AN AXIOMATIC APPROACH TO FORCING IN A GENERAL SETTING

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Abstract. The technique of *forcing* is almost ubiquitous in set theory, and it seems to be based on technicalities like the concepts of genericity, forcing names and their evaluations, and on the recursively defined forcing predicates, the definition of which is particularly intricate for the basic case of atomic first order formulas. In his [3], the first author has provided an axiomatic framework for set forcing over models of ZFC that is a collection of guiding principles for extensions over which one still has *control* from the *ground model*, and has shown that these axiomatics necessarily lead to the the usual concepts of genericity and of forcing extensions, and also that one can infer from them the usual recursive definition of forcing predicates. In this paper, we present a more general such approach, covering both class forcing and set forcing, over various base theories, and we provide additional details regarding the formal setting that was outlined in [3].

1. Introduction

We will provide a set of guiding principles with respect to class forcing, that avoid the usual technicalities connected with any of the standard setups for (class) forcing, and show that they induce the common concepts of genericity and of class generic extensions, and that they yield the usual recursive definitions of forcing predicates. We will consider various base theories (Gödel-Bernays set theory GB, Kelley-Morse set theory KM, and some of their variants), and also deduce the somewhat simpler axioms for the special case of set forcing over ZFC and some of its weakenings in Section 7. In this introductory section, we want to provide a rough description of our guiding principles, which will be followed with formal definitions in Sections 2–6.

We require forcing extensions to be based on preorders [1] in a ground model – let us fix such a transitive ground model $\mathcal{M} \in V$ for this discussion, and a class preorder $\mathbb{P}$ from $\mathcal{M}$. We require $\mathcal{M}$ to satisfy the axioms of some axiom system T for set theory that is suitable for class forcing (either Gödel-Bernays set theory GB, Kelley-Morse set theory KM, or some variants of those). We think of conditions (elements) of $\mathbb{P}$ as having partial information on properties of our extensions. We require that stronger conditions have more such information, and that any particular forcing extension is based on a choice of filter on $\mathbb{P}$. We think of such a filter as a selection of conditions which have *correct* information about our

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[1]A preorder is a reflexive and transitive binary relation.

extension, and we will refer to such conditions as being *correct*. The motivation for using a *filter* of conditions could be explained as follows.

- If we consider the information that a condition q has to be correct, then any weaker condition p has less information than q, and this information should therefore also be correct. This corresponds to the upwards closure property of filters.
- If p and q are correct conditions, we consider the information that is jointly collected by p and q to be correct. We require that there is a condition that collects this joint information and that we consider to be correct. This clearly corresponds to the property of a filter that any two of its elements are compatible, as witnessed by yet another element of the filter.

We require that for any condition $p \in \mathbb{P}$, there exists a filter G of correct conditions of which p is an element. Given any particular such filter G, we require the *generic extension* $\mathcal{M}[G]$ to contain $\mathcal{M}$ as a subset and G as an element. There are a number of natural axioms which make sure that we have *ground model control* over our generic extensions, in a sufficiently simple way. One necessary requirement for this is that elements of $\mathcal{M}[G]$ are connected to elements of the ground model so that the latter serve as a sort of *name* for the former. We require the existence of a definable and highly absolute relation on our ground model, that was called the $\mathbb{P}$-membership relation in [3]. It is supposed to relate to partial knowledge about the membership relation in forcing extensions. If a and b are elements of $\mathcal{M}$ and $p \in \mathbb{P}$, we say that a is an element of b according to p, and write $a \in_p b$ in case the triple $\langle p, a, b\rangle$ stands in this relation. [2] We want to define a membership relation for $\mathcal{M}[G]$, letting the object denoted by a be an element of the object denoted by b in case a is an element of b according to some correct condition (that is, $\exists p \in G\ a \in_p b$). In order to be able to obtain a transitive model as our forcing extension, we thus require the relation $\exists p \in \mathbb{P}\ \ a \in_p b$ to be well-founded. The relation $\exists p \in G\ a \in_p b$ will usually not be extensional, but we nevertheless obtain a transitive $\in$-structure (which will serve as our generic extension $\mathcal{M}[G]$) as the image of the homomorphism that is our *evaluation map* F_G, recursively defined by setting $F_G(b) = \{F_G(a) \mid a \in_G b\}$ for every $b \in \mathcal{M}$.

In order to be able to show that $\mathcal{M}[G]$ is well-defined and satisfies the axioms of T, we will need to require a strong form of set-likeness: we ask that for any $b \in \mathcal{M}$, $\{\langle a, p\rangle \mid a \in_p b\}$ is a set in $\mathcal{M}$.

Furthermore, we also require the existence of forcing predicates in $\mathcal{M}$, individually for each first order formula, and also for each second order formula in the case of KM. We do not require any particular defining instances for these predicates, we only require them to be connected to truth in generic extensions by the following two axioms (these requirements correspond to what is usually known as the *forcing theorem* in a standard class forcing setup):

- Whatever holds in $\mathcal{M}[G]$ is forced by some condition in G.
- Whatever is forced by some condition $p \in G$ holds true in $\mathcal{M}[G]$.

[2]This relation corresponds to the relation that $\langle a, p\rangle \in b$ in a standard forcing setup, given that a, b are usual forcing names.

To show that $\mathcal{M}[G] \models T$, an additional axiom is needed, which states that the $\mathbb{P}$-membership relation has *high degrees of freedom*, in the sense that for any relation S on $\mathcal{M} \times P$ in $\mathcal{M}$, we find $b \in \mathcal{M}$ for which $\{\langle a, p\rangle \mid a \in_p b\} = S$. [3]

Finally, we will have to assume that our class forcing notion $\mathbb{P}$ is pretame or tame, which are technical conditions on $\mathbb{P}$ that are equivalent to the preservation of the axioms from T in a standard class forcing setup.

2. The basic setup

We want to verify our results for models of the base theory that is Gödel-Bernays set theory GB, and also for some of its variants. In Section 8, we will also consider the stronger theory KM, and we will consider yet another strengthening of it in Section 9. These are usually presented as theories in a two-sorted language, with variables for sets and for classes, and their models will be of the form $\mathcal{M} = \langle M, \mathcal{C}\rangle$, where M denotes the domain of sets, and $\mathcal{C}$ denotes the domain of classes of $\mathcal{M}$.

Let $\mathcal{L}(\in)$ denote the collection of first order formulas in the language with the $\in$-predicate in which we additionally allow for second order variables, and atomic formulas of the form $x \in X$, where x is a first order variable and X is a second order variable. We consider equality between first or second order elements to abbreviate the statement that they have the same elements. The axioms of GB are given by the axioms of ZF for sets, allowing class parameters in the axiom schemes of Separation and Replacement (that is, allowing for formulas from $\mathcal{L}(\in)$ in which second order variables are replaced by second order parameters from $\mathcal{C}$), together with the class axiom of fist order class comprehension, that is comprehension for classes using $\mathcal{L}(\in)$-formulas with second order parameters from $\mathcal{C}$. If $\mathcal{M} \models GB$ (or any of its variants), we usually use lowercase letters to denote first order elements of $\mathcal{M}$, that is elements of M, and uppercase letters to denote second order elements of $\mathcal{M}$, that is elements of $\mathcal{C}$. Note that by the separation axiom, we have $M \subseteq \mathcal{C}$.

