplus_\alpha)$ denote the following forcing poset:

If $\operatorname{card}\alpha = \theta^+$ is a successor cardinal, $q^{**} \in C(G^\oplus_\alpha)$ iff

- q^{**} is a closed, bounded subset of $[\theta, \operatorname{card}\alpha)$ and
- $\forall \eta \in q^{**} \ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

If $\operatorname{card}\alpha$ is inaccessible, q^{**} is a condition in $C_\alpha(G^\oplus_\alpha)$ iff

- q^{**} is a closed, bounded set of cardinals below $\operatorname{card}\alpha$ and
- $\forall \eta \in q^{**} \ g(\operatorname{ot} f_\alpha[\eta]) = g(\alpha)$.

Conditions in $C(G^\oplus_\alpha)$ are ordered by end-extension (in both cases). If $\operatorname{card}\alpha > \omega$ is regular, we let $P_{\alpha+1} = P^\oplus_\alpha * C(G^\oplus_\alpha)$. If $p(\alpha) = (p(\alpha)(0), p(\alpha)(1))$, we denote $p(\alpha)(0)$ by (p_α, f_α) and denote $p(\alpha)(1)$ by p^{**}_α. We write $p{\restriction}\alpha^\oplus$ to denote $p{\restriction}\alpha^\frown p(\alpha)(0) \in P^\oplus_\alpha$. For a condition $p \in P$ (or some P_α), we call $\{\gamma \colon p_\gamma \neq \check{\mathbf{1}}\}$ the string support of p and denote it by S-supp(p), we call $\{\gamma \colon p^{**}_\gamma \neq \check{\mathbf{1}}\}$ the club support of p and denote it by C-supp(p).

We finished the definition of the successor stages of our forcing. It remains to define its limit stages. Assume α is a limit ordinal and P_γ is defined for $\gamma < \alpha$, T is the inverse limit of $\langle P_\gamma \colon \gamma < \alpha\rangle$ and $p \in T$. Then $p \in P_\alpha$ if

1. if α is regular, S-supp(p) is bounded below α and

2. for every regular θ, $\mathrm{card}(\text{C-supp}(p) \cap \theta^+) < \theta$. [2]

Let P be the direct limit of $\langle P_\alpha \colon \alpha \in \mathrm{Ord}\rangle$. We usually assume conditions to satisfy the following properties (possible as a dense subset of conditions does):

A1. $\forall\gamma\ \mathbf{1}_{P_\gamma^\oplus} \Vdash p_\gamma^{**} \in C(G_\gamma^\oplus)$.

A2. $\text{C-supp}(p) \subseteq \text{S-supp}(p)$.

We will at some points have to temporarily cease from assumption A1. We will explicitly mention whenever we do so.

Claim 8.1 (String Extendibility) *Assume f is a function with domain $d \subseteq \alpha$ such that for every $\gamma \in d$ $f(\gamma)$ is a P_γ-name which is forced by the trivial condition to equal either 0 or 1. Assume d is bounded below every regular cardinal. Then any given $p \in P_\alpha$ with* $\text{S-supp}(p) \cap d = \emptyset$ *can be extended to $q \le p$ such that $\Vdash_{P_\gamma} q_\gamma = f(\gamma)$ whenever $\gamma \in d$.* □

Definition 8.2 (strategically closed part of a condition) *Given a cardinal $\eta < \alpha$ and $p \in P_\alpha$, we define $u_\eta(p) \in P_\alpha$ as follows:*

- $(u_\eta(p))(\gamma)(0) = \begin{cases} \mathbf{1} & \textit{if } \gamma < \eta \\ p(\gamma)(0) & \textit{otherwise} \end{cases}$

- $(u_\eta(p))^{**}_\gamma = \begin{cases} \mathbf{1} & \textit{if } \gamma < \eta^+ \\ p^{**}_\gamma & \textit{otherwise} \end{cases}$

and call $u_\eta(p)$ the η^+-strategically closed part of p. We let $u_\eta(P_\alpha) := \{u_\eta(p) \colon p \in P_\alpha\}$ and call it the η^+-strategically closed part of P_α.

Note:

- The fact that $u_\eta(p) \in P_\alpha$ uses assumption A1.
- We may think of $u_\eta(p)$ as the condition extracting from p its choice of bits and bijections in the interval $[\eta, \eta^+)$ and everything at and above η^+.
- The same definition applies to $p \in P_\alpha^\oplus$. It is usually the case that definitions and statements refering to some condition in P_α will have a natural equivalent for $P_\alpha^\oplus$, explicit mention of which will be omitted most of the time.

The following claim will often be tacitly used. It was repeatedly used in [4] and [2] and in slightly different context in [3], but no proof was given in those papers.

Claim 8.3 *If $p \in P_\alpha$, $\eta < \alpha$ is a cardinal and $q \le p$ then there is $r \le q$ such that $q \le r$ (i.e. q and r are* equivalent*) and $u_\eta(r) \le u_\eta(p)$. Moreover if p and q satisfy A1 and A2, so does r.*

[2]The former condition is the reason why we called our forcing "Easton-like" earlier on.

Proof: Assume $p \in P_\alpha$, $\eta < \alpha$ is a cardinal and $q \leq p$. We want to construct $r \leq q$ such that $u_\eta(r) \leq u_\eta(p)$. We define r by induction on $i < \alpha$. For $i < \eta$, let $r(i) = q(i)$.

Assume now that $i \geq \eta$ and $r{\restriction}i$ is defined, $r{\restriction}i \leq q{\restriction}i$ and $u_\eta(r{\restriction}i) \leq u_\eta(p{\restriction}i)$. If $p(i)(0) = \check{\mathbf{1}}$, let $r(i)(0) = q(i)(0)$, let $r(i)(0) = p(i)(0)$ otherwise. If $p(i)(0) = \check{\mathbf{1}}$, $r{\restriction}i \Vdash r(i)(0) = q(i)(0) \leq q(i)(0)$ and $u_\eta(r{\restriction}i) \Vdash r(i)(0) = q(i)(0) \leq \check{\mathbf{1}}$. Otherwise, $r{\restriction}i \leq q{\restriction}i \Vdash p(i)(0) = q(i)(0)$ and so $r{\restriction}i \Vdash r(i)(0) = p(i)(0) \leq q(i)(0)$. Also, $u_\eta(r{\restriction}i) \Vdash r(i)(0) = p(i)(0) \leq p(i)(0)$.

If $i < \eta^+$, let $r_i^{**} = q_i^{**}$. If $i \geq \eta^+$, assume that $r{\restriction}i^\oplus$ is defined, $r{\restriction}i^\oplus \leq q{\restriction}i^\oplus$ and $u_\eta(r{\restriction}i^\oplus) \leq u_\eta(p{\restriction}i^\oplus)$. Let

$$r_i^{**} = \begin{cases} q_i^{**} & \text{if } r{\restriction}i^\oplus \in G \\ p_i^{**} & \text{otherwise} \end{cases}.$$

Then $r{\restriction}i^\oplus \Vdash r_i^{**} = q_i^{**} \leq q_i^{**}$. Let A be a maximal antichain below $u_\eta(r{\restriction}i^\oplus)$ that refines $r{\restriction}i^\oplus$, i.e. for every $a \in A$ either $a \leq r{\restriction}i^\oplus$ or $a \perp r{\restriction}i^\oplus$. If $a \leq r{\restriction}i^\oplus$, then $a \Vdash r_i^{**} = q_i^{**} \leq p_i^{**}$. If $a \perp r{\restriction}i^\oplus$, then $a \Vdash r_i^{**} = p_i^{**} \leq p_i^{**}$. Hence $u_\eta(r{\restriction}i^\oplus) \Vdash r_i^{**} \leq p_i^{**}$.

