Ideal Topologies

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Ideals

Let $\kappa$ be a cardinal (most of the time: regular and uncountable). An ideal on $\kappa$ is a collection of small subsets of $\kappa$.

**Definition 1**

A collection $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ is an ideal (on $\kappa$) if:

- $\emptyset \in \mathcal{I}$, $\kappa \notin \mathcal{I}$,
- $\forall A, B \ A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$, and
- $\forall A, B \ A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

We will also demand our ideals to be non-principal, that is $\{\alpha\} \in \mathcal{I}$ for every $\alpha < \kappa$, and we demand them to be closed under $<\kappa$-unions.

Examples: the bounded ideal, the nonstationary ideal $\text{NS}_\kappa$, ...

Note that our additional demands imply that any ideal contains the bounded ideal.
Cantor spaces

Let $\mathcal{P}(\kappa) \approx \kappa^2 = \{g \mid g: \kappa \to 2\}$. This collection is usually given a topology based on bounded ideal: The $\kappa$-Cantor space is the set $\kappa^2$ with the topology given by the basic open sets (which are also easily seen to be closed)

$$[f] = \{g \in \kappa^2 \mid f \subseteq g\}$$

for $f \in <\kappa^2 = \bigcup_{\alpha < \kappa} \alpha^2$.

However, we would obtain the same topology if we took as basic open sets all sets of the form $[f]$ where $f$ is a partial function from $\kappa$ to 2 of size less than $\kappa$, i.e. a function with domain in the bounded ideal.

If $\kappa = \omega$, this is certainly the most natural topology on the space $\omega^2$. However, in particular if $\kappa > \omega$, we can equally consider topologies based on ideals other than $\text{bd}_\kappa$. 
Let $\mathcal{I}$ be an ideal on $\kappa$.

**Definition 2**

The $\mathcal{I}$-topology is the topology with the basic open sets of the form $[f]$ where $\text{dom}(f) \in \mathcal{I}$ (as before, each $[f]$ is also closed).

- Open sets are (as always) arbitrary unions of basic open sets, and we call open sets in the $\mathcal{I}$-topology $\mathcal{I}$-open sets, and similarly use $\mathcal{I}$-closed, ...
- Note that the $\mathcal{I}$-topology refines the bounded topology: it has more open sets (and thus also more closed sets, ...).
- In case $\mathcal{I} = \text{NS}_\kappa$, the basic open sets are thus induced by functions with non-stationary domain. We call the resulting topology the *nonstationary topology* (on $\kappa$).
Basic cardinality observations

In the bounded topology on $\kappa^2$, one usually assumes $2^{<\kappa} = \kappa$, and then there are $\kappa$-many basic open sets, and $2^\kappa$-many open sets (while there are $2^{2^\kappa}$-many subsets of $\kappa^2$). If $\mathcal{I}$ contains an unbounded subset of $\kappa$ however, we get the maximal possible number of open sets:

**Observation 3**
Assume that $\mathcal{I}$ contains an unbounded subset $A$ of $\kappa$. Then,

1. there are $2^\kappa$-many disjoint $\mathcal{I}$-basic open sets with union $\kappa^2$, and
2. there are $2^{2^\kappa}$-many $\mathcal{I}$-open sets.
Tall ideals

Many natural properties of ideals correspond to prominent examples of subsets of our spaces to be topologically simple.

Tallness is a very natural property of ideals:

Definition 4
An ideal $\mathcal{I}$ is *tall* if every unbounded set has an unbounded subset in $\mathcal{I}$.

Observation 5
$\text{NS}_\kappa$ is tall.
On the collection of unbounded sets

Let \( \text{ub}_\kappa \subseteq \kappa^2 \) denote the collection of unbounded subsets of \( \kappa \).

**Observation 6**

\( \mathcal{I} \) is tall if and only if \( \text{ub}_\kappa \) is \( \mathcal{I} \)-open.