We will also consider the strengthenings of GB that are obtained by adding the axiom of choice (GBc) in the form of the statement that every set can be well-ordered, or (GBC) the axiom that there is a well-order of all sets in order-type Ord (or equivalently, a set-like global well-order), as well as the axiom systems GB^-, GBc^- and GBC^-, which are obtained from GB, GBc or GBC respectively by removing the powerset axiom and using the axiom scheme of Collection rather than Replacement. We fix a base theory T to be one of the above theories.

We next provide the definition of a *class forcing generic framework*, which will be the basic formal concept in our approach. As in [6], we will use the more general notion of preorders rather than (the perhaps more common restriction to) partial orders, dropping the requirement of antisymmetry. If $\vec{A}$ is a finite sequence, we will abbreviate the property that all sequents of $\vec{A}$ are elements of $\mathcal{C}$ by $\vec{A} \in \mathcal{C}$, and a similar comment applies to statements of the form $\vec{a} \in M$ when $\vec{a}$ is a finite sequence.

Definition 2.1. *A* class forcing generic framework *is a tuple of the form*

$$\left\langle \mathcal{M}, \mathbb{P}, R, \left(\Vdash^{\vec{A}}_{\varphi}\right)_{\varphi \in \mathcal{L}(\in), \vec{A} \in \mathcal{C}}, \mathfrak{G} \right\rangle \text{ with the following properties.}$$

[3]This axiom could be seen to demand that the elements of $\mathcal{M}$ reflect all forcing names from the standard forcing setup.

- *$\mathcal{M}$ is a transitive set-size model of T: M is transitive, $\bigcup\mathcal{C} \subseteq M$, and $\mathcal{M}$ is a set such that $\mathcal{M} \models T$.*
- *$\mathbb{P} = \langle P, \leq\rangle$ is a preorder such that both P and $\leq$ are in $\mathcal{C}$.*
- *The* $\mathbb{P}$-membership relation *R is a relation on $P \times M \times \mathcal{C}$ that is definable over $\mathcal{M}$ by an $\mathcal{L}(\in)$-formula $\varphi(p, a, B)$ with first order variables p and a, and a second order variable B, so that for $p \in P$, $a \in M$ and $B \in \mathcal{C}$ we have $R(p, a, B)$ if and only if $\varphi(p, a, B)$.* [4]

  We also require φ to be absolute for transitive models of T containing $\mathcal{C}$, and we denote the property $R(p, a, B)$ as $a \in_p B$. [5]
- *$\mathfrak{G}$ is a second order unary predicate on P, i.e. a unary predicate on $\mathcal{P}(P)$, and we require that $\mathfrak{G}(G)$ implies that $G \subseteq \mathbb{P}$ is a filter. If $\mathfrak{G}(G)$ holds, we say that G is a* generic filter*, or a* $\mathbb{P}$-generic filter *on M. Whenever we quantify over G in the following, we assume that we quantify over G's such that $\mathfrak{G}(G)$ holds.*
- *For every $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, $\Vdash_{\varphi}^{\vec{A}} \in \mathcal{C}$ is a predicate (which we also call a* forcing relation for *φ) on $P \times M^m$, where m denotes the number of free first order variables of φ.*

  If $\langle q, a_0, \ldots, a_{m-1}\rangle \in \Vdash_{\varphi}^{\vec{A}}$, we also write $q \Vdash \varphi(a_0, \ldots, a_{m-1}, \vec{A})$.

3. The basic axioms

In this section, we present our basic axioms.

(1) **Existence of generic filters:** $\forall p \in P\, \exists G\; p \in G$. [6]
(2) **Well-Foundedness:** The binary relation $\exists p \in \mathbb{P}\;\; a \in_p b$ on M is well-founded.
(3) **Growth of Information:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$, for all $\vec{a} \in M$, and $p, q \in P$, if $p \Vdash \varphi(\vec{a}, \vec{A})$ and $q \leq p$, then $q \Vdash \varphi(\vec{a}, \vec{A})$.

Assume that G is such that $\mathfrak{G}(G)$ holds. Define a relation $\in_G$ on $\mathcal{M}$ by letting $a \in_G B$ if $\exists p \in G\;\; a \in_p B$. Using axiom (2), this relation on $\mathcal{M}$ is well-founded, and since $\mathcal{M} \in V$, it is clearly set-like in V. We may thus recursively define our *evaluation function* F_G along the relation $\in_G$, letting $F_G(A) = \{F_G(b) \mid b \in_G A\}$ for each $A \in \mathcal{C}$. [7] Let $\mathcal{M}[G]$ denote the $\in$-structure on the transitive set $F_G[\mathcal{C}]$: [8] That is, let $\mathcal{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$, where $M[G] = F_G[M]$, and $\mathcal{C}[G] = F_G[\mathcal{C}]$.

The next two axioms state that a natural form of the forcing theorem holds, that is based on our forcing relations. Given a finite tuple $\vec{A} = \langle A_i \mid i < n\rangle \in \mathcal{C}$, let $F_G(\vec{A}) = \langle F_G(A_i) \mid i < n\rangle$.

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[4] Note that by our choice of the domain of R, only *sets* from M can stand in $\mathbb{P}$-membership relation to sets or classes from $\mathcal{C}$.

[5] In a standard forcing setup, this would correspond to the property that $\langle a, p\rangle \in B$.

[6] By our above convention, we tacitly require here that $\mathfrak{G}(G)$ holds, i.e. that G is generic.

[7] It may seem like we are taking some sort of transitive collapse of the structure $\langle \mathcal{M}[G], \in_G\rangle$, however note that there is no reason to assume that $\in_G$ is extensional, or that $\in_G$ can be factorized in order to obtain an extensional relation.

[8] For the moment, this notation is somewhat ambiguous, for $\mathcal{M}[G]$ may not only depend on $\mathcal{M}$ and on G. We will later however show this to be the case under additional assumptions.

(4) **Truth Lemma:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, all $\vec{a} \in M$ and all G,

$$\mathcal{M}[G] \models \varphi(F_G(\vec{a}), F_G(\vec{A})) \text{ iff } \exists p \in G\ p \Vdash \varphi(\vec{a}, \vec{A}).$$

(5) **Definability Lemma:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, all $\vec{a} \in M$ and $p \in P$,

$$p \Vdash \varphi(\vec{a}, \vec{A}) \text{ iff } \forall G \ni p\ \ \mathcal{M}[G] \models \varphi(F_G(\vec{a}), F_G(\vec{A})).$$ [9]

Our next axiom $(\bar{6})$ states that within M, a weak form of set-likeness holds for the $\mathbb{P}$-membership relation. We will later replace it by the stronger axiom (6). [10]

$(\bar{6})$ **Weak Set-Likeness:** If $b \in M$, then $\{a \mid \exists p \in P\ a \in_p b\} \in M$.

We will also introduce two additional axioms, axioms (7) and (8), later on in our paper. For the moment, we introduce two other additional axioms, stating that all elements of M have a name in M, and that there is a (class) name for our generic filters. They will later be replaced by the stronger axiom (7) which will imply both these axioms.

(*) **Names for ground model objects:**
- $\forall a \in M\, \exists \check{a} \in M\, \forall G\ F_G(\check{a}) = a$, and
- $\forall A \in \mathcal{C}\ \exists \check{A} \in \mathcal{C}\ \forall G\ F_G(\check{A}) = A$.

