A QUASI LOWER BOUND ON THE CONSISTENCY STRENGTH OF PFA

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Abstract. A long-standing open question is whether supercompactness provides a lower bound on the consistency strength of the Proper Forcing Axiom (PFA). In this article we establish a quasi lower bound by showing that there is a model with a proper class of subcompact cardinals such that PFA (indeed the weaker statement that PFA holds for $(2^{\aleph_0})^+$-linked forcings) fails in all of its proper forcing extensions. Neeman obtained such a result assuming the existence of "fine structural" models containing very large cardinals, however the existence of such models remains open. We show that Neeman's arguments go through for a similar notion of "L-like" model and establish the existence of L-like models containing very large cardinals. The main technical result needed is the compatibility of Local Club Condensation with Acceptability in the presence of very large cardinals, a result which constitutes further progress in the outer model programme.

The *core model programme* (initiated by Jensen, see Steel's [16] for a survey) has had considerable success in establishing lower bounds on the consistency strength of set-theoretic statements, up to the level of Woodin cardinals. But the consistency strength of the Proper Forcing Axiom (PFA) is conjectured to be that of a supercompact cardinal, for which no core model theory is currently available. It is therefore worthwhile to consider *quasi lower bounds* on the consistency strength of PFA and the main result of this paper is that a proper class of subcompact cardinals serves as such a quasi lower bound:

Theorem 1. *Assuming the consistency of a proper class of subcompact cardinals, it is consistent that there is a proper class of subcompact cardinals, but* PFA *(even restricted to posets which are* $(2^{\aleph_0})^+$*-linked) holds in no proper extension*[1] *of the universe.*

What exactly is meant by a *quasi lower bound*?

The necessary ingredients are

- the desired set-theoretic principle φ for which we want to obtain a quasi-lower bound result

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[1]A proper extension of the universe is an extension of the universe which preserves the stationarity of S for every stationary $S \subseteq [\gamma]^{\aleph_0}$ for all γ. In particular, every proper forcing extension of the universe is a proper extension of the universe.

- a collection A of assumptions on the ground model (such as being "L-like")
- the type of extensions B of the universe to be considered (such as all proper extensions)
- a large cardinal type C (this will be our quasi-lower bound)
- a (larger) large cardinal type D (an upper bound for the consistency of φ)

The statement that the existence of a large cardinal (or large cardinals) of type C is a (strict) quasi lower bound for φ with respect to extensions of type B in fact consists of three parts:

(a) If $\mathbf{V}$ satisfies the assumptions in A and φ holds in an extension of type B, then $\mathbf{V}$ contains a large cardinal larger (in terms of consistency strength) than those of type C.
(b) Assuming large cardinals, the assumptions in A are consistent with the existence of a large cardinal (or large cardinals) of type D.
(c) An upper bound result: given a universe with a large cardinal (or large cardinals) of type D, φ holds in an extension of type B.

(a) and (b) give the following corollary:

(d) Assuming the consistency of large cardinals, it is consistent that there is a large cardinal (or there are large cardinals) of type C, but φ fails in all extensions of the universe of type B.[2]

Theorem 1 is (d) of a result of this form, where φ is PFA$(2^{\aleph_0})^+$-linked, A are some assumptions about being $\mathbf{L}$-like that will later be provided in full detail, B refers to all proper extensions of the universe, our quasi-lower bound C is a proper class of subcompacts and D is a supercompact cardinal. For the proof of this quasi-lower bound result, (a) is a variant of Neeman's [14], (b) is the main work of this paper, which constitutes an advance in the *outer model programme* and (c) is Baumgartner's classic result that if there is a supercompact cardinal, then PFA holds in a proper forcing extension. Note that in light of (c), (b) rules out the possibility that it is the assumptions in A that imply φ to fail in all extensions of type B, and not the lack of large cardinals (stronger than those of type C) in $\mathbf{V}$.

The aim of the *outer model programme* (see [9]) is to show that large cardinal properties can be preserved when forcing desirable features of Gödel's constructible universe. In [10], Local Club Condensation was shown to be consistent with the existence of an ω-superstrong cardinal. The main work (b) of the present paper strengthens this result by demanding that the witnessing predicate for Local Club Condensation be acceptable:

[2]An additional requirement could be made:

(b*) Assuming a large cardinal (or large cardinals) of type C, the assumptions in A are consistent with the existence of a large cardinal (or large cardinals) of type C.