Summing up, $r{\restriction}(i+1) \leq q{\restriction}(i+1)$ and $u_\eta(r{\restriction}(i+1)) \leq u_\eta(p{\restriction}(i+1))$. The last statement of the claim is immediate from the definition of r. □

Definition 8.4 (small part of a condition)
If $\eta < \alpha$ is a cardinal and $p \in P_\alpha$, we define $l_\eta(p)$ as follows:

- $(l_\eta(p))(\gamma)(0) = \begin{cases} \mathbf{1} & \text{if } \alpha > \gamma \geq \eta \\ p(\gamma)(0) & \text{otherwise} \end{cases}$
- $(l_\eta(p))^{**}_\gamma = \begin{cases} \mathbf{1} & \text{if } \alpha > \gamma \geq \eta^+ \\ p^{**}_\gamma & \text{otherwise} \end{cases}$

and call $l_\eta(p)$ the η-sized part of p. $l_\eta(p)$ is in general not a condition in P_α. Note also that $l_\eta(p)$ complements $u_\eta(p)$ in the sense that it carries exactly all information about p not contained in $u_\eta(p)$.

Definition 8.5 (stable below η^+) *Assume $\langle p^i \colon i < \delta\rangle$ is a decreasing sequence of conditions in $P_{<\alpha}$ of limit length $\delta < \eta^+$, $\eta < \alpha$ a cardinal. We say that $\langle p^i \colon i < \delta\rangle$ is stable below η^+ iff*

- *$\langle l_\eta(p^i) \colon i < \delta\rangle$ is eventually constant or*
- *η is singular and for every cardinal $\mu < \eta$, $\langle l_\mu(p^i) \colon i < \delta\rangle$ is eventually constant.*

Definition 8.6 *If θ is a regular uncountable cardinal and $\theta \leq \gamma_0 < \gamma_1 < \theta^+$, then there is a club $C_{\{\gamma_0,\gamma_1\}} \subseteq \theta$ such that for every $\eta \in C_{\{\gamma_0,\gamma_1\}}$*

- *$f_{\gamma_i}[\eta] \supseteq \eta$ for $i \in \{0,1\}$ and*
- *$f_{\gamma_0}[\eta]$ is a proper initial segment of $f_{\gamma_1}[\eta]$.*

For $\gamma \in [\theta, \theta^+)$, we let C_γ be the club $\{\eta < \theta \colon f_\gamma[\eta] \supseteq \eta\}$. Whenever $v \subseteq [\theta, \theta^+)$ is of size less than θ and at least 2, we let

$$C_v := \bigcap_{\{\gamma_0,\gamma_1\} \subseteq v} C_{\{\gamma_0,\gamma_1\}}.$$

In any of the above cases, we call C_v the separating club for v.

Definition 8.7 (Strategic Belowness)
Assume $\alpha' \leq \alpha$, θ is regular, $p \in P_\alpha$ and $q \leq p\restriction\alpha'$. We say that q is strategically below p at θ if $\text{C-supp}(p) \cap [\theta, \theta^+) = \emptyset$, *if $\theta \geq \alpha'$ or all of the following hold:*

(i) $\forall\gamma \in \text{C-supp}(p)\cap[\theta,\theta^+)$ *below α', $q\restriction\gamma$ forces that p_γ has a $P_{\sup(\text{S-supp}(q)\cap\theta)}$-name,*

(ii) $\forall\gamma \in \text{C-supp}(p)\cap[\theta,\theta^+)$ *below α', $q\restriction\gamma^\oplus$ forces* $\max q^{**}_\gamma > \sup(\text{S-supp}(p)\cap\theta)$ *and* $\sup(\text{S-supp}(q)\cap\theta) > \max p^{**}_\gamma$,

(iii) $\sup(\text{S-supp}(q)\cap\theta)$ *is greater than some element of $C_{\text{C-supp}(p)\cap[\theta,\theta^+)}$ greater than* $\sup(\text{S-supp}(p)\cap\theta)$ *and*

(iv) if θ is inaccessible, $\sup(\text{S-supp}(q)\cap\theta) > \text{card}(\text{C-supp}(p)\cap[\theta,\theta^+))$.

If $\eta < \alpha' \leq \alpha$, η is a cardinal and $q \leq p\restriction\alpha'$, we say that q is η^+-strategically below p if for every regular $\theta > \eta$, q is strategically below p at θ. It is immediate that if $\eta_0 < \eta_1$ are both cardinals and q is ${\eta_0}^+$-strategically below p then q is ${\eta_1}^+$-strategically below p.

Note: The common case will be when $\alpha' = \alpha$ in the above. If $p \in P_\alpha$, $q \in P_{\alpha'}$, $\alpha' < \alpha$ and q is η^+-strategically below p, then q is η^+-strategically below $p\restriction\alpha'$. The reverse direction of this implication will usually not hold, as in general Clauses (iii) and (iv) get weaker as α gets smaller.

Claim 8.8 (Persistence of Strategic Belowness)

- *If $\alpha < \alpha^*$, $p, q \in P_{\alpha^*}$ and q is η^+-strategically below p, then $q\restriction\alpha$ is η^+-strategically below $p\restriction\alpha$.*
- *For $p, q, r \in P_\alpha$ and a cardinal $\eta < \alpha$, if q is η^+-strategically below p and $r \leq q$, then r is η^+-strategically below p.*
- *For $p, q, r \in P_\alpha$ and a cardinal $\eta < \alpha$, if $q \leq p$ and r is η^+-strategically below q, then r is η^+-strategically below p.*

Proof: Follows straightforward from definition 8.7. □

Notation: Assume $\langle s^i : i < \delta\rangle$ is a decreasing sequence of conditions in S_α. Then $\langle s^i : i < \delta\rangle$ is eventually constant and we denote it's limit by $\bigcup_{i<\delta} s^i$. Given a decreasing sequence of conditions $\langle p^i : i < \delta\rangle$ in P_α of limit length δ, we say that $r = \langle r(\delta) : \delta < \alpha\rangle$ is the componentwise union of $\langle p^i : i < \delta\rangle$ if for every $\gamma < \alpha$, $r(\gamma) = ((r_\gamma, f_\gamma), r^{**}_\gamma)$ where $f_\gamma = f^r_\gamma = f^{p^i}_\gamma$ whenever p^i specifies a bijection from $\text{card}\,\gamma$ to γ and

$$r_\gamma = \bigcup_{i<\delta} p^i_\gamma \text{ and } r^{**}_\gamma = \bigcup_{i<\delta} (p^i)^{**}_\gamma.$$

r is usually not a condition in P_α as the r^{**}_γ are not necessarily names for closed sets, but the supports of r can be calculated as if r were a condition by letting

$$\text{S-supp}(r) = \{\gamma : r_\gamma \neq \check{\mathbb{1}}\} = \bigcup_{i<\gamma} \text{S-supp}(p^i)$$

and

$$\text{C-supp}(r) = \{\gamma \colon r^{**}_\gamma \neq \check{\mathbf{1}}\} = \bigcup_{i<\gamma} \text{C-supp}(p^i).$$

Definition 8.9 (Strategic lower bound) *Given a cardinal $\eta < \alpha$ and a sequence $\langle p^i \colon i < \delta\rangle$ of conditions in P_α of limit length $\delta < \eta^+$ which is stable below η^+, form their componentwise union r.* S-supp(r) *is bounded below every regular cardinal,* C-supp$(r) \cap \theta^+$ *has size less than θ for every regular θ. We would like to obtain a condition $q \in P_\alpha$ with the following properties for every $\gamma \in$* C-supp(r)*, $\gamma \geq \eta^+$:*

(1) $q{\restriction}\gamma^\oplus \Vdash q_{\text{ot}\, f_\gamma[\sup r^{**}_\gamma]} = r_\gamma$.