(**) **Name for generic filters:** $\exists \dot{G} \in \mathcal{M}\ \ \forall G\ F_G(\dot{G}) = G$.

Given $a \in M$ and $A \in \mathcal{C}$, we will use $\check{a}$ and $\check{A}$ to denote names for a and A as provided by axiom (*) above, and we will use $\dot{G}$ to denote a (class) name for G as provided by axiom (**) above.

4. Forcing predicates and density

Our axioms (1)–(5) together with the axioms (*) and (**) suffice to verify some of the basic properties of forcing, and in particular to verify that the forcing predicates satisfy their usual defining clauses, by arguments that are similar to the arguments of [3, Section 4]. For the sake of completeness, and for the benefit of our readers, we would nevertheless like to present some of these arguments here. The first step will be to verify an auxiliary result on the forcing of negated statements.

Lemma 4.1. *For all $\varphi \in \mathcal{L}(\in)$ and $\vec{A} \in \mathcal{C}$, we have that*

$$p \Vdash \neg\varphi(\vec{a}, \vec{A}) \text{ iff } \forall q \leq p\ q \not\Vdash \varphi(\vec{a}, \vec{A}).$$

Proof. Let us assume that

(i) $p \Vdash \neg\varphi(\vec{a}, \vec{A})$.

By axiom (5), equivalently

(ii) $\forall G \ni p\ \ \mathcal{M}[G] \models \neg\varphi(F_G(\vec{a}), F_G(\vec{A}))$.

By axiom (4), this is equivalent to

(iii) $\forall G \ni p\, \forall q \in G\ q \not\Vdash \varphi(\vec{a}, \vec{A})$.

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[9] Note that we already required the forcing relations to be definable in our basic setup, however this axiom connects them with their intended meaning, and it thus seems justified to consider it to be our version of the *definability lemma*.

[10] We provide this weaker form here in order to be able to show that this form suffices for the results of Section 5.

We want to argue that this in turn is equivalent to our desired statement that

(iv) $\forall q \leq p\ q \not\Vdash \varphi(\vec{a}, \vec{A})$.

Thus, assume first that (iii) holds, and let $q \leq p$. By axiom (1), we may pick a generic filter $G \ni q$, which will thus also contain p as an element. By (iii), we thus have that $q \not\Vdash \varphi(\vec{a}, \vec{A})$, as desired.

Conversely, assume that (iv) holds. Let G be a generic filter that contains p as an element, and assume for a contradiction that there is $r \in G$ such that $r \Vdash \varphi(\vec{a}, \vec{A})$. Since G is a filter, we may pick q below both p and r. By axiom (3), it follows that $q \Vdash \varphi(\vec{a}, \vec{A})$, contradicting (iv). □

We are now ready to show that our axioms imply generic filters to intersect all dense classes in $\mathcal{C}$.

Lemma 4.2. *Let $D \in \mathcal{C}$ be such that D is dense in $\mathbb{P}$. If G is a generic filter, then G intersects D.*

Proof. Let G be a generic filter and assume for a contradiction that $G \cap D = \emptyset$. Making use of axioms (*) and (**), it follows that

$$\mathcal{M}[G] \models \neg \exists x\ x \in F_G(\check{D}) \cap F_G(\dot{G}).$$

By axiom (4), we may thus find $p \in G$ such that

$$p \Vdash \neg \exists x\ x \in \check{D} \cap \dot{G}.$$

By Lemma 4.1, equivalently

$$\forall q \leq p\ q \not\Vdash \exists x\ x \in \check{D} \cap \dot{G}.$$

Since D is dense, we may fix $q \leq p$ in D. But then,

$$q \Vdash \check{q} \in \check{D} \cap \dot{G},$$

contradicting the above. □

We next need another auxiliary result on open dense sets (which could easily be extended to arbitrary dense sets, but the current version is sufficient for our purposes). We say that a subset A of a preorder $\mathbb{P}$ is *open* if it is downward closed, that is if $p \in A$ and $q \leq p$, then also $q \in A$.

Lemma 4.3. *If $D \subseteq \mathbb{P}$ is open, $D \in \mathcal{C}$, then D is dense below p if and only if*

$$(\dagger)\ \forall G \ni p\, D \cap G \neq \emptyset.$$

Proof. Assume first that (†) holds. Let $r \leq p$, and using axiom (1), let G be a generic filter with $r \in G$. It follows that also $p \in G$, and thus using (†), we obtain $s \in D \cap G$. Since D is open and G is a filter, we obtain q below both r and s that is an element of $D \cap G$, showing that D is dense below p.

On the other hand, assume that D is dense below p, and let G be a generic filter containing p as an element. Let E be the dense set of conditions which are either below p and in D, or incompatible to p. By Lemma 4.2, it follows that $G \cap E \neq \emptyset$. Since $p \in G$ and G is a filter, it thus follows that $G \cap D \neq \emptyset$, as desired. □

It is now possible to show, as in [3], that the usual defining clauses for the forcing relation can be recovered from our basic axioms. The only additional clause that we need is the one for the elementhood relation with respect to classes. Its proof is very similar to the one for the elementhood relation with respect to sets, however we would like to provide one sample argument here in this paper for the benefit of our readers, and we will refer them to [3] for the other clauses below (note that since $M \subseteq \mathcal{C}$, the below lemma also covers the case of the elementhood relation with respect to sets). Let $a \,\overline{\in}_p\, B$ if and only if $\exists q \geq p \; a \in_q B$.

Lemma 4.4. *$p \Vdash a \in A$ iff $\forall r \leq p \, \exists s \leq r \, \exists x \; [x \,\overline{\in}_s\, A \,\wedge\, s \Vdash a = x]$.*

Proof. Let us assume that

(i) $p \Vdash a \in A$.

By axiom (5), this is equivalent to

(ii) $\forall G \ni p \; F_G(a) \in F_G(A)$.

By the definition of F_G and of $\in_G$, this in turn is equivalent to

(iii) $\forall G \ni p \, \exists x \in M \; [F_G(a) = F_G(x) \,\wedge\, \exists q \in G \; x \in_q A]$.

Using axiom (5) once again, we obtain the following equivalence.

(iv) $\forall G \ni p \, \exists x \in M \; [\exists r \in G \; r \Vdash a = x \,\wedge\, \exists q \in G \; x \in_q A]$.

Now we make use of axiom (3), equivalently obtaining that

(v) $\forall G \ni p \, \exists s \in G \, \exists x \in M \; [s \Vdash a = x \,\wedge\, x \,\overline{\in}_s\, A]$.

Using Lemma 4.3 yields our final desired equivalence:

(vi) $\forall r \leq p \, \exists s \leq r \, \exists x \in M \; [x \,\overline{\in}_s\, A \,\wedge\, s \Vdash a = x]$.

□

In the next lemma, we list the remaining results with respect to the forcing predicates obeying their usual defining clauses, which are shown exactly as in [3, Section 5], simply carrying along additional second order predicates (in case there are any). For the detailed arguments to verify these properties, we refer the interested reader to [3]. Let us remark that apart from the $\in$-relation, we consider the $\neq$-relation as our second basic relation in the below, however $a \neq b$ could be seen as abbreviating $\neg(a = b)$, and $a \notin b$ should be seen as abbreviating $\neg(a \in b)$.