**Proof:** First, assume that \( \mathcal{I} \) is a tall ideal. Let \( c_i^A \) denote the constant function with domain \( A \) and value \( i \). Then,

\[
\text{ub}_\kappa = \bigcup \{ [c_1^A] \mid A \in \mathcal{I} \cap \text{ub}_\kappa \} \text{ is } \mathcal{I} \text{-open.}
\]

Assume \( \text{ub}_\kappa \) is \( \mathcal{I} \)-open. Given \( A \in \text{ub}_\kappa \), there is an \( \mathcal{I} \)-basic open set \([f] \subseteq \text{ub}_\kappa \) with \( A \in [f] \). Since \([f] \subseteq \text{ub}_\kappa \), \( f \) takes value 1 on some \( B \in \text{ub}_\kappa \) with \( B \subseteq \text{dom}(f) \in \mathcal{I} \). Hence, \( B \subseteq A \) is unbounded, as desired. \( \square \)

A more intricate argument shows that for no \( \mathcal{I} \) is \( \text{ub}_\kappa \) an \( \mathcal{I}-F_\sigma \) set (a \( \kappa \)-union of \( \mathcal{I} \)-closed sets). Hence, if \( \mathcal{I} \) is tall, then there is an \( \mathcal{I} \)-open set that is not \( \mathcal{I}-F_\sigma \).
The Club subsets of $\kappa$

Let $\text{Club}_\kappa$ denote the collection of club subsets of $\kappa$.

A similar argument as for $\text{ub}_\kappa$ shows: If $\mathcal{I} = \text{NS}_\kappa$, then $\text{Club}_\kappa$ is not $\mathcal{I}$-$F_\sigma$.

However, as soon as $\mathcal{I}$ contains a stationary subset of $\kappa$, we have the following contrasting result:

Observation 7

$\mathcal{I}$ contains a stationary subset of $\kappa$ if and only if $\text{Club}_\kappa$ is $\mathcal{I}$-closed.

Observation 8

$\text{Club}_\kappa$ is $\mathcal{I}$-open if and only if $\mathcal{I}$ contains the set of all limit ordinals, and for every nonstationary set $N$ of limit ordinals, there is a regressive function $f : N \rightarrow \kappa$ such that

$$\bigcup_{\alpha \in N} [f(\alpha), \alpha) \in \mathcal{I}.$$
Stationary tallness

Stationary tallness relates to $\text{NS}_\kappa$ as does tallness to $\text{bd}_\kappa$:

**Definition 9**

$I$ is stationary tall if every stationary set $S$ has a stationary subset in $I$.

**Observation 10**

If $I$ contains a club subset $C$ of $\kappa$, then $I$ is stationary tall.

*Proof:* If $S$ is stationary, $S \cap C \subseteq C \in I$ is stationary. □

An ideal $I$ is *maximal* if whenever $A$ and $B$ are disjoint subsets of $\kappa$, at least one of them is in $I$.

**Observation 11**

Every maximal ideal is stationary tall.

*Proof:* Assume that $S$ is a stationary subset of $\kappa$. Write $S$ as disjoint union of two stationary sets $S_0 \cup S_1$, using Solovay’s theorem. One of them has to be in $I$ by maximality. □
$\mathcal{C}_\kappa$ denotes the collection of subsets of $\kappa$ that contain a club. Usually, the club filter is the standard example of a complicated set – in the bounded topology, it is not Borel (Halko-Shelah).

Observation 12

$\mathcal{I}$ is stationary tall if and only if $\mathcal{C}_\kappa$ is $\mathcal{I}$-closed.

Observation 13

$\mathcal{I}$ contains a club subset of $\kappa$ if and only if $\mathcal{C}_\kappa$ is $\mathcal{I}$-open.
Non-$\mathcal{I}$-Borel sets

However, the Halko-Shelah result generalizes to the nonstationary topology. $\mathcal{I}$-Borel sets are (iteratively) generated from the $\mathcal{I}$-open sets by taking $\kappa$-unions and complements.