(2) $q{\restriction}\gamma^\oplus \Vdash q^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$.

Other components of q should be equal to the respective components of r. If such q exists, we call q the η^+-strategic lower bound for $\langle p^i \colon i < \delta\rangle$. Whenever we want to apply the above, we will be in a situation where each $\sup r^{**}_\gamma$ *will have been decided by any lower bound of $\langle p^i{\restriction}\gamma^\oplus \colon i < \delta\rangle$ to equal an actual ordinal value (and is not just a name for an ordinal). It is immediate from the definitions that if our desired q exists as a condition in P_α, then q is a greatest lower bound for $\langle p^i \colon i < \delta\rangle$.*

Claim 8.10 (Existence of strategic lower bounds)
Assume $\eta < \alpha$ is a cardinal, $\langle p^i \colon i < \delta\rangle$ is a sequence of conditions in P_α of limit length $\delta < \eta^+$ which is stable below η^+ such that p^{i+1} is η^+-strategically below p^i for all $i < \delta$. Then the η^+-strategic lower bound q for $\langle p^i \colon i < \delta\rangle$ exists.

Proof: By induction on $\alpha \geq \eta^+$. If $\alpha = \eta^+$, the claim follows by stability of $\langle p^i \colon i < \delta\rangle$ below η^+. For any $\gamma < \alpha$, given that the claim holds within P_γ, it immediately follows that it holds within $P^\oplus_\gamma$. We want to show the claim holds for α, i.e. show that the η^+-strategic lower bound q^α for $\langle p^i \colon i < \delta\rangle$ exists. Inductively, for $\gamma < \alpha$, let q^γ be the η^+-strategic lower bound for $\langle p^i{\restriction}\gamma \colon i < \delta\rangle$, let $q^{\gamma^\oplus}$ be the η^+-strategic lower bound for $\langle p^i{\restriction}\gamma^\oplus \colon i < \delta\rangle$. We will also use that if $\gamma_0 < \gamma_1 < \alpha$, then $q^{\gamma_1}{\restriction}\gamma_0 \leq q^{\gamma_0}$. Thus we also have to show that if $\gamma < \alpha$, then $q^\alpha{\restriction}\gamma \leq q^\gamma$. Let r be the componentwise union of $\langle p^i \colon i < \delta\rangle$. We first show that the sequence $\langle p^i \colon i < \delta\rangle$ has the property that for every regular $\theta \in [\eta^+, \alpha)$, either C-supp$(p^i) \cap [\theta, \theta^+) = \emptyset$ for all $i < \delta$ or the following hold:

(i) sup(S-supp$(r) \cap \theta$) $>$ sup(S-supp$(p^i) \cap \theta$) for all $i < \delta$,

(ii) for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, $q^{\gamma^\oplus} \Vdash \sup r^{**}_\gamma =$ sup(S-supp$(r) \cap \theta$),

(iii) for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, f_γ[sup(S-supp$(r) \cap \theta$)] $\supseteq$ sup(S-supp$(r) \cap \theta$),

(iv) for $\gamma_0 < \gamma_1$ both in C-supp$(r) \cap [\theta, \theta^+)$, f_{γ_0}[sup(S-supp$(r) \cap \theta$)] is a proper initial segment of f_{γ_1}[sup(S-supp$(r) \cap \theta$)]

(v) for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, q^γ forces that r_γ has a $P_{\sup(\text{S-supp}(r) \cap \theta)}$-name.

(vi) if θ is inaccessible, sup(S-supp$(r) \cap \theta$) $\geq$ card(C-supp$(r) \cap [\theta, \theta^+)$).

Properties (i) and (ii) immediately follow from Property (ii) in Definition 8.7. Properties (iii) and (iv) follow as Property (iii) in Definition 8.7 implies that for every regular $\theta \in [\eta^+, \alpha)$, $\sup(\text{S-supp}(r) \cap \theta)$ belongs to $C_{\text{C-supp}(r)\cap[\theta,\theta^+)}$. Property (v) follows from Property (i) in Definition 8.7, Property (vi) follows from Property (iv) in Definition 8.7.

Now we show, using (i)-(vi), that we can form the η^+-strategic lower bound q for $\langle p^i : i < \delta\rangle$ as in definition 8.9: Assume $\theta \in [\eta^+, \alpha)$ is regular, $\operatorname{card}\gamma = \theta$. Given (i)-(iv), $q^{\gamma^\oplus}$ decides $\sup r^{**}_\gamma$ and forces ot $f_\gamma[\sup r^{**}_\gamma] \geq \sup(\text{S-supp}(r) \cap \theta)$ to be distinct from ot $f_\xi[\sup r^{**}_\xi]$ for every $\xi < \gamma$. By (v), q^γ forces that r_γ has a $P_{\sup r^{**}_\gamma}$-name, allowing us to satisfy (1) as in definition 8.9. (2) in definition 8.9 can obviously be satisfied. Finally (vi) implies that $\text{S-supp}(q) \setminus \text{S-supp}(r)$ (and hence $\text{S-supp}(q)$) is bounded below every regular cardinal and hence q actually is a condition in P_α. $\Box$

Note: To be exact, note that we assumed our conditions p to satisfy property A1: $\forall\gamma\ \mathbf{1}_{P_{\gamma^\oplus}} \Vdash p^{**}_\gamma \in C(G^\oplus_\gamma)$. This will usually not be the case for q as obtained above. But, as can be seen from the construction, it will be the case that

$$\forall\gamma\ u_\eta(q)\restriction\gamma^\oplus \Vdash q^{**}_\gamma \in C(G^\oplus_\gamma).$$

Thus we may replace q by an equivalent and η^+-strategically equivalent q' satisfying A1, where we say that q and q' are η^+-strategically equivalent iff $u_\eta(q') \leq u_\eta(q)$ and $u_\eta(q) \leq u_\eta(q')$.

Claim 8.11 (Induced Strategic Belowness)
Assume $\eta < \alpha$ is a cardinal, α is a limit ordinal, $p, q \in P_\alpha$, $\langle \alpha_j : j < \operatorname{cof}\alpha\rangle$ is cofinal in α and increasing with $\alpha_0 > \eta$ such that for every $j < \operatorname{cof}\alpha$, $q\restriction\alpha_j$ is η^+-strategically below p. Then q is η^+-strategically below p.

Proof: Immediate from definition 8.7. $\Box$

Claim 8.12 (Existence of induced strategic lower bounds)
Assume $\eta < \alpha$ is a cardinal, α is a limit ordinal, $\kappa = \operatorname{card}\alpha$, $\langle p^i : i < \delta\rangle$ is a sequence of conditions of limit length $\delta < \eta^+$ in P_α, $\langle \alpha_j : j < \operatorname{cof}\alpha\rangle$ is cofinal in α and increasing such that $\alpha_0 > \eta$ and:

- *$\forall i < \delta$ there exists $n < \operatorname{cof}\alpha$ such that $p^{i+1}\restriction\alpha_n$ is η^+-strategically below p^i and $p^{i+1}[\alpha_n, \alpha) = p^i[\alpha_n, \alpha)$.*
- *$\forall j < \operatorname{cof}\alpha$ there are unboundedly many $i < \delta$ for which there exists $n \geq j$ s.t. $p^{i+1}\restriction\alpha_n$ is η^+-strategically below p^i.*

Then the η^+-strategic lower bound for $\langle p^i : i < \delta\rangle$ exists and is η^+-strategically below p^0.