Lemma 4.5. *Let $\varphi \in \mathcal{L}(\in)$. Let $\vec{a}$ and $\vec{A}$ be finite tuples from M and from $\mathcal{C}$ respectively. Then the following hold true.*

- $p \Vdash \varphi(\vec{a}, \vec{A})$ *iff* $\forall q \leq p \, \exists r \leq q \; r \Vdash \varphi(\vec{a}, \vec{A})$.
- $p \Vdash \varphi \vee \psi(\vec{a}, \vec{A})$ *iff* $\forall r \leq p \, \exists q \leq r \; [q \Vdash \varphi(\vec{a}, \vec{A}) \,\vee\, q \Vdash \psi(\vec{a}, \vec{A})]$.
- $p \Vdash \exists x \varphi(\vec{a}, \vec{A})$ *iff* $\forall r \leq p \, \exists q \leq r \, \exists x \in M \; q \Vdash \varphi(\vec{a}, \vec{A})$.
- $p \Vdash \varphi \,\wedge\, \psi(\vec{a}, \vec{A})$ *iff* $p \Vdash \varphi(\vec{a}, \vec{A}) \,\wedge\, p \Vdash \psi(\vec{a}, \vec{A})$.
- $p \Vdash a \neq b$ *iff* $\forall r \leq p \, \exists q \leq r \, \exists c$
$$(c \,\overline{\in}_q\, a \,\wedge\, q \Vdash c \notin b) \vee (c \,\overline{\in}_q\, b \,\wedge\, q \Vdash c \notin a).$$

The existence of forcing relations for atomic formulas is usually a problematic aspect in the case of class forcing (for a detailed account of this see [6]). The fundamental difference in our setup is that the definability of forcing predicates is already an integral assumption. In the above, we only show that any single step, reducing forcing statements about atomic formulas to ones for names of lower rank, proceeds as usual. We thus avoid the usual problem of having to obtain the definability of the forcing predicates for atomic formulas by recursion (which a priori is a recursion on classes, and is thus not guaranteed by the axioms of GBC, see [6], or also [5]). Such recursion is an axiom of KM on the other hand, and this is the reason why every notion of class forcing satifies the forcing theorem in KM (see [6], [5], or [1]).

5. Preserving the axioms of set theory

In this section, let us assume the additional axiom

$$(***)\ \textbf{Preservation of axioms:}\quad \forall G\ M[G]\models T.$$

Building on the terminology from [3, Section 3], let us say that a *class forcing generic extension* is a class forcing generic framework together with a particular choice of generic filter G.

Theorem 5.1. *Given a particular class forcing generic extension, assuming axioms (1) to (5), and also axioms ($\bar{6}$), (*), (**) and (***) to hold, $\mathcal{M}[G]$ is actually well-defined in the sense that it depends only on $\mathcal{M}$ and on G, and is in fact the $\subseteq$-smallest model $\mathcal{N}$ of T that contains $\mathcal{C}\cup\{G\}$ as a subset if any such model $\mathcal{N}$ exists.* [11]

Proof. If $\mathcal{N}=\langle N,\mathcal{D}\rangle\models T$ is transitive such that $\mathcal{C}\cup\{G\}\subseteq\mathcal{D}$, then we can construct $M[G]$ within $\mathcal{N}$: Working in $\mathcal{N}$, define the relation $\in_G$ on M by letting $a\in_G b$ if $\exists p\in G\ a\in_p b$. By our absoluteness assumption on the $\mathbb{P}$-membership relation, this definition of $\in_G$ is absolute between $\mathcal{N}$ and V. This relation on M is well-founded by axiom (2), and set-like by axiom ($\bar{6}$). We thus may define the restriction of F_G to M in $\mathcal{N}$: Let $Q(f,b)$ be the formula asserting that f is a function whose domain d includes b as an element and is closed under $\in_G$-predecessors, and for every $a\in d$ we have $f(a)=\{f(c)\mid c\in_G a\}$. Now F_G is defined by letting $F_G(b)=f(b)$ if there is an $f\in N$ such that $Q(f,b)$ holds, and is $\emptyset$ otherwise. By axioms (2) and ($\bar{6}$), for any $b\in M$, such function $f\in N$ actually exists. We thus have $M[G]\subseteq N$.

If $A\in\mathcal{C}$, we obtain $F_G(A)=\{F_G(a)\mid a\in_G A\}\in\mathcal{D}$ by using first order class comprehension in $\mathcal{N}$, and thus we also have $\mathcal{C}[G]=F_G[\mathcal{C}]\subseteq\mathcal{D}$. It thus follows that $\mathcal{M}[G]\subseteq\mathcal{N}$, and by axioms (*) and (**), $\mathcal{M}[G]$ therefore is the $\subseteq$-smallest model of T that contains all classes from $\mathcal{C}$ and the generic filter G as one of its classes, if any such model $\mathcal{N}$ exists, as desired. □

One of the classical results about set forcing is that it preserves the axioms of either ZF^-, ZF or ZFC, and this easily extends to any of the second order theories that we consider in this paper. However the situation is completely different for class forcing, for any of the second order theories that we consider in this paper may easily be destroyed by class forcing. For example, simply consider the notion of forcing which adds a function from ω to Ord using finite conditions. This notion of class forcing will provide us with a class function mapping ω surjectively onto Ord in any of its generic extensions, clearly yielding GB^- to fail. A key notion in this context is that of *pretameness*, which was implicit in ealier work of A. Zarach and of M. Stanley, and which was isolated by S. Friedman in [4].

Definition 5.2. $\mathbb{P}$ *is* pretame *(for $\mathcal{M}$) if for every $p\in\mathbb{P}$ and every sequence $\langle D_i\mid i\in I\rangle\in\mathcal{C}$ of dense subclasses of $\mathbb{P}$ with $I\in M$, there is $q\leq p$ and $\langle d_i\mid i\in I\rangle\in M$ such that for every $i\in I$, $d_i\subseteq D_i$ is predense below q in $\mathbb{P}$.*

[11] If such model $\mathcal{N}$ does not exist, this means that for this particular choice of ground model $\mathcal{M}$ and forcing notion $\mathbb{P}$, our collection of axioms is inconsistent. This may well be the case if $\mathbb{P}$ is not pretame for $\mathcal{M}$, as will be discussed below.

M. Stanley showed that pretameness is actually equivalent to the property that forcing with $\mathbb{P}$ preserves the axioms of GB^- in a standard class forcing setup (see [7, Theorem 3.1]), and it is shown in [7, Theorem 1.12] that pretameness is equivalent to a large number of desirable niceness features of forcing notions. It therefore seems very natural to restrict ones attention to pretame notions of forcing in the context of class forcing. Pretame notions of forcing also preserve the axiom of choice, and the existence of a global wellorder in order-type Ord: regarding the axiom of choice, if x is a set in a generic extension $\mathcal{M}[G]$ by a pretame forcing notion $\mathbb{P}$, we pick a $\mathbb{P}$-name $\dot{x}$ for x, and a wellorder $\prec$ of $X = \{\sigma \mid \exists p \in P\ \langle\sigma, p\rangle \in \dot{x}\}$. We then construct a wellorder of x as follows: given two of its elements y and z, we let $y \lhd z$ if the $\prec$-least $\mathbb{P}$-name for y in X (that is, the $\prec$-least $\sigma \in X$ so that $\sigma^G = y$) is $\prec$-below any $\mathbb{P}$-name for z in X. The argument for the preservation of the existence of a global wellorder in order-type Ord is essentially the same, and presented within [7, Theorem 3.1].