**Proposition 14**

If $\mathcal{I} = \text{NS}_\kappa$, then $C_\kappa$ is not $\mathcal{I}$-Borel.

Assuming that $2^{<\kappa} = \kappa$, we can construct a Bernstein set, and such a set can easily be shown to not be $\mathcal{I}$-Borel.

**Proposition 15**

If $2^{<\kappa} = \kappa$, then there is a non-$\mathcal{I}$-Borel set (for any $\mathcal{I}$).
Ideal topologies are in fact particular instances of tree forcing topologies.

**Definition 16**

- A \( \kappa \)-tree is a subset of \( 2^{<\kappa} \) closed under initial segments.
- A *branch* through a \( \kappa \)-tree \( T \) is some \( x \in 2^{\kappa} \) such that \( x \upharpoonright \alpha \in T \) for every \( \alpha < \kappa \). \( [T] \subseteq 2^{\kappa} \) denotes the set of all branches through \( T \).
- A *tree forcing* notion \( P \) on \( \kappa \) is a notion of forcing in which conditions are \( \kappa \)-trees, including the full tree \( 2^{<\kappa} \), ordered by inclusion.
- Such a forcing notion \( P \) is *topological* if for any two \( R, S \in P \) and any \( x \in [R] \cap [S] \), there is \( T \in P \) such that \( x \in [T] \subseteq [R] \cap [S] \).
- If \( P \) is a topological notion of tree forcing on \( \kappa \), we let the \( P \)-*topology* be the topology on \( 2^{\kappa} \) generated by the basic open sets of the form \( [T] \), for conditions \( T \in P \).
Example: $\kappa$-Cohen forcing

The conditions in $\kappa$-Cohen forcing are the elements of $2^{<\kappa}$, ordered by reverse inclusion. But we can also identify $\kappa$-Cohen forcing with a tree forcing notion: Given $s \in 2^{<\kappa}$, let

$$T_s = \{ t \in 2^{<\kappa} \mid t \subseteq s \lor s \subseteq t \}.$$ 

It is easy to see that $\kappa$-Cohen forcing corresponds to the tree forcing notion consisting of conditions $T_s$ for $s \in 2^{<\kappa}$, and that the topology generated by $\kappa$-Cohen forcing (when viewed as a tree forcing notion on $\kappa$) is the standard bounded topology on $2^\kappa$. 
Grigorieff forcing

**Definition 17**

Let \( \kappa \) be an infinite cardinal and let \( \mathcal{I} \) be an ideal on \( \kappa \). \( G_\mathcal{I} \), Grigorieff forcing with the ideal \( \mathcal{I} \) is the notion of forcing consisting of conditions which are partial functions \( p \) from \( \kappa \) to 2 such that \( \text{dom}(p) \in \mathcal{I} \), ordered by inclusion.

We can view \( G_\mathcal{I} \) as a tree forcing by identifying a condition \( p \in G_\mathcal{I} \) with the tree \( T \) on \( 2^{<\kappa} \) which we inductively construct as follows:

\( \emptyset \in T \). Given \( t \in T \) of order-type \( \alpha \), let \( t \hat{\sim} 0 \in T \) if \( p(\alpha) \neq 1 \), and let \( t \hat{\sim} 1 \in T \) if \( p(\alpha) \neq 0 \) (these are both supposed to include the cases when \( \alpha \) is not in the domain of \( p \)). At limit levels \( \alpha \), we extend every branch through the tree constructed so far.

It is easy to see that these two forcings are isomorphic. Then, if \( T \) is the tree on \( 2^{<\kappa} \) corresponding to the condition \( p \in G_\mathcal{I} \), we have \([T] = [p]\). Hence, the \( G_\mathcal{I} \)-topology is exactly the \( \mathcal{I} \)-topology, and \( G_\mathcal{I} \) is topological.
**Definition 18**

Let $\kappa$ be a regular uncountable cardinal. $\kappa$-Silver forcing (or $\kappa$-club Silver forcing) $\nabla_\kappa$ is the notion of forcing consisting of conditions $p$ which are partial functions from $\kappa$ to 2 such that the complement of the domain of $p$ is a club subset of $\kappa$.