Proof: By Claims 8.8 and 8.10, we know that for every $j < \operatorname{cof}\alpha$, the η^+-strategic lower bound for $\langle p^i\restriction\alpha_j : i < \delta\rangle$ exists and denote it by q^j. Let p^δ be the componentwise union of the q^j, $j < \operatorname{cof}\alpha$, and note that whenever $j < k < \operatorname{cof}\alpha$, $q^k \leq q^j$ and for every γ of regular cardinality, $\langle (q^j)^{**}_\gamma : j < \operatorname{cof}\alpha\rangle$ is eventually constant. It is thus easily seen that p^δ is a condition in P_α extending each p^i. The final statement of the claim follows by claims 8.8 and 8.11. $\Box$

Definition 8.13 (reducing a dense set) *If D is a dense subset of P_α and $\eta < \alpha$ is a cardinal, we say that q reduces D below η if for every $r \in P_\alpha$ with $u_\eta(r) \leq u_\eta(q)$, there is $s \leq r$ with $u_\eta(s) = u_\eta(r)$ and s meets D in the sense that $\exists d \in D\ s \leq d$.*

Definition 8.14 (equivalent dense set) *If P is a notion of forcing and $D \subseteq P$ we say that D is an* equivalent dense subset of P *if for every $p \in P$ there is $d \in D$ so that $d \leq p$ and $p \leq d$, i.e. p and d are* equivalent.

The central technical theorem of our paper at its core will establish that our iteration P is Δ-distributive. Before stating that theorem, we will provide the reader with the definition of Δ-distributivity, which is originally given in [1] and restated here in a less general version, slightly adapted to our iteration P:

Definition 8.15 *We say P_α is Δ-distributive if whenever $\langle D_i : i < \operatorname{card} \alpha\rangle$ are dense subsets of P_α and $p \in P_\alpha$, there is $q \leq p$ which reduces D_i below i^+ for every i, where we let $i^+ = \omega$ for finite i.*

Now we adapt this definition to the context of class forcing:

Definition 8.16 *We say that P is Δ-distributive at κ if whenever $\langle D_i : i < \kappa\rangle$ is a definable sequence of dense classes of P and $p \in P$, then there is $q \leq p$ which reduces D_i below i^+ for every i. We say that P is Δ-distributive if P is Δ-distributive at κ for every uncountable cardinal κ.*

Theorem 8.17 *Suppose $\omega \leq \eta < \alpha$, $\eta \in$* **Card** *and $\kappa = \operatorname{card} \alpha$. Then the following hold:*

1. *[Strategic Successors, Strategic Closure]*
   If $\alpha^* \geq \alpha$, $p \in P_{\alpha^*}$, then for any $q \leq p{\restriction}\alpha$ there exists $r \leq q$ which is η^+-strategically below p. If η is regular we can additionally ensure that $l_\eta(r) = l_\eta(q)$, therefore $u_\eta(P_\alpha)$ and $u_\eta(P_\alpha^\oplus)$ are both η^+-strategically closed.

2. *[Early Information]*
   If $p \in P_\alpha$, then there is $q \leq p$ so that $q{\restriction}i^\oplus$ forces that q_i^{**} has a P_γ-name for some $\gamma < \operatorname{card} i$ whenever $i \in$ C-supp(q), $i \geq \eta^+$ and a P_γ-name for some $\gamma < \nu$ if $\operatorname{card} i = \nu^+$ and $\nu \geq \eta$ is singular. Moreover there is such q for which q_i has a P_γ-name for some $\gamma < \operatorname{card} i$ whenever $\operatorname{card} i \geq \eta$ is singular or equal to ω. If q satisfies all of the above, we say that q has early information above η. If $\eta = \omega$, we say that q has early information. If η is regular, we can ensure that $l_\eta(q) = l_\eta(p)$ in the above.

3. *[Smallness of the iteration]*
   If α is regular, P_α has a dense subset of size α. Otherwise P_α has a dense subset of size α^+.

4. *[Chain Condition]*
   Assume η is regular. If J is an antichain of P_α such that $u_\eta(p) \parallel u_\eta(q)$ whenever p and q are in J, then $|J| \leq \eta$.

5. *[Reducing dense sets]*

- *Assume η is regular and $\langle D_i \colon i < \eta\rangle$ is a collection of dense subsets of P_α. Then any condition in P_α can be strengthened to a condition q with the same η-sized part so that for every $i < \eta$, q reduces D_i below η.*
- *Assume $\eta \leq \alpha$ is singular and $\langle D_i \colon i < \eta\rangle$ is a collection of dense subsets of P_α. Then for any $\zeta < \eta$, any condition in P_α can be strengthened to a condition q with the same ζ-sized part so that for every $i < \eta$ there exists $\eta_i < \eta$ so that q reduces D_i below η_i.*
- *P_α is Δ-distributive.*

6. *[Early names]*
   - *Assume η is regular and $\dot f$ is a P_α-name for an ordinal-valued function with domain η. Then any condition in P_α can be strengthened to a condition q with the same η-sized part forcing that for every $i < \eta$, there is a maximal antichain of size at most η below q deciding $\dot f(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. We say that q reduces $\dot f$ below η. In particular, such q forces that $\dot f$ has a P_γ-name for some $\gamma < \eta^+$.*
   - *Let $\eta \leq \alpha$ be a singular cardinal. Let $\dot f$ be a P_α-name for an ordinal-valued function with domain η. Then for any $\zeta < \eta$, any condition in P_α can be strengthened to a condition q with the same ζ-sized part, forcing that for every $i < \eta$, there is a maximal antichain of size less than η below q deciding $\dot f(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. We say that q reduces $\dot f$ below η. In particular, such q forces that $\dot f$ has a P_η-name.*
7. *[Preservation of the* GCH*]*
   After forcing with P_α, GCH *holds.*
8. *[Covering, Preservation of Cofinalities]*
   For every cardinal θ, for every $p \in P_\alpha$ and every P_α-name $\dot x$ for a set of ordinals of size θ there is a set X in $\mathbf{V}$ of size θ and an extension q of p such that $q \Vdash \dot x \subseteq X$. Therefore forcing with P_α preserves all cofinalities.
9. *[Club Extendibility]*
   If $I \subseteq \alpha$ is s.t. $\operatorname{card}(I \cap \theta^+) < \theta$ for every regular θ, $I \subseteq \bigcup_{\theta\ regular}[\theta, \theta^+)$ and $\langle \bar\delta^i \colon i \in I\rangle$ is s.t. $\bar\delta_i < \operatorname{card} i$ for every $i \in I$, then for every $p \in P_\alpha$, there is $q \leq p$ s.t. $\forall i \in I\ q\restriction i^\oplus \Vdash \max q_i^{**} \geq \bar\delta_i$. Moreover if $\eta < \operatorname{card}\min I$ is regular, there is such q with $l_\eta(q) = l_\eta(p)$.

Proof: By induction on α.