However, pretame notions of forcing need not preserve the powerset axiom. This can easily be observed by considering the pretame notion of forcing that adds a proper class of Cohen subsets of ω. Since in any of its extensions, the powerset of ω would have to be a proper class, it cannot exist as a set. For the preservation of the powerset axiom, we need the notion of tameness, which was introduced by S. Friedman in [4]. Tameness is shown in [4, Proposition 2.20 and Theorem 2.21] to be equivalent to the preservation of the axioms of GB in a standard forcing setup. This yields the assumption of tameness for our preorder $\mathbb{P}$ to be a natural axiom in case we want to preserve the powerset axiom.

Definition 5.3.
- *A* predense $\leq p$ partition *is a pair $\langle D_0, D_1\rangle$ such that $D_0 \cup D_1$ is predense below p and such that $p_0 \in D_0 \wedge p_1 \in D_1$ implies that $p_0 \perp p_1$.*
- *Suppose that $\langle\langle D_0^i, D_1^i\rangle \mid i \in a\rangle$ and $\langle\langle E_0^i, E_1^i\rangle \mid i \in a\rangle$ are sequences of predense $\leq p$ partitions. We say that they are* equivalent $\leq p$ *if for each $i \in a$,*
$$\{q \in P \mid q \text{ meets } D_0^i \iff q \text{ meets } E_0^i\}$$
is dense below p.
- *$\mathbb{P}$ is* tame *(for $\mathcal{M} \models GB$) if $\mathbb{P}$ is pretame (for $\mathcal{M}$) and for each $a \in M$ and $p \in P$, there is $q \leq p$ and $\alpha \in \mathrm{Ord}(M)$ s.t. whenever*
$$\vec{D} = \langle\langle D_0^i, D_1^i\rangle \mid i \in a\rangle \in M$$
is a sequence of predense $\leq q$ partitions, then
$$\{r \in P \mid \vec{D} \text{ is equivalent } \leq r \text{ to some } \vec{E} = \langle\langle E_0^i, E_1^i\rangle \mid i \in a\rangle \in V_\alpha^M\}$$
is dense below q.

Given a countable transitive model $\mathcal{M}$ of T [12] and a pretame, or tame (in case T contains the powerset axiom) notion of class forcing $\mathbb{P} \in \mathcal{M}$, interpreting $a \in_p B$ as $\langle a, p\rangle \in B$, it is straightforward to verify axioms (1)–(3), ($\bar{6}$), and also (*) and (**) with respect to $\mathcal{M}$ and $\mathbb{P}$. For the forcing theorem, as described in axioms (4) and (5), see [7] and [6]. (***) follows by the results of Stanley and of Friedman mentioned above. This shows that in this case, $\mathcal{M}[G]$ is the usual forcing extension

[12]This is supposed to mean that both M and $\mathcal{C}$ are countable.

of $\mathcal{M}$ obtained by adjoining the generic filter G, for this forcing extension will satisfy T by our assumption of either pretameness or tameness.

6. Axiomatizations that do not assume the preservation of the axioms of set theory

Alternatively, we can replace ($\bar{6}$), (*), (**) and (***) by the following axioms, that is in particular we can derive the preservation of the axioms of set theory from natural axioms rather than requiring it. Axiom (6) is a strengthining of axiom ($\bar{6}$), which seems to be a natural requirement in the context of class forcing, and could be seen to essentially say that names for sets correspond to set-sized objects in a standard class forcing setup. Axiom (7) essentially says that we can freely (from the perspective of M) construct names in M. Axiom (8) is (pre)tameness, and is only required in the case of class forcing (for it is trivial in the case of set forcing).

(6) **Strong Set-Likeness:** If $b \in M$, then $\{\langle a,p\rangle \mid a \in_p b\} \in M$.

(7) **Existence of names:** If $S \in \mathcal{C}$ is a relation on $M \times \mathbb{P}$, then there exists $B \in \mathcal{C}$ such that

$$S(a,p) \text{ if and only if } a \in_p B.$$

Moreover, if $S \in M$, then we obtain the above B to be in M, and there is a map Γ that is first order definable over M which on input $S \in M$ yields one such witnessing $B \in M$. [13]

(8)
- **Pretameness:** $\mathbb{P}$ is pretame (for $\mathcal{M}$).
- **Tameness:** If T contains the power set axiom, then we make the stronger requirement that $\mathbb{P}$ is tame (for $\mathcal{M}$).

Lemma 6.1. *The axioms (*) and (**) can be derived from axiom (7).*

Proof. Essentially, axiom (7) allows us to construct analogues of the usual canonical names for ground model objects and for the generic filter. That is, using axiom (7), by recursion on rank, for $b \in M$, we define $\check{b}$ to be such that $x \in_p \check{b}$ if and only if $p = 1$ and x is of the form $\check{a}$ for some $a \in b$, and we do this in a definable way, making use of the map Γ. Note that using Γ, we in fact obtain a strong form of axiom (*): we have the extra property that the map from b to $\check{b}$ described above is definable over M. We then use this to define $\dot{G}$ to be such that $x \in_p \dot{G}$ if and only if $p \in P$ and $x = \check{p}$. [14] □

The proof of the following theorem mostly proceeds similar to the proof of [7, Theorem 3.1 (1)]. The verification of the powerset axiom in $\mathcal{M}[G]$ proceeds similar to the proof of [4, Theorem 2.21].

Theorem 6.2. *The axiom (***) can be derived from the axioms (1)–(8).*

Proof. Since $\mathcal{M}[G]$ is a transitive $\in$-structure, it clearly satisfies Regularity and Extensionality. Using axiom (7), it is easy to see that $M[G]$ satisfies Pairing, and by axiom (*), it satisfies Infinity.

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[13] This additional definability assumption is of course redundant when T yields the existence of a global well-order.

[14] Let us comment on the case of class forcing with the axiom of choice, but without the existence of a global well-order: In that case, without the additional definability assumption in axiom (7), we could still find a name $\check{b}$ for each $b \in M$, however it seems that constructing the name $\dot{G}$ would require us to make a proper class of choices.

Let us treat the union axiom: Let $a \in M[G]$ be given; we need to show that for some $b \in M[G]$, $\bigcup a \subseteq b$. Let $X = \{c \mid \exists p\ c \in_p a\} \in M$ by axiom $(\bar{6})$. Let $Y = \{d \mid \exists c \in X\, \exists q\ d \in_q c\} \in M$ by axiom $(\bar{6})$. Using axiom (7), let $\dot{b} \in M$ be such that $d \in_r \dot{b}$ if and only if $d \in Y$ and $r = 1$. Using the definition of F_G, it is straightforward to check that $b = F_G(\dot{b})$ is as desired.

Let us verify first order class comprehension in $\mathcal{M}[G]$. If $\varphi \in \mathcal{L}(\in)$ and $B \in \mathcal{C}$, using axiom (7), let $A \in \mathcal{C}$ be such that $a \in_p A$ if and only if $p \Vdash \varphi(a, B)$. It then follows that $F_G(A) = \{x \in M[G] \mid \mathcal{M}[G] \models \varphi(x, F_G(B))\}$.