Note that $\nabla_\kappa$ is a dense subset of Grigorieff forcing with $\text{NS}_\kappa$. This yields that $\nabla_\kappa$ can be viewed as a $\kappa$-tree forcing notion. In fact, whenever $p$ is a condition in $\mathcal{G}_{\text{NS}_\kappa}$ and $x \in 2^\kappa$ is such that $p \subseteq x$, then $p$ can be extended to a condition $q \subseteq x$ in $\nabla_\kappa$. This easily yields that those two notions of forcing generate the same topology, and hence that the $\nabla_\kappa$-topology is exactly the nonstationary topology.
Unsurprisingly, combinatorial properties of tree forcing notions $P$ yield properties of their corresponding topologies. For example, if $P$ is $<\kappa$-distributive, then the $P$-topology yields a $\kappa$-Baire space (i.e., the intersection of $\kappa$-many open dense sets of that space is nonempty).

Friedman, Khomskii and Kulikov (Regularity Properties of the generalized Reals, Annals of Pure and Applied Logic, 2016) investigated such consequences of a slight strengthening of Axiom A for $\kappa$-tree forcing notions. If $\kappa$ is inaccessible, the classical proof that Silver forcing satisfies Axiom A also shows that $\mathbb{V}_\kappa$ satisfies this strong form of Axiom A. We are going to show that a more intricate argument yields the same result under the assumption of $\diamondsuit_\kappa$ – note that by results of Shelah, $\diamondsuit_\kappa$ holds whenever $\kappa > \omega_1$ is a successor cardinal for which $2^{<\kappa} = \kappa$. This will allow us to infer results on the nonstationary topology on $2^\kappa$ for many cardinals $\kappa$ (namely, all regular cardinals $\kappa > \omega_1$ that satisfy $2^{<\kappa} = \kappa$, and also for $\kappa = \omega_1$ in case $\diamondsuit_{\omega_1}$ holds).
Axiom $A^*$

The following slight strengthening of Axiom $A$ for $\kappa$-tree forcing notions was introduced by Friedman, Khomskii and Kulikov:

**Definition 19**

A notion $\langle P, \leq \rangle$ of tree forcing on $\kappa$ satisfies Axiom $A^*$ if there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$ with $\leq_0 = \leq$, satisfying:

1. $q \leq_\beta p$ implies $q \leq_\alpha p$ (i.e., $\leq_\beta \subseteq \leq_\alpha$) for all $\alpha \leq \beta$.

2. If $\langle p_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions in $P$ and $\lambda \leq \kappa$, satisfying that $p_\beta \leq_\alpha p_\alpha$ for all $\alpha < \beta < \lambda$, then there is $q \in P$ such that $q \leq_\alpha p_\alpha$ for all $\alpha < \lambda$.

3. For all $p \in P$, all $D$ that are dense below $p$ in $P$, and all $\alpha < \kappa$, there exists $E \subseteq D$ of size at most $\kappa$, and $q \leq_\alpha p$ such that $E$ is predense below $q$, and such that additionally $[q] \subseteq \bigcup \{[r] \mid r \in E\}$. 

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Theorem 20 [Friedman-Khomskii-Kulikov]
If a tree forcing notion $P$ satisfies Axiom $A^*$, then the nowhere dense sets in the $P$-topology are closed under $\kappa$-unions, i.e., all $P$-meager sets are $P$-nowhere dense.

Corollary 21
If $\kappa$ is inaccessible and $I = NS_\kappa$, then $I$-meager $\equiv I$-nowhere dense.

Definition 22
$X \subseteq 2^\kappa$ satisfies the property of Baire in the $P$-topology in case $X$ can be written in the form $X = O \Delta M$, where $O$ is $P$-open, and $M$ is $P$-meager.