Proof of 1 and 2: Starting from p and q as in the statement of 1, we will find $r \leq q$ which is η^+-strategically below p and has early information above η and thus prove 1 and 2 simultaneously. We distinguish several cases for α assuming that $\eta < \kappa$, as 1 is immediate and 2 is easy otherwise. Note that (iii) and (iv) in Definition 8.7 are always easy to satisfy by choosing r such that $\sup(\operatorname{supp}(r) \cap \theta)$ is sufficiently large whenever $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$ and $\theta \in (\eta, \alpha)$ is regular. We will thus ignore (iii) and (iv) in the following and concentrate only on making (i) and (ii) in Definition 8.7 hold.

Case 1: $\alpha = \beta + 1$ is a successor ordinal Using 6 inductively, if $\operatorname{card} \beta$ is regular, strengthen q to q^* s.t. $q^* \restriction \beta$ forces that $(q^*)_\beta = q_\beta$ has a $P_{\sup \text{S-supp}(q^*) \cap \kappa}$-name by first reducing q_β below η and then sufficiently increasing S-supp(q^*). If $\operatorname{card} \beta$ is singular, reduce q_β below η, which ensures that $(q^*)_\beta = q_\beta$ has a P_γ-name for some $\gamma < \eta$. Also make sure that $q^* \restriction \beta^\oplus$ reduces q^{**}_β below η and let $(q^*)^{**}_\beta = q^{**}_\beta$. Now we use 1 and 2 inductively to find $r \leq q^*$ such that $r \restriction \beta$ is η^+-strategically below p and has early information above η. Choose δ such that

- $\delta > \eta, \sup(\text{S-supp}(p) \cap \kappa)$,
- $q^* \restriction \beta^\oplus$ forces that $\delta > \sup(q^*)^{**}_\beta$ and
- $\operatorname{ot} f_\beta[\delta] > \sup(\text{S-supp}(r) \cap \kappa)$.

Let $r_\beta = (q^*)_\beta$, $r^{**}_\beta = (q^*)^{**}_\beta \cup \{\delta\}$ and let $r_{\operatorname{ot} f_\beta[\delta]}$ be a $P_{\operatorname{ot} f_\beta[\delta]}$-name which is forced by $r \restriction \beta$ to equal r_β. Then $r \leq q$ is η^+-strategically below p and has early information above η, as desired.

Case 2: α is a limit ordinal, $\operatorname{cof} \alpha = \kappa$ If κ is singular, 1 is trivial. To show 2 holds, first ensure that q_β has a P_γ-name for some $\gamma < \kappa$ for every $\beta \in \text{S-supp}(q) \cap [\kappa, \alpha)$ using 6 inductively and 1. 2 then follows using 2 inductively. Assume κ is regular and let $\bar{\alpha} = \sup(\text{C-supp}(q) \cap \alpha) < \alpha$. Use 1 and 2 inductively to find $r \leq q$ such that $r \restriction \bar{\alpha}$ is η^+-strategically below p, has early information above η and $r[\bar{\alpha}, \alpha) = q[\bar{\alpha}, \alpha)$. Then $r \leq q$ is η^+-strategically below p and has early information above η, as desired.

Case 3: α is a limit ordinal, $\operatorname{cof} \alpha < \kappa$ Let $\eta^* = \max\{\eta, \operatorname{cof} \alpha\}$. Let $\langle \alpha_i : i < \operatorname{cof} \alpha \rangle$ be an increasing sequence that is cofinal in α with $\alpha_0 > (\eta^*)^+$. We build a decreasing sequence of conditions $\langle q^i : i \leq \operatorname{cof} \alpha \rangle$ as follows.

- Let q^0 be such that $q^0 \restriction \alpha$ is η^+-strategically below q.
- Given q^i, let q^{i+1} be so that $q^{i+1} \restriction \alpha_i$ is $(\eta^*)^+$-strategically below q^i, has early information above η^* and $q^{i+1}[\alpha_i, \alpha) = q^i[\alpha_i, \alpha)$.
- If $\delta \leq \operatorname{cof} \alpha$ is a limit ordinal, let q^δ be the $(\eta^*)^+$-strategic lower bound of $\langle q^i : i < \delta \rangle$, which exists by Claim 8.12.

$q^{\operatorname{cof} \alpha} \leq q$ is $(\eta^*)^+$-strategically below p by Claim 8.12 and has early information above η^*, hence by our assumption on q^0 above, $q^{\operatorname{cof} \alpha}$ is η^+-strategically below p. We may choose $r \leq q^{\operatorname{cof} \alpha}$ such that $r \restriction \alpha_0$ has early information above η and $r[\alpha_0, \alpha) = q^{\operatorname{cof} \alpha}[\alpha_0, \alpha)$. Then r is as desired.

Proof of 3: We prove that $D_\alpha := \{p \in P_\alpha : (\forall \theta\ \theta \text{ is a singular cardinal} \rightarrow \forall \gamma \in \text{S-supp}(p) \cap [\theta, \theta^+)\ \exists \xi < \theta\ p_\gamma \text{ has a } P_\xi\text{-name}) \wedge (\forall \theta \in \mathbf{Card}\ \exists \gamma\ \text{S-supp}(p) \cap [\theta, \theta^+) = [\theta, \gamma))\}$ has an equivalent dense subset E_α of size α if α is regular and of size α^+ if α is singular. Note that D_α itself is dense in P_α by 2.

If α is regular, conditions in P_α have bounded support below α, thus the claim follows by 3 inductively.

If $\alpha = \beta + 1$ is a successor ordinal, assume $p \in D_\alpha$ and D_β has an equivalent dense subset E_β of size α^+ inductively. If κ is regular, p_β can be identified with an antichain of E_β below $p\restriction\beta$. Since for any two elements a_0, a_1 of such an antichain, $u_\kappa(a_0) \parallel u_\kappa(a_1)$, such an antichain will have size at most κ using 4 inductively, thus there are α^+-many possible choices for p_β. p_β^{**} can be identified with a collection of less than κ-many antichains of E_β below $p\restriction\beta$, each element-wise paired with ordinals below κ, thus using similar arguments as before, there are α^+-many possible choices for p_β^{**}. If $\operatorname{card}\beta$ is singular, p_β has a P_γ-name for some $\gamma < \operatorname{card}\beta$ and hence there are less than α-many possible choices for p_β in this case. This yields that P_α has a dense subset of size $alpha^+$.

If α is singular and $p \in D_\alpha$, we can modify p to an equivalent p' such that for every $\gamma < \alpha$, $p'\restriction\gamma \in E_\gamma$. Hence P_α has a dense subset of size $\prod_{\gamma<\alpha} \gamma^+ \leq \alpha^+$.

Proof of 4: Assume J is an antichain of P_α such that whenever p and q are in J, $u_\eta(p) \parallel u_\eta(q)$. We may assume that all conditions in J are from E_α and have early information. Assume for a contradiction that J has size at least η^+. By 3 inductively, $p\restriction\eta$ is the same for η^+-many conditions in J and thus we may assume it is the same for all conditions in J. By GCH and a Δ-system argument, there is $W \subseteq J$ of size η^+ and a size less than η subset A of η^+ such that C-supp$(p) \cap$ C-supp$(q) \cap [\eta, \eta^+) = A$ whenever $p \neq q$ are both in W. But using that GCH holds after forcing with P_η by 7 inductively, it follows that for η^+-many conditions p in W, $\langle p(i)(1) \colon i \in A\rangle$ is the same (modulo equivalence). But - using the assumption that $u_\eta(p) \parallel u_\eta(q)$ - any two such conditions are compatible, thus W (and hence also J) is not an antichain.