We now show that $\mathcal{M}[G]$ satisfies Collection. Let $a \in M$, $A \in \mathcal{C}$, and assume that $\mathcal{M}[G] \models \forall x \in F_G(a)\, \exists y\ \langle x, y\rangle \in F_G(A)$. We need to find $b \in M$ such that $\mathcal{M}[G] \models \forall x \in F_G(a)\, \exists y \in F_G(b)\ \langle x, y\rangle \in F_G(A)$.

By axiom (4), we may pick some $p \in G$ such that $p \Vdash \forall x \in a\, \exists y\ \langle x, y\rangle \in A$. By axiom (6), $X = \{\langle c, r\rangle \mid c \in_r a\} \in M$. For each $\langle c, r\rangle \in X$, let

$$D_{\langle c,r\rangle} = \{s \in \mathbb{P} \mid [s \leq p, r \,\wedge\, \exists d \in M\ s \Vdash \langle c, d\rangle \in A] \,\vee\, s \perp r\} \in \mathcal{C}\,.$$

By Lemma 4.5, it follows that each $D_{\langle c,r\rangle}$ is dense below p in $\mathbb{P}$. Using that $\mathbb{P}$ is pretame by axiom (8), there is $q \in G$ below p and there is a sequence $\langle d_{\langle c,r\rangle} \mid \langle c, r\rangle \in X\rangle \in M$ such that $d_{\langle c,r\rangle} \subseteq D_{\langle c,r\rangle}$ and $d_{\langle c,r\rangle}$ is predense below q in $\mathbb{P}$ for each $\langle c, r\rangle \in X$.

Using Collection in M, there is a set $Y \in M$ such that for each $\langle c, r\rangle \in X$ and for each $s \in d_{\langle c,r\rangle}$ that is compatible with r, there is $d \in Y$ such that $s \Vdash \varphi(c, d, A)$. Using axiom (7), let $b \in M$ be such that $d \in_s b$ if and only if $d \in Y$ and there is $\langle c, r\rangle \in X$ such that $s \in d_{\langle c,r\rangle}$ and $s \Vdash \varphi(c, d, A)$. Then, by construction, $\mathcal{M}[G] \models \forall x \in F_G(a)\, \exists y \in F_G(b)\ \langle x, y\rangle \in F_G(A)$, as desired.

Let us next show that $\mathcal{M}[G]$ satisfies Separation. Let $a \in M$ and $A \in \mathcal{C}$. We need to find $b \in M$ such that $\mathcal{M}[G] \models F_G(b) = F_G(a) \cap F_G(A)$. By axiom (6), $X = \{\langle c, r\rangle \mid c \in_r a\} \in M$. For each $\langle c, r\rangle \in X$, let $D_{\langle c,r\rangle} = \{q \leq r \mid q \Vdash c \in A\} \in \mathcal{C}$. Using that $\mathbb{P}$ is pretame by axiom (8), we may pick $p \in G$ and a sequence $\langle d_{\langle c,r\rangle} \mid \langle c, r\rangle \in X\rangle \in M$ such that $d_{\langle c,r\rangle} \subseteq D_{\langle c,r\rangle}$ and $d_{\langle c,r\rangle}$ is predense below p in $\mathbb{P}$ for each $\langle c, r\rangle \in X$. Using axiom (7), let $b \in M$ be such that $c \in_q b$ if and only if there is $r \in \mathbb{P}$ such that $\langle c, r\rangle \in X$, $q \in d_{\langle c,r\rangle}$ and $q \Vdash c \in A$. It clearly follows that b is as desired.

Let us argue that the axiom of choice is preserved. Let $a \in M$. We have to find a well-order $\lhd$ of $F_G(a)$ in $M[G]$. Using axiom $(\bar{6})$, let $X = \{c \mid \exists r \in P\ c \in_r a\} \in M$, and let $\prec$ be a well-order of X, using the axiom of choice in M. Now given x and y in $F_G(a)$, we simply let $x \lhd y$ if and only if there is $\dot{x} \in X$ such that $x = F_G(\dot{x})$ and for all $\dot{y} \in X$ such that $y = F_G(\dot{y})$, we have $\dot{x} \prec \dot{y}$. Using that comprehension holds in $\mathcal{M}[G]$, we thus obtain $\lhd \in M[G]$.

Let's argue that if $\mathcal{M}$ has a set-like global well-order $\prec \in \mathcal{C}$, then we can find a set-like global well-order $\lhd \in \mathcal{C}[G]$. Note that $M \in \mathcal{C} \subseteq \mathcal{C}[G]$ by axiom (*). But by our absoluteness assumptions, the restriction of F_G to M is in $\mathcal{C}[G]$. We thus simply let $x \lhd y$ if and only if there is $\dot{x} \in M$ such that $x = F_G(\dot{x})$ and for all $\dot{y} \in M$ such that $y = F_G(\dot{y})$, we have $\dot{x} \prec \dot{y}$. Using that first order class comprehension holds in $\mathcal{M}[G]$, we thus obtain $\lhd \in \mathcal{C}[G]$.

Finally, we argue that assuming $\mathbb{P}$ to be tame, the powerset axiom is preserved. It suffices to show that $\mathcal{P}(a)$ exists in $M[G]$ for every $a \in M$: If $a \in M[G]$, using axiom ($\bar{6}$), let $X = \{c \mid \exists r \in P\ c \in_r a\} \in M$. Using that the restriction of F_G to M is in $\mathcal{C}[G]$, and using that replacement holds in $\mathcal{M}[G]$, there is a surjection from X onto a in $M[G]$, and this surjection naturally induces a surjection from $\mathcal{P}(X)$ onto $\mathcal{P}(a)$ in $\mathcal{C}[G]$. Using replacement in $\mathcal{M}[G]$ once again, if $\mathcal{P}(X)^{M[G]} \in M[G]$, it follows that $\mathcal{P}(a) \in M[G]$.

Therefore, let $a \in M$. Using tameness and Lemma 4.2, let $p \in G$ and $\alpha \in \mathrm{Ord}(M)$ be such that whenever $\vec{d} = \langle\langle d_0^i, d_1^i\rangle \mid i \in a\rangle \in M$ is a sequence of predense $\leq p$ partitions, there is $r \in G$, $r \leq p$ and $\vec{e} \in V_\alpha^M$ such that $\vec{d}$ and $\vec{e}$ are equivalent $\leq r$. Now for any $\sigma \in M$ such that $\sigma^G \subseteq a$, consider the sequence of classes $\vec{D} = \langle\langle D_0^i, D_1^i\rangle \mid i \in a\rangle$ defined by letting $D_0^i = \{q \mid q \Vdash \check{i} \notin \sigma\}$ and $D_1^i = \{q \mid q \Vdash \check{i} \in \sigma\}$. By pretameness, we may pick a predense $\leq p$ partition $\vec{d} = \langle\langle d_0^i, d_1^i\rangle \mid i \in a\rangle \in M$ such that $d_0^i \subseteq D_0^i$ and $d_1^i \subseteq D_1^i$ for every $i \in a$. By our assumptions, we find $\vec{e} \in V_\alpha^M$ such that

$$i \in \sigma^G \iff G \cap e_1^i \neq \emptyset,$$

and therefore using axiom (7), letting $\sigma_0 \in M$ be such that $c \in_p \sigma_0$ if and only if $c = \check{i}$ for some $i \in a$ and $p \in e_1^i$, it follows that $\sigma^G = {\sigma_0}^G$. Making use of the map Γ described in axiom (7), we have a definable choice of such σ_0's in M given $\vec{e} \in V_\alpha^M$, and we thus obtain a set $\Sigma \in M$ of such σ_0's. Using axiom (7) once again, let $\pi \in M$ be such that $c \in_p \pi$ if and only if $c = \sigma_0$ for some $\sigma_0 \in \Sigma$ and $p = 1$. This clearly yields π^G to be the powerset of a in $\mathcal{M}[G]$, as desired. $\square$