We will also verify (b*) for the quasi lower bound result of the present paper. (b*) gives the stronger corollary

(d*) Assuming the consistency of a large cardinal (or large cardinals) of type C, it is consistent that there is a large cardinal (or there are large cardinals) of type C, but φ fails in all extensions of the universe of type B.

Theorem 2. *Local Club Condensation and Acceptability are simultaneously consistent with the existence of an ω-superstrong cardinal.*

Acceptability is discussed in Section 1 below, where also the definition of Local Club Condensation is provided. κ is ω-superstrong iff it is the critical point of an embedding $j\colon \mathbf{V} \to \mathbf{M}$ with $V_{j^\omega(\kappa)} \subseteq \mathbf{M}$.

In [14], Neeman proved the following:

Theorem 3 (Neeman)**.** [14] *Suppose* $\mathbf{V}$ *is a proper extension of a fine structural model* $\mathbf{M}$ *and* $\mathrm{PFA}((2^{\aleph_0})^+\text{-linked})$ *holds in* $\mathbf{V}$. *Then* $[\kappa, \kappa^+)$ *is* Σ^2_1*-indescribable in* $\mathbf{M}$*, where* $\kappa = (\omega_2)^{\mathbf{V}}$.

We will not define "fine structural" here as we will not need it. A forcing P is κ-linked iff there is a function $f : P \to \kappa$ such that p, q are compatible whenever $f(p) = f(q)$. For $[\kappa, \kappa^+)$ to be Σ^2_1-indescribable is a light strengthening of subcompactness of κ:

Definition 4. [13] $[\kappa, \kappa^+)$ *is* Σ^2_1*-indescribable if for every* $Q \subseteq H_{\kappa^+}$ *and every first order formula* φ *with one free variable, whenever there exists* $B \subseteq H_{\kappa^{++}}$ *such that* $(H_{\kappa^{++}}, \in, B) \models \varphi(Q)$*, there exists a cardinal* $\bar\kappa < \kappa$ *and* $\bar Q \subseteq H_{\bar\kappa^+}$ *such that*

- $\exists \bar B\ (H_{\bar\kappa^{++}}, \in, \bar B) \models \varphi(\bar Q)$ *and*
- *there exists an elementary embedding* $\pi\colon (H_{\bar\kappa^+}, \in, \bar Q) \to (H_{\kappa^+}, \in, Q)$ *with* $\pi \restriction \bar\kappa = \mathrm{id}$.

Definition 5 (Jensen, see [5])**.** $\square$ on the singular cardinals *denotes the principle that there is a sequence* $\langle C_\alpha \colon \alpha \text{ a singular cardinal}\rangle$*, such that for each* α*,* C_α *is club in* α *consisting only of cardinals and its order-type is smaller than* α*, the limit points of* C_α *are singular cardinals and* $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$ *whenever* $\bar\alpha$ *is a limit point of* C_α.

As a variant of Neeman's theorem, we have the following result, which constitutes part (a) of our quasi-lower bound result :

Theorem 6. *If* $\mathbf{V}$ *is a proper extension of a model* $\mathbf{M}$*,* $\mathbf{M}$ *satisfies Local Club Condensation, Acceptability,* $\square$ *on the singular cardinals and* $\square_\lambda$ *for every singular* λ *and* $\mathbf{V}$ *satisfies* $\mathrm{PFA}((2^{\aleph_0})^+\text{-linked})$*, then there is a* Σ^2_1*-indescribable gap* $[\kappa, \kappa^+)$ *in* $\mathbf{M}$.

No fine structural models in the sense of [14] containing large cardinals stronger than Woodin limits of Woodin cardinals have been constructed so far, reducing the import of Neeman's theorem. Our work removes this defect, not by constructing fine structural models in the original sense of Neeman, but by constructing models containing very large cardinals with properties sufficiently close to those required by Neeman in [14] for his line of argument to go through.