Proof of 5:

Claim 8.18 *Assume $p \in P_\alpha$, D is a dense subset of P_α and $\nu < \alpha$ is regular. Then there is $q \leq p$ s.t. $l_\nu(q) = l_\nu(p)$ and q reduces D below ν.*

Proof: Build a decreasing sequence of conditions in P_α below p as follows: Let $p^0 = p$. Choose q^0 so that $q^0 \leq p^0$ and $q^0 \in D$. By possibly passing to an equivalent condition, we may also ensure that $u_\nu(q^0) \leq u_\nu(p^0)$. At stage $j+1$, let $p^{j+1} \leq p^0$ be any condition incompatible to all q^k, $k \leq j$, such that $u_\nu(p^{j+1}) = u_\nu(q^j)$ if such exists and choose q^{j+1} such that:

- $q^{j+1} \leq p^{j+1}$,
- $q^{j+1} \in D$ and
- $u_\nu(q^{j+1})$ is chosen according to the strategy for ν^+-strategic closure below $\langle u_\nu(q^k) \colon k \leq j\rangle$.

At limit stages $j < \nu^+$, let $p^j \leq p^0$ be a condition which is incompatible to all q^k, $k < j$ so that for all $k < j$, $u_\nu(p^j) \leq u_\nu(q^k)$ if such exists. Note that a p^j satisfying the latter condition can always be found by the strategic choice of the $u_\nu(q^k)$. Choose $q^j \leq p^j$ so that $q^j \in D$ and $u_\nu(q^j) \leq u_\nu(p^j)$. Proceed until at some stage j no condition p^j as above can be chosen. By 4, this will be the case for some $j < \nu^+$. We can then find $q \in P_\alpha$ so that $u_\nu(q) \leq u_\nu(q^k)$ for every $k < j$ and $l_\nu(q) = l_\nu(p)$. By our construction, q reduces D below ν. $\square$

Using the claim for $\nu = \eta$, the case of regular η follows immediately, applying 1 once more. For the case of $\eta \leq \alpha$ singular, choose a continuous, cofinal in η, increasing sequence $\langle \eta_i \colon i < \operatorname{cof}\eta\rangle$ of cardinals where each η_{i+1} is regular and

$\eta_0 > \operatorname{cof}\eta$. Build a sequence of conditions $\langle q^i \colon i < \operatorname{cof}\eta\rangle$ so that $q^{i+1} = q^i$ for limit ordinals i and otherwise q^{i+1} reduces the first η_i-many given dense sets below η_i, $l_{\eta_i}(q^{i+1}) = l_{\eta_i}(q^i)$ and $u_{\eta_i}(q^{i+1})$ is chosen according to the strategy for $(\eta_i)^+$-strategic closure of $u_{\eta_i}(P_\alpha)$ for each $i < \operatorname{cof}\eta$. At limit stages $i \leq \operatorname{cof}\eta$, we may take lower bounds of the conditions obtained so far using stability of the obtained sequence of conditions below η_i together with $(\eta_i)^+$-strategic closure of $u_{\eta_i}(P_\alpha)$ provided by 1.

Proof of 6: Apply 5 to reduce the dense sets D_i of conditions which decide $\dot f(i)$, $i < \eta$.

Proof of 7 and 8: These follow from Δ-distributivity of P_α, see [1], Lemma 2.10 and Lemma 2.13.

Proof of 9: Given $p \in P_\alpha$, $I \subseteq \alpha$ and $\langle \bar\delta^i \colon i \in I\rangle$ as in the statement of the claim, let $p' \leq p$ be such that for every θ with $I \cap [\theta, \theta^+) \neq \emptyset$, we have that $\sup(\operatorname{supp}(p') \cap \theta) \geq \sup(\{\bar\delta^i \colon i \in I \cap [\theta,\theta^+)\})$. Now let $q \leq p'$ be η^+-strategically below p' (or ω_1-strategically below p' if no $\eta < \operatorname{card}\min I$ is specified). It follows that q is as desired. If $\eta < \operatorname{card}\min I$ is regular, we may easily ensure that $l_\eta(q) = l_\eta(p)$ in the above.

Corollary 8.19 *P preserves* ZFC*, cofinalities, cardinals and the* GCH*.*

Proof: By Lemma 2.23 of [1], Δ-distributivity of P implies that P is tame and hence preserves ZFC and cofinalities. GCH preservation is immediate from Theorem 8.17, Clauses 7 and 6. □

Note: For every i of regular cardinality, $\bigcup_{p\in G} p_i^{**}$ is club in $\operatorname{card} i$ for any P-generic G. This is immediate from theorem 8.17, 9 above.

Claim 8.20 *P forces Local Club Condensation.*

Proof: We will verify the equivalent form of Local Club Condensation introduced in Lemma 4.1. Let G be P-generic. Let A be the generic predicate obtained from G, i.e. $\alpha \in A \leftrightarrow \exists p \in G\ p{\restriction}\alpha \Vdash p_\alpha = 1$. Note that $\mathbf{V}[G]{=}\mathbf{L}[A]$ as any set of ordinals in $\mathbf{V}$ is coded into A. We claim that $\langle M_\alpha \colon \alpha \in \mathrm{Ord}\rangle$ witnesses Local Club Condensation in $\mathbf{V}[G]$ with $M_\alpha{=}L_\alpha[A]$. First assume α has regular uncountable cardinality κ. Note that for $\beta \in \alpha \setminus \kappa$ we have $A(\beta) = A(\operatorname{ot} f_\beta[\delta])$ for all δ in the club $\bigcup_{p\in G} p_\beta^{**} \subseteq \kappa$. It follows that for a club C of $\delta < \kappa$, $A(\beta) = A(\operatorname{ot} f_\beta[\delta])$ and moreover $f_\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \kappa$; this is seen using Lemma 1.1. Let, as in Lemma 4.1, F denote the function $(f,x) \mapsto f(x)$ whenever $f \in M_\alpha$ is a function with $x \in \operatorname{dom}(f)$. Let $M_\alpha^* = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, F, \ldots)$ be a Skolemized structure for a countable language and for any $X \subseteq \alpha$ let $M_\alpha^*(X)$ be the least substructure of M_α^* containing X as a subset. Consider the continuous chain $\langle M_\alpha^*(f_\alpha[\delta]) \colon \delta \in D\rangle$, where D consists of all elements δ of C s.t. $\delta = f_\alpha[\delta] \cap \kappa$ and $f_\alpha[\delta] = M_\alpha^*(f_\alpha[\delta]) \cap \mathrm{Ord}$. Then $M_\alpha^*(f_\alpha[\delta])$ condenses for each $\delta \in D$.

It remains to verify Local Club Condensation for α when α has singular cardinality κ. Suppose that $\beta \geq \alpha$ and $\dot S \in \mathbf{V}$ is a P_β-name for a structure

$(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, F, \ldots)$ for a countable language in $\mathbf{L}[A]$ such that the $\dot{S}$-closure of κ is all of M_α, with F as above. We show that any condition $p \in P_\beta$ has an extension q^* which forces that there is a continuous chain $\langle Y_\gamma \colon \gamma \in C\rangle$ of condensing substructures of $\dot{S}$ whose domains $\langle y_\gamma \colon \gamma \in C\rangle$ have union M_α such that $\langle y_\gamma \cap \mathrm{Ord} \colon \gamma \in C\rangle$ belongs to the ground model, where C is a closed unbounded subset of $\mathbf{Card} \cap \kappa$, each y_γ has cardinality γ and contains γ as a subset. Choose C to be any club subset of $\mathbf{Card} \cap \kappa$ of ordertype $\operatorname{cof} \kappa$ whose minimum is either ω or a singular cardinal and is at least $\operatorname{cof} \kappa$. Choose some large (w.r.t. β), regular ν.