7. Set forcing

The axioms for set forcing, which are essentially as in [3], can easily be derived from the case of class forcing over models of *GBc* that was treated in this paper. In addition, we will also consider the base theories ZF$^-$, ZFC$^-$ and ZF, which correspond to the second order theories GB^-, GBc^- and GB. Let T thus denote our base theory, which will either be ZF$^-$, ZFC$^-$, ZF or ZFC. Our terminology will be slightly different from that of [3]. In this section, let $\mathcal{L}(\in)$ denote the collection of first order formulas.

Definition 7.1. *A* set forcing generic framework *is a tuple of the form*

$$\left\langle M, \mathbb{P}, R, (\Vdash_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G} \right\rangle \text{ with the following properties.}$$

- *M is a transitive set-size model of T.*
- *$\mathbb{P} = \langle P, \leq\rangle \in M$ is a preorder.*
- *The* $\mathbb{P}$-membership relation *R is a relation on $P \times M^2$ that is definable over M by an $\mathcal{L}(\in)$-formula.*

  We also require φ to be absolute for transitive models of T containing M, and we denote the property $R(p, a, b)$ as $a \in_p b$. [15]
- *$\mathfrak{G}$ is a second order unary predicate on P, i.e. a unary predicate on $\mathcal{P}(P)$, and we require that $\mathfrak{G}(G)$ implies that $G \subseteq \mathbb{P}$ is a filter. If $\mathfrak{G}(G)$ holds, we say that G is a* generic filter, *or a* $\mathbb{P}$-generic filter *on M. Whenever we quantify over G in the following, we assume that we quantify over G's such that $\mathfrak{G}(G)$ holds.*
- *For every $\varphi \in \mathcal{L}(\in)$, $\Vdash_\varphi \in \mathcal{C}$ is a predicate (which we also call a* forcing relation for φ*) on $P \times M^m$, where m denotes the number of free variables of φ. If $\langle q, a_0, \ldots, a_{m-1}\rangle \in \Vdash_\varphi$, we also write $q \Vdash \varphi(a_0, \ldots, a_{m-1})$.*

[15] In a standard forcing setup, this would correspond to the property that $\langle a, p\rangle \in b$.

Let us briefly provide the list of corresponding axioms for set forcing.

(S1) **Existence of generic filters:** $\forall p \in P \, \exists G \; p \in G$.
(S2) **Well-Foundedness:** The binary relation $\exists p \in \mathbb{P} \;\; a \in_p b$ on M is well-founded.
(S3) **Growth of Information:** For all $\varphi \in \mathcal{L}(\in)$, for all $\vec{a} \in M$, and $p, q \in P$, if $p \Vdash \varphi(\vec{a})$ and $q \leq p$, then $q \Vdash \varphi(\vec{a})$.

Define the relation $\in_G$ on M by letting $a \in_G b$ if $\exists p \in G \; a \in_p b$. Recursively define our *evaluation function* F_G along the relation $\in_G$, letting $F_G(a) = \{F_G(b) \mid b \in_G a\}$ for each $a \in M$. Let $M[G]$ denote the $\in$-structure on the transitive set $F_G[M]$.

(S4) **Truth Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in M$ and all G,

$$M[G] \models \varphi(F_G(\vec{a})) \text{ iff } \exists p \in G \; p \Vdash \varphi(\vec{a}).$$

(S5) **Definability Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in M$ and $p \in P$,

$$p \Vdash \varphi(\vec{a}) \text{ iff } \forall G \ni p \; M[G] \models \varphi(F_G(\vec{a})).$$

(S6) **Set-Likeness:** If $b \in M$, then $\{a \mid \exists p \in P \; a \in_p b\} \in M$.
(S*) **Names for ground model objects:** $\forall a \in M \, \exists \check{a} \in M \, \forall G \; F_G(\check{a}) = a$.
(S**) **Name for generic filters:** $\exists \dot{G} \in M \; \forall G \; F_G(\dot{G}) = G$.
(S***) **Preservation of axioms:** $\forall G \; M[G] \models T$.

We can use these axioms to deduce the following version of Theorem 5.1. A *set forcing generic extension* is a set forcing generic framework together with a particular choice of generic filter G.

Theorem 7.2. *Given a particular set forcing generic extension, assuming axioms (S1) to (S6), and also axioms (S*), (S**) and (S***) to hold, $M[G]$ is actually well-defined in the sense that it depends only on M and on G, and is in fact the $\subseteq$-smallest model N of T that contains $M \cup \{G\}$ as a subset.*

Proceeding toward a set forcing analogue of the results of Section 6, we introduce one further axiom.

(S7) **Existence of names:** If $s \in M$ is a relation on $M \times \mathbb{P}$, then there exists $b \in \mathcal{M}$ such that

$$s(a, p) \text{ if and only if } a \in_p b.$$

In case the axiom of choice is not contained in T, we additionally assume that there is a map Γ that is first order definable over M which on input s yields one such witnessing $b \in M$.

The arguments from Section 6 then also yield the following.

Theorem 7.3. *Axioms (S*), (S**) and (S***) can be deduced from axioms (S1)–(S7).*

8. Kelley-Morse set theory

Kelley-Morse set theory KM extends the axioms of GBC by the scheme of second order class comprehension. Let $\mathcal{L}^2(\in)$ denote the collection of all second order formulas in the language with the $\in$-predicate.

Definition 8.1. *A* class forcing generic framework for *KM is a tuple of the form*

$$\left\langle \mathcal{M}, \mathbb{P}, R, \left(\Vdash_{\varphi}^{\vec{A}}\right)_{\varphi\in\mathcal{L}^2(\in),\vec{A}\in\mathcal{C}}, \mathfrak{G} \right\rangle$$

with the same properties as those of a class forcing generic framework in Definition 2.1, except that the last item from that definition has to be changed to the following.

- *For every $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}^2(\in)$ for which the number of free second order variables corresponds to the length of $\vec{A}$, $\Vdash_{\varphi}^{\vec{A}} \in \mathcal{C}$ is a predicate (which we also call a* forcing relation for *φ) on $P \times M^m$, where m denotes the number of free first order variables of φ.*
  If $\langle q, a_0, \ldots, a_{m-1}\rangle \in \Vdash_{\varphi}^{\vec{A}}$, we also write $q \Vdash \varphi(a_0, \ldots, a_{m-1}, \vec{A})$.

Axioms (1)–(5) are the same as in Section 3, except that $\mathcal{L}(\in)$ has to be replaced by $\mathcal{L}^2(\in)$ throughout, and that in axioms (4) and (5), we have to refer to the number of free second order variables, rather than just the number of second order variables. Axioms ($\bar{6}$), (*) and (**) are exactly as in Section 3.