Theorem 23 [Friedman-Khomskii-Kulikov]
If $\kappa$ is inaccessible and every $\Delta^1_1$-subset of $2^\kappa$ satisfies the property of Baire (in the bounded topology) – which is consistent relative to ZFC – then it does so also in the $\nabla_\kappa$-topology, i.e., the nonstationary topology on $2^\kappa$. 
Axiom $A^*$, once again

Let me remind you once again about Axiom $A^*$:

**Definition 24**

A notion $\langle P, \leq \rangle$ of tree forcing on $\kappa$ satisfies Axiom $A^*$ if there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$ with $\leq_0 = \leq$, satisfying:

1. $q \leq_\beta p$ implies $q \leq_\alpha p$ (i.e., $\leq_\beta \subseteq \leq_\alpha$) for all $\alpha \leq \beta$.

2. If $\langle p_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions in $P$ and $\lambda \leq \kappa$, satisfying that $p_\beta \leq_\alpha p_\alpha$ for all $\alpha < \beta < \lambda$, then there is $q \in P$ such that $q \leq_\alpha p_\alpha$ for all $\alpha < \lambda$.

3. For all $p \in P$, all $D$ that are dense below $p$ in $P$, and all $\alpha < \kappa$, there exists $E \subseteq D$ of size at most $\kappa$, and $q \leq_\alpha p$ such that $E$ is predense below $q$, and such that additionally $[q] \subseteq \bigcup \{[r] \mid r \in E\}$. 

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**Theorem 10**

If $\Diamond_\kappa$ holds, then $\forall = \forall_\kappa$ satisfies Axiom $A^*$.

**Proof:** For any $\alpha < \kappa$ and $p, q \in \forall$, let $q \leq_\alpha p$ if $q \leq p$ and the first $\alpha$-many elements of the complements of the domains of $p$ and of $q$ are the same. It is clear (or at least easy to check) that Items (1) and (2) in Definition 5 are thus satisfied, and we only have to verify Item (3).

Let $p \in \forall$, let $\alpha < \kappa$, and let $D \subseteq \forall$ be dense below $p$. We need to find $q \leq_\alpha p$ and $E \subseteq D$ of size at most $\kappa$ such that $E$ is predense below $q$. Fix a $\Diamond_\kappa$-sequence $\langle A_i \mid i < \kappa \rangle$: $\forall A \subseteq \kappa \{i < \kappa \mid A \cap i = A_i\}$ is a stationary subset of $\kappa$.

We inductively construct a decreasing sequence $\langle p_i \mid i \leq \kappa \rangle$ of conditions in $\forall$ with $p_i = p$ for $i \leq \alpha$, and a sequence $\langle \alpha_i \mid i < \kappa \rangle$ of ordinals with the property that $\langle \alpha_j \mid j \leq i \rangle$ enumerates the first $(i + 1)$-many elements of $\kappa \setminus \text{dom}(p_i)$ for every $i \leq \kappa$, as follows. Let $\langle \alpha_i \mid i \leq \alpha \rangle$ enumerate the first $\alpha + 1$-many elements of the complement of the domain of $p$. 


Assume that we have constructed $p_i$ for some $i \geq \alpha$, and also $\alpha_j$ for $j \leq i$.

Using that $D$ is dense below $p$, let $q_i^0 \leq p_i$ be such that

- $q_i^0(\alpha_j) = A_i(j)$ for all $j < i$,
- $q_i^0(\alpha_i) = 0$, and
- $q_i^0 \in D$,

and let $q_i^1 \leq q_i^0 \upharpoonright (\text{dom}(q_i^0) \setminus \{\alpha_i\})$ be such that

- $q_i^1(\alpha_i) = 1$, and
- $q_i^1 \in D$.

Let $p_{i+1} = q_i^1 \upharpoonright (\text{dom}(q_i^1) \setminus \{\alpha_j \mid j \leq i\})$, and note that $p_{i+1} \leq_i p_i$.