Let $p^0 = p$. We may assume $\text{C-supp}(p^0) \cap [\theta^+, \theta^{++}) \neq \emptyset$ for every $\theta \in C$. Given p^i, let $\langle M^i_\theta \colon \theta > \min C, \text{C-supp}(p^i) \cap [\theta, \theta^+) \neq \emptyset\rangle$ be a sequence of domains of elementary submodels of H_ν such that each M^i_θ has size less than θ, is transitive below θ and contains θ, p^i, $\dot{S}$ and $\langle M^j_\theta \colon j < i\rangle$ as elements. Moreover make sure that $M^i_{\theta_0} \subseteq M^i_{\theta_1}$ whenever $\theta_0 < \theta_1$ and that $M^i_{\gamma^+} = \bigcup_{\delta \in C \cap \gamma} M_\delta$ whenever γ is a limit point of C. Latter is possible as $\min C \geq \operatorname{cof} \kappa$ and we may thus sufficiently enlarge the $M^i_{\delta^+}$, $\delta \in C \cap \gamma$, after choosing $M^i_{\gamma^+} \supseteq \bigcup_{\delta \in C \cap \gamma} M^i_{\delta^+}$ in the first place. Choose $p^{i+1} \leq p^i$ such that p^{i+1} reduces every dense subset of P_β in M^i_θ below $\operatorname{card} M^i_\theta$, is ω_1-strategically below p^i and such that $\sup(\text{S-supp}(p^{i+1}) \cap \theta) \geq \operatorname{card}(M^i_\theta)$ and $\geq M^i_\theta \cap \theta$ whenever $\text{C-supp}(p^i) \cap [\theta, \theta^+) \neq \emptyset$.

Let r be the componentwise union of $\langle p^i \colon i < \omega\rangle$, let q be the ω_1-strategic lower bound. Let $y_\gamma := \bigcup_{i<\omega} M^i_{\gamma^+}$ for every $\gamma \in C$. We have obtained the following properties for every $\gamma \in C$:

(1) y_γ is transitive below γ^+,

(2) $y_\gamma \cap [\gamma, \gamma^+) = \text{S-supp}(r) \cap [\gamma, \gamma^+)$,

(3) $y_\gamma \cap [\gamma^+, \gamma^{++}) = \text{C-supp}(r) \cap [\gamma^+, \gamma^{++})$,

(4) q forces that the $\dot{S}$-closure of y_γ intersected with Ord equals y_γ and

(5) q forces that $A \cap y_\gamma$ has a $P_{y_\gamma \cap \gamma^+}$-name.

(6) $\langle y_\gamma \colon \gamma \in C\rangle$ is continuous and increasing.

(1) is immediate as each $M^i_{\gamma^+}$ is transitive below γ^+, (2) and (3) follow by easy density and elementarity arguments. For (4), it suffices to show that the $\dot{S}$-closure of $M^i_{\gamma^+}$ intersected with the ordinals is forced by q to be contained in $M^{i+2}_{\gamma^+}$ for every $i < \omega$: We required that $M^i_{\gamma^+} \in M^{i+1}_{\gamma^+}$. Thus $D = \{t \in P_\beta \colon t \Vdash (\dot{S}\text{-closure of } M^i_{\gamma^+}) \cap \mathrm{Ord} \text{ is covered by a ground model set of size } \gamma\}$ is dense in P_β using clause 8 of Theorem 8.17, contained (as an element) in $M^{i+1}_{\gamma^+}$ and will thus be hit by p^{i+2}; (4) now follows as $p^{i+2} \in M^{i+2}_{\gamma^+}$: using elementarity, p^{i+2} forces that we can cover the $\dot{S}$-closure of $M^i_{\gamma^+}$ by a set in $M^{i+2}_{\gamma^+}$ of size γ; as $\gamma \subseteq M^{i+2}_{\gamma^+}$, this covering set will be contained (as a subset) in $M^{i+2}_{\gamma^+}$. (5) follows similar to (4) , using easy density arguments. (6) is immediate by our requirements on the M^i_θ.

Let π_γ be the collapsing map of y_γ. If $\xi \in y_\gamma \cap [\gamma^+, \gamma^{++})$, f_ξ is a bijection from γ^+ to ξ, hence $f_\xi \restriction (y_\gamma \cap \gamma^+)$ is a bijection from $y_\gamma \cap \gamma^+$ to $y_\gamma \cap \xi$ by elementarity, i.e. $\pi_\gamma(\xi) = \mathrm{ot}(f_\xi[y_\gamma \cap \gamma^+])$, therefore $q(\pi_\gamma(\xi)) = r(\xi)$. Now extend q to q^* such that for every $\xi \in y_\gamma$, $\xi \geq \gamma^{++}$, we have $q^*(\pi_\gamma(\xi)) = r(\xi)$; this is possible since if γ is inaccessible, $\sup(\text{S-supp}(r) \cap \gamma) = \mathrm{card}\, y_\gamma$ and whenever $\text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset$ and θ is inaccessible, $\sup(r^{**}_\zeta) = \sup(\text{S-supp}(r) \cap \theta) > \sup(C \cap \theta)^+$ for every $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$ by easy density arguments, hence when we form q out of r and have to set $q(\mathrm{ot}\, f_\zeta[\sup(r^{**}_\zeta)])$ to be equal to $q(\zeta)$ for $\zeta \in \text{C-supp}(r) \cap [\theta, \theta^+)$, we do not make any new requirements in the interval $[\gamma, \gamma^+)$ - note that $\mathrm{ot}\, f_\zeta[\sup(r^{**}_\zeta)] \geq \sup(r^{**}_\zeta)$. We thus made sure q^* forces Condensation for y_γ for every $\gamma \in C$. □

Theorem 8.21 *Local Club Condensation is consistent with the existence of an ω-superstrong cardinal.*

Proof: Assume κ is ω-superstrong, witnessed by the embedding $j\colon \mathbf{V} \to \mathbf{M}$. Let P be the Local Club Condensation forcing as defined at the beginning of this section. We want to show that forcing with P may preserve the ω-superstrength of κ. Let P^* denote the $\mathbf{M}$-version of P (using the definition of P in $\mathbf{M}$). Note that for every $n < \omega$, $P_{j^n(\kappa)} = P^*_{j^n(\kappa)}$. We want to find a $\mathbf{V}$-generic $G \subseteq P$ and an $\mathbf{M}$-generic $G^* \subseteq P^*$ such that $j''G \subseteq G^*$ and $V[G]_{j^\omega(\kappa)} \subseteq M[G^*]$. After finding a suitable $P_{j^\omega(\kappa)}$-generic $G_{j^\omega(\kappa)}$, we will let $G^*_{j^\omega(\kappa)}$ be $G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$. We will let G^* be the filter generated by $G^*_{j^\omega(\kappa)}$ together with the image of G under j. $\mathbf{V}[G]_{j^\omega(\kappa)} \subseteq \mathbf{M}[G^*]$ follows as every element of $\mathbf{V}[G]_{j^\omega(\kappa)}$ has a P-name in $\mathbf{V}_{j^n(\kappa)}$ for some $n < \omega$ by Clause 6 of Theorem 8.17. We have to show the following:

1. $G^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over $\mathbf{M}$.
2. G^* is P^*-generic over $\mathbf{M}$.
3. We can choose $G_{j^\omega(\kappa)}$ in such a way that $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$.