Related work has been done independently by Matteo Viale and Christoph Weiß in [17], using completely unrelated techniques:

Definition 7. [17] *Let* P *be a notion of forcing.* P *is a standard iteration of length* κ *if*

- P *is the direct limit of an iteration* $\langle P_\alpha \colon \alpha < \kappa\rangle$ *that takes direct limits stationarily often.*

- *P_α has size less than κ for all $\alpha < \kappa$.*

Theorem 8 (Viale, Weiß)**.** [17] *Suppose that κ is inaccessible and* PFA *can be forced by a standard iteration of length κ that collapses κ to ω_2. Then κ is strongly compact. Moreover, if the iteration is proper then κ is supercompact.*

In fact, their result applies not to just standard iterations, but to arbitrary extensions which satisfy the κ-covering and κ-approximation properties. Note that these results are incomparable with ours: They obtain full supercompactness from PFA, whereas we only obtain a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ (already from a fragment of PFA), a notion strictly weaker than supercompactness. On the other hand, we do not need to assume covering or approximation properties.

This paper is organised as follows. In Section 1 we discuss binary functional predicates and Acceptability and the relationship of the latter to GCH. In Section 2 we prove – as a kind of warm-up – that Acceptability and Stationary Condensation (introduced in [10]) are simultaneously consistent with very large cardinals. The heart of the paper is Section 3, where we prove the same for Acceptability and Local Club Condensation. After a discussion of smaller large cardinals and Strong Condensation for ω_2 in Sections 4 and 5, we provide our quasi lower bound result in Section 6. Sections 7 and 8 show that Local Club Condensation implies certain $\diamondsuit$ principles and is strictly stronger than Stationary Condensation. We end with Open Questions in Section 9.

1. Binary Functional Predicates, Acceptability and the GCH

Definition 9. *We say that A is a binary functional predicate (bfp) on the ordinals if $A\colon \mathbf{Ord} \times \mathbf{Ord} \to \mathbf{Ord}$ is s.t. $\forall \alpha \in \mathbf{Ord}$ $\mathrm{range}(A{\restriction}(\alpha \times \alpha)) \subseteq \alpha$.*

We say that A is a bfp on α if $A\colon \alpha \times \alpha \to \alpha$ and for every $\beta < \alpha$, $\mathrm{range}(A{\restriction}(\beta \times \beta)) \subseteq \beta$. If $\alpha < \beta$, we let $[\alpha, \beta)^{\urcorner} := (\beta \times [\alpha, \beta)) \cup ([\alpha, \beta) \times \beta)$ and say that A is a bfp on $[\alpha, \beta)$ if $\mathrm{dom}(A) = [\alpha, \beta)^{\urcorner}$ and for every $\gamma < \beta$, $\mathrm{range}(A{\restriction}(\gamma \times \gamma)) \subseteq \gamma$. We say that A is a bfp if A is a bfp on $\mathbf{Ord}$, A is a bfp on α for some ordinal α or A is a bfp on $[\alpha, \beta)$ for ordinals $\alpha < \beta$. For a bfp A, we write $A{\restriction}\alpha$ instead of $A{\restriction}(\alpha \times \alpha)$ and we write $A{\restriction}[\alpha, \beta)$ instead of $A{\restriction}[\alpha, \beta)^{\urcorner}$. We write $A(\alpha)$ for $A{\restriction}\{\alpha\} = A{\restriction}[\alpha, \alpha + 1)$. We will sometimes be sloppy about domains or restrictions of bfps when they are either obvious from context or irrelevant.

Definition 10. *If A is a bfp, we define the hierarchy $\langle L_\alpha[A] \colon \alpha \in \mathbf{Ord}\rangle$ verbatim as one would do for a standard unary predicate A: $L_0[A] = \emptyset$, $L_{\alpha+1}[A]$ is the set of all $y \subseteq L_\alpha[A]$ which are definable by a first-order formula using parameters and allowed to refer to $A{\restriction}(\alpha \times \alpha)$ over $L_\alpha[A]$ and $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ for limit ordinals α. We let $\mathbf{L}[A] = \bigcup_{\alpha\in\mathbf{Ord}} L_\alpha[A]$.*

Before turning to Acceptability, we now want to introduce the principle of Local Club Condensation in order to be able to make an important remark towards the end of this section. If $\mathcal{B}$ has domain B and is a substructure of some structure on $L_\alpha[A]$, we say that $\mathcal{B}$ *condenses* or that $\mathcal{B}$ *has Condensation* iff $(B, \in, A)$ is isomorphic to some $(L_{\bar{\alpha}}[A], \in, A)$. We also say that B

condenses or that B has Condensation in this case. We say that A codes $\mathbf{M}$ iff $\mathbf{M} = \mathbf{L}[A]$. For any set X, $\operatorname{card} X$ denotes the cardinality of X. Reformulated in the context of models of the form $\mathbf{L}[A]$ for a bfp A, Local Club Condensation (originally introduced in [10]) is defined as follows:

Local Club Condensation for $\mathbf{L}[A]$ is the principle that if α has uncountable cardinality κ and $\mathcal{A} = (L_\alpha[A], \in, A, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \leq \gamma < \kappa \rangle$ of condensing substructures of $\mathcal{A}$ whose domains B_γ have union $L_\alpha[A]$, each B_γ has cardinality $\operatorname{card} \gamma$ and contains γ as a subset. We say that A witnesses Local Club Condensation (in $\mathbf{V}$) if A codes $\mathbf{V}$ and Local Club Condensation for $\mathbf{L}[A]$ holds.

In the following, we will often use the fact that there is a reasonable definition of ordered pairs in $\mathbf{L}[A]$ which does not increase rank, i.e. an injective, definable and very absolute operation $(.\,,.)\colon \mathbf{L}[A]^2 \to \mathbf{L}[A]$ such that for every ordinal α, $a, b \in L_\alpha[A]$ implies $(a, b) \in L_\alpha[A]$. That this can be done in $\mathbf{L}$ is shown in [2]. That this can be generalized to $\mathbf{L}[A]$ is easily observed. When we use ordered pairs in the following, we will usually assume (tacitly) that they are coded in this way. In particular when we view functions as sets of ordered pairs, those ordered pairs will also be coded in such a way. Hence for every ordinal α, $A{\restriction}(\alpha \times \alpha) \in L_{\alpha+1}[A]$. Another fact which we will use and a proof of which can again be found in [2] is that a satisfaction predicate for L_α can be defined (uniformly in α) over L_β whenever $\beta > \alpha$. That this can be generalized to $\mathbf{L}[A]$ is again an easy observation. Yet another fact of importance will be that each L_α has a canonical definable wellorder thus allowing for Skolem Hulls to be taken within every L_α. This is again shown in [2] and can easily be generalized to $\mathbf{L}[A]$, giving rise to definable wellorders for each $L_\alpha[A]$ and thus allowing us to take Skolem Hulls within every $L_\alpha[A]$. Finally, we will also use that Gödel's Condensation Lemma holds not only for limit levels of the $\mathbf{L}$-hierarchy, but also for it's successor levels, i.e. for every ordinal α, if $M \prec (L_\alpha, \in)$, then M is isomorphic to $(L_{\bar\alpha}, \in)$ for some $\bar\alpha \leq \alpha$. A proof of this fact can again be found in [2]. This easily generalizes to the following: For every ordinal α, if $M \prec (L_\alpha[A], \in, A)$, then M is isomorphic to $(L_{\bar\alpha}[\bar A], \in, \bar A)$ for some $\bar\alpha \leq \alpha$ and some $\bar A$.

Definition 11. *If A is a bfp, we say that there is or there appears a new subset of δ in $L_{\gamma+1}[A]$ iff there is $x \subseteq \delta$ so that $x \in L_{\gamma+1}[A] \setminus L_\gamma[A]$. We say that all subsets of δ appear before ν in M, where M is either $L[A]$ or $L_\alpha[A]$ for some ordinal $\alpha > \nu$, iff for all $x \subseteq \delta$*

$$x \in M \to x \in L_\nu[A].$$

Definition 12. *If A is a bfp, we say that A is acceptable iff for any ordinals $\gamma \geq \delta$, if there is a new subset of δ in $L_{\gamma+1}[A]$, then*

$$H^{L_{\gamma+1}[A]}(\delta) = L_{\gamma+1}[A].^3$$

[3] $H^M(X)$ denotes the Skolem Hull of X in M. We often write $L_\xi[A]$ to denote the structure $(L_\xi[A], \in, A{\restriction}(\xi \times \xi))$. In particular, Skolem Hulls of the form $H^{L_\xi[A]}(x)$ always denote Skolem Hulls in the structure $(L_\xi[A], \in, A{\restriction}(\xi \times \xi))$.

We say that A witnesses Acceptability (for $\mathbf{M}$*) iff A is acceptable