Let $\alpha_{i+1}$ be the least element of $\kappa \setminus \text{dom}(p_{i+1})$ above $\alpha_i$. 
For limit ordinals $i \leq \kappa$, let $p_i = \bigcup_{j<i} p_j$, and if $i < \kappa$, let $\alpha_i = \bigcup_{j<i} \alpha_j$ be the least element of $\kappa \setminus \text{dom}(p_i)$. Let $q = p_\kappa$, and let $E = \{q_i^0 \mid i < \kappa\} \cup \{q_i^1 \mid i < \kappa\}$. To verify Axiom A, we want to show that $E$ is predense below $q$.

Thus, let $r \leq q$ be given. Using the properties of our diamond sequence, pick $i < \kappa$ such that $i \geq \alpha$, and such that for all $j < i$ with $\alpha_j \in \text{dom}(r)$, $A_i(j) = r(\alpha_j)$. Pick $\delta \in \{0, 1\}$ such that $r(\alpha_i) = \delta$ in case $\alpha_i \in \text{dom}(r)$. Then, $q_i^\delta$ is compatible to $r$, as desired.

In order to check the additional property for Axiom $A^*$, note that any extension $s$ of $q$ to a total function from $\kappa$ to 2 can be treated in the same way as $r$ above, yielding some $i < \kappa$ and $\delta \in \{0, 1\}$ such that $s \in [q_i^\delta]$. □
So what does Axiom $A^*$ have to do with meager sets?

In order to properly connect topics, let me present the following result:

**Lemma 26 [Friedman-Khomskii-Kulikov]**

If a $\kappa$-tree forcing notion $P$ satisfies Axiom $A^*$ (the proof uses quite a bit less), then every $P$-meager set is $P$-nowhere dense.

**Proof:** Let $\{A_i \mid i < \kappa\}$ be a collection of $P$-nowhere dense sets. We need to show that $\bigcup_{i<\kappa} A_i$ is $P$-nowhere dense. For every $i < \kappa$, let $D_i$ be the dense subset $D_i = \{p \mid [p] \cap A_i = \emptyset\}$ of $P$, using that $A_i$ is $P$-nowhere dense. Using Axiom $A^*$, construct $\langle p_i \mid i \leq \kappa \rangle$ and $\langle E_i \subseteq D_i \mid i < \kappa \rangle$, such that for all $i < j \leq \kappa$,

- $p_j \leq_i p_i$, and
- $[p_i] \subseteq \bigcup \{[p] \mid p \in E_i\}$.

Hence, for every $i < \kappa$, $[p_\kappa] \subseteq \bigcup \{[p] \mid p \in D_i\}$. In particular, $[p_\kappa] \cap A_i = \emptyset$ for all $i < \kappa$, hence $\bigcup_{i<\kappa} A_i$ is $P$-nowhere dense. □
We will need the following, the forward direction of which is immediate:

**Lemma 27 [Friedman-Khomskii-Kulikov]**

If $P$ is a topological notion of forcing that satisfies Axiom $A^*$, then $X \subseteq 2^\kappa$ satisfies the Baire property in the $P$-topology if and only if

$$\forall T \in P \ \exists S \leq T \ ([S] \subseteq X \ \lor \ [S] \cap X = \emptyset).$$

In particular, for $\mathcal{I} = \text{NS}_{\kappa}$, $X \subseteq \kappa$ satisfies the $\mathcal{I}$-Baire property if every $\mathcal{I}$-basic open set $[f]$ contains an $\mathcal{I}$-basic open set $[g]$ such that either $[g] \subseteq X$ or $[g] \cap X = \emptyset$. 
On the Baire property

Quite similar arguments as for $P$-meager $\equiv P$-nowhere dense (without the intermediate principle of Axiom $A^*$) show the following, where the case of inaccessible $\kappa$ is implicit in Friedman-Khomskii-Kulikov:

**Theorem 28**

If $\kappa$ is inaccessible or $\diamondsuit_\kappa$ holds, then every comeager set, i.e., every $\kappa$-intersection of open dense subsets of $2^\kappa$ in the bounded topology, contains a dense set that is open in the nonstationary topology.