We will assume 3 for the moment and proof 1 and 2 using 3. We will then proof 3 without using either 1 or 2. Assume that j is given by an ultrapower embedding, which means that every element of $\mathbf{M}$ is of the form $j(f)(a)$ where f has domain $H_{j^\omega(\kappa)}$ and a belongs to $H_{j^\omega(\kappa)}$.

Proof of 1: Suppose $D \in \mathbf{M}$ is dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where $\mathrm{dom}(f) = V_{j^\omega(\kappa)}$ and $a \in V_{j^{n+1}(\kappa)}$ for some $n \in \omega$. Choose $p \in G_{j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is dense on $P_{j^\omega(\kappa)}$. The existence of p follows from Clause 5 of Theorem 8.17, using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$. Then $j(p)$ belongs to $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$ by 3 and reduces D below $j^{n+1}(\kappa)$. Hence $E := \{q \in P_{j^{n+2}(\kappa)}\colon q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D\}$ is dense below $j(p){\restriction}j^{n+2}(\kappa)$ in $P_{j^{n+2}(\kappa)}$. Since $G_{j^{n+2}(\kappa)}$ contains $j(p){\restriction}j^{n+2}(\kappa)$ and is $P_{j^{n+2}(\kappa)}$-generic over $\mathbf{M}$, $G_{j^{n+2}(\kappa)} \cap E \neq \emptyset$. Choose q in that intersection. Then $q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D \cap G^*_{j^\omega(\kappa)}$.

Proof of 2: Like 1, using that $j''G \subseteq G^*$ as an immediate consequence of 3.

Proof of 3: We will specify a master condition $q \in P_{j^\omega(\kappa)}$ so that $q \in G_{j^\omega(\kappa)}$ ensures $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$. Let $\dot{G}$ be the canonical name in $\mathbf{V}$ for the $P_{j^\omega(\kappa)}$-generic. We define r by letting, for all $\gamma \geq j(\kappa)$:

$r(\gamma)(0) = \bigcup_{p \in \dot{G}} j(p)(\gamma)(0)$ and $r^{**}_\gamma = \bigcup_{p \in \dot{G}} j(p)^{**}_\gamma$.

As we did earlier, we write S-supp(r) for $\{\gamma\colon r(\gamma)(0) \neq \check{\mathbf{1}}\}$ and C-supp(r) for $\{\gamma\colon r^{**}_\gamma \neq \check{\mathbf{1}}\}$. It is easily observed that S-supp(r) is bounded below every regular cardinal and that card(C-supp$(r) \cap \theta^+) < \theta$ for every regular cardinal θ. We want to form q out of r by setting, for every $\gamma \in$ C-supp(r):

- $q\restriction\gamma^\oplus \Vdash q^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$,
- If $\gamma \geq j(\kappa)^+$, choose $q_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]}$ such that $q\restriction\gamma \Vdash q_{\mathrm{ot}\, f_\gamma[\sup r^{**}_\gamma]} = r_\gamma$,

We also set $q(\gamma)(0) = r(\gamma)(0)$ for γ in S-supp(r) and let components other than the above have value $\check{\mathbf{1}}$. The following Claim will finish the proof of Theorem 8.21:

Claim 8.22 *1. $q \in P_{j^\omega(\kappa)}$.*

2. q extends $j(p)$ whenever $p\restriction\kappa = \mathbf{1}$.

3. Whenever $p \leq q$, $p \in G$, then $p \leq j(p)$; hence if $p \in G_{j^\omega(\kappa)}$, then $j(p) \in G_{j^\omega(\kappa)}$, i.e. $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$.

Proof of 1: We want to define, for every cardinal $\theta \geq j(\kappa)^+$ with C-supp$(r) \cap [\theta, \theta^+) \neq \emptyset$ a model M_θ: Choose some large (w.r.t. $j^\omega(\kappa)$), regular (in $\mathbf{M}$) $\nu \in$ range(j), fix a well-ordering R of $H_{j^{-1}(\nu)}$ and let M_θ be the Skolem Hull of sup(S-supp$(r) \cap \theta) \cup$ (C-supp$(r) \cap [\theta, \theta^+)$) in $H_\nu{}^{\mathbf{M}}$ according to $j(R)$.

Claim 8.23 *For all θ with* C-supp$(r) \cap [\theta, \theta^+) \neq \emptyset$,

- $M_\theta \cap \theta =$ sup(S-supp$(r) \cap \theta) = \sup r^{**}_\theta$.
- $M_\theta \cap [\theta, \theta^+) =$ C-supp$(r) \cap [\theta, \theta^+)$.

Proof: For the first statement, assume $\xi \in M_\theta$, $\xi < \theta$. Then ξ can be defined using finite sets of parameters $S_0 \subseteq$ sup(S-supp$(r) \cap \theta$) and $S_1 \subseteq$ C-supp$(r) \cap [\theta, \theta^+)$. Choose $p \in G$ so that $S_0 \subseteq$ S-supp$(j(p) \cap \theta)$ and $S_1 \subseteq$ C-supp$(j(p)) \cap [\theta, \theta^+)$. Let $t \leq p$ in G be such that whenever C-supp$(p) \cap [\rho, \rho^+) \neq \emptyset$, sup(S-supp$(t) \cap \rho) \geq \sup(H^{H_{j^{-1}(\nu)}}$(sup(S-supp$(p) \cap \rho) \cup$ (C-supp$(p) \cap [\rho, \rho^+)$)) $\cap \rho)$. It follows that $\xi <$ sup(S-supp$(j(t)) \cap \theta <$ sup(S-supp$(r) \cap \theta$), which is equal to $\sup r^{**}_\theta$ by the usual arguments. The proof of the second statement is similar. $\square$

Let π_θ denote the collapsing map of M_θ and note that for $\gamma \in$ C-supp$(r) \cap [\theta, \theta^+)$, $\pi_\theta(\gamma) = \mathrm{ot}\, f_\gamma(\sup r^{**}_\gamma)$. By the usual arguments, it follows that our above definition of q has no conflicting requirements and q has appropriate supports in order to be a condition in $P_{j^\omega(\kappa)}$.

Proof of 2: Observe that C-supp$(r) \cap [j(\kappa), j(\kappa^+)) = j''[\kappa, \kappa^+)$ and $\sup r_{j(\kappa)^{**}} = \kappa$. Hence $\pi_{j(\kappa)}(\gamma) = j^{-1}(\gamma)$ for $\gamma \in$ C-supp$(r) \cap [j(\kappa), j(\kappa^+))$. This, together with the ususal argument at cardinals $> j(\kappa)$ implies 2.

Proof of 3: Assume $p \leq q$. Then $p \leq j(p)$ as $p\restriction\kappa = j(p)\restriction\kappa$ and $p[\kappa, j^\omega(\kappa)) \leq q \leq j(p)[\kappa, j^\omega(\kappa))$. $\square_{\text{Claim 8.22}}$ $\square_{\text{Theorem 8.21}}$

Note: Many other (smaller) large cardinal properties can be preserved while forcing with P, for example measurable cardinals.

9 A possible future application

See [3], where this is turned into an actual application.

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