We can then use the above axioms to show that the forcing predicates obey their usual defining clauses as in Section 4, except that we have to consider additional types of formulas. As in the case of *GBC* we can avoid atomic formulas expressing (in)equality for classes, for we may consider those to be defined in terms of the $\in$-relation. But we need to treat second order quantification, extending Lemma 4.5 by the following clause. It is verified essentially by the same argument as for first order quantification in [3], however we would like to provide the argument for the benefit of our readers.

Lemma 8.2. $p \Vdash \exists X \varphi(\vec{a}, \vec{A})$ *iff* $\forall r \leq p\, \exists q \leq r\, \exists X \in \mathcal{C}\;\; q \Vdash \varphi(\vec{a}, \vec{A})$.

Proof. Let us assume that

(i) $p \Vdash \exists X \varphi(X, \vec{a}, \vec{A})$.

By axiom 5, this is equivalent to

(ii) $\forall G \ni p\;\; \mathcal{M}[G] \models \exists X \varphi(X, F_G(\vec{a}), F_G(\vec{A}))$.

By the definition of $\mathcal{M}[G]$, this is in turn equivalent to

(iii) $\forall G \ni p\; \exists X \in \mathcal{C}\;\; \mathcal{M}[G] \models \varphi(F_G(\dot{X}), F_G(\vec{a}), F_G(\vec{A}))$.

Making use of axiom 4, we equivalently obtain

(iv) $\forall G \ni p\; \exists X \in \mathcal{C}\;\; \exists q \in G\; q \Vdash \varphi(\dot{X}, \vec{a}, \vec{A})$.

Applying Lemma 4.3 yields equivalence to

(v) $\forall r \leq p\; \exists q \leq r\; \exists X \in \mathcal{C}\;\; q \Vdash \varphi(X, \vec{a}, \vec{A})$,

as desired. □

We can now let axiom (***) be as in Section 5, and verify Theorem 5.1 exactly as in that section also when $T = KM$. It is shown in [1] that tame forcing preserves KM.

Let axiom (6) be exactly as in Section 6. Let axiom (7) be exactly as in Section 6, except that we can ignore its additional definability assumption, for the axioms of KM include the existence of a global well-order. Let axiom (8) be the assumption that $\mathbb{P}$ is tame (for $\mathcal{M}$), as in Section 6, noting that the axioms of KM include the power set axiom.

As in Section 6, we would like to show that our axioms (1)–(8) imply axioms (*), (**) and (***). For (*) and (**), this is shown exactly as in Lemma 6.1. The verification of (***) proceeds exactly as in Theorem 6.2, however we have to additionally verify second order class comprehension in $\mathcal{M}[G]$. This is done by essentially the same easy argument as for first order class comprehension from the proof of Theorem 6.2.

Proposition 8.3. *$\mathcal{M}[G]$ satisfies second order class comprehension.*

Proof. If $\varphi \in \mathcal{L}^2(\in)$ and $B \in \mathcal{C}$, using axiom (7), let $A \in \mathcal{C}$ be such that $a \in_p A$ if and only if $p \Vdash \varphi(a, B)$. It then follows that $F_G(A) = \{x \in M[G] \mid \mathcal{M}[G] \models \varphi(x, F_G(B))\}$. □

9. Extensions of KM

The results of the previous section can also be extended to base theories beyond KM. As a sample result, let us finally investigate the case of the base theory T that is KM together with the axiom of class choice (CC), which was first studied by A. Mostowski and W. Marek in [8].

Definition 9.1.
- *Given $Y \in \mathcal{C}$ and $\alpha \in \mathrm{Ord}$, we let $Y_\alpha = \{x \mid \langle \alpha, x\rangle \in Y\}$.*
- CC *is the statement that whenever $\varphi \in \mathcal{L}^2(\in)$, $A \in \mathcal{C}$, and for every ordinal α there is some $X \in \mathcal{C}$ such that $\varphi(\alpha, X, A)$ holds, then there is $Y \in \mathcal{C}$ such that for every ordinal α, we have $\varphi(\alpha, Y_\alpha, A)$.*

Note that in the above, we could equivalently use arbitrary sets rather than just ordinals as indices, for our base theory KM provides us with a global well-order in order-type Ord. We will tacitly make use of this in the below.

It is shown in [2] that tame forcing preserves $KM + \mathrm{CC}$. All results from the previous section thus apply to the stronger base theory $T = KM + \mathrm{CC}$ as well – we only need to verify that axioms (1)–(8) from Section 8 imply the preservation of CC. Our proof of this is based on the argument from [2]. Given $x, y \in M$, let $\{\check{x,y}\}$ be defined by letting $a \in_p \{\check{x,y}\}$ if and only if $a \in \{x, y\}$ and $p = 1$. Let $\langle \check{x,y}\rangle$ be defined by letting $a \in_p \langle \check{x,y}\rangle$ if and only if $a \in \{\{\check{x,x}\}, \{\check{x,y}\}\}$ and $p = 1$. Thus, $\langle \check{x,y}\rangle$ is a canonical name for the ordered pair $\langle F_G(x), F_G(y)\rangle$ in our setup.

Proposition 9.2. $\mathcal{M}[G] \models \mathrm{CC}$.

Proof. Let $A \in \mathcal{C}$. As a first step, we will argue that a strong form of Lemma 8.2 holds, namely whenever $p \Vdash \exists X \varphi(X, A)$, then there is $X \in \mathcal{C}$ such that $p \Vdash \varphi(X, A)$. Using (second order) class comprehension in $\mathcal{M}$, let $D \in \mathcal{C}$ be the dense class of all $q \le p$ for which there is some $X_q \in \mathcal{C}$ such that $q \Vdash \varphi(X_q, A)$. Let A be a maximal antichain in D. Using CC in $\mathcal{M}$, for every $q \in A$, we may pick some $X_q \in \mathcal{C}$ such that $q \Vdash \varphi(X_q, A)$. Now we obtain our desired $X \in \mathcal{C}$ making use of axiom (7), letting $a \in_r X$ if and only if

$$r \le q \wedge a \in \mathrm{dom}(X_q) \wedge r \Vdash a \in X_q.$$

Now suppose that $\mathcal{M}[G] \models \forall\alpha\, \exists X\ \varphi(\alpha, X, F_G(A))$. By axiom (4), there is $p \in G$ such that $p \Vdash \forall\alpha\, \exists X\ \varphi(\alpha, X, A)$. For any $\alpha \in \mathrm{Ord}^{\mathcal{M}}$, by the above, we find some $X^\alpha \in \mathcal{C}$ such that $p \Vdash \varphi(\alpha, X^\alpha, A)$. Using that our forcing relations are in $\mathcal{C}$, and using CC in $\mathcal{M}$, we may obtain $X \in \mathcal{C}$ such that $\forall\alpha\ p \Vdash \varphi(\alpha, X_\alpha, A)$. Making use of axiom (7), define $Y \in \mathcal{C}$ by letting $a \in_r Y$ if and only if for some

$\alpha \in \mathrm{Ord}^{\mathcal{M}}$, $a = \langle \check{\alpha, x} \rangle$ and r is such that $x \in_r X_\alpha$. It follows that for all $\alpha \in \mathrm{Ord}$, $\mathcal{M}[G] \models \varphi(\alpha, F_G(X)_\alpha, F_G(A))$, as desired. $\square$

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