This allows us to show the following, again due to Friedman et al. in the case of inaccessible $\kappa$ (and the proof below is essentially theirs):

**Theorem 29**

If $\kappa$ is inaccessible or $\diamondsuit_\kappa$ holds, and every $\Delta^1_1$-subset of $2^\kappa$ has the Baire property (both of the latter can be forced by adding $\kappa^+$-many Cohen subsets of $\kappa$), then it does so also in the nonstationary topology.
Proof of Theorem 29:

Let $P$ denote $\kappa$-Silver forcing, let $\mathcal{I} = \text{NS}_\kappa$. Let $A \in \Delta^1_1$, and let $f \in P$. We need to find an $\mathcal{I}$-open subset of $[f]$ that is either contained in or disjoint from $A$. Let $C$ denote the club subset of $\kappa$ that is the complement of the domain of $f$, and enumerate $C$ in increasing order as $\langle c_\gamma \mid \gamma < \kappa \rangle$. Let $\varphi$ denote the natural order-preserving bijection between $2^{<\kappa}$ and extensions of $f$ by bounded functions: Given $s \in 2^\alpha$ with $\alpha < \kappa$, let $\varphi(s)$ be the $\subseteq$-minimal $g \in P$ such that $g$ extends $f$ and $g(c_\gamma) = s(\gamma)$ for every $\gamma < \alpha$. Let $\varphi^*$ be the induced homeomorphism between $2^\kappa$ and $[f]$. Let $A' = \varphi^*[A]$, which is again a $\Delta^1_1$-subset of $2^\kappa$, using that $\Delta^1_1$ is closed under continuous preimages. Hence, $A'$ has the Baire property, by our assumption. This means that either $A'$ is meager, or it is comeager in some basic open set $[s]$ of the bounded topology on $2^\kappa$. If $A'$ is meager, Theorem 28 yields an $\mathcal{I}$-open set $[t]$ that is disjoint from $A'$. If $A'$ is comeager in $[s]$, applying Theorem 28 relativized to $[s]$, we find an $\mathcal{I}$-open set $[t] \subseteq A' \cap [s]$. But then, in either case, $(\varphi^*)^{-1}[[t]] \subseteq [f]$ is an $\mathcal{I}$-open set that is either disjoint from or contained in $A$, as desired. □
A further result – Comparing notions of meagerness

Let $\mathcal{I} = \text{NS}_\kappa$.

Observation 30
If $[f]$ is an $\mathcal{I}$-basic open set, with $\text{dom}(f)$ of size $\kappa$, then $[f]$ is meager (in fact, nowhere dense) in the bounded topology. Thus, there is always a meager set that is not $\mathcal{I}$-meager.

Observation 31
Every set of size less than $2^\kappa$ is $\mathcal{I}$-meager. Hence, if $\text{non}(\mathcal{M}_\kappa) < 2^\kappa$, then there is an $\mathcal{I}$-meager set that is not meager.

Theorem 32
If $\kappa$ is inaccessible or $\diamondsuit_\kappa$ holds, and the reaping number $r(\kappa) = 2^\kappa$, then there is an $\mathcal{I}$-meager set which does not have the Baire property (and thus in particular is not meager) in the bounded topology.
Open Questions

Question 33
Is there a proper $\mathcal{I}$-Borel hierarchy? If so, what is its length and structure?

We have answered the following positively whenever $\kappa$ is inaccessible or $\Diamond_\kappa$ holds.

Question 34
- Does $\kappa$-Silver forcing satisfy Axiom $A^*$ whenever $\kappa$ is regular and uncountable?
- If $\kappa$ is regular and uncountable, and $\mathcal{I} = \text{NS}_\kappa$, are $\mathcal{I}$-meager sets always $\mathcal{I}$-nowhere dense?

We know the following holds for many $\kappa$, at least under certain assumptions on generalized cardinal invariants.

Question 35
Let $\mathcal{I} = \text{NS}_\kappa$. Is there always an $\mathcal{I}$-meager set that is not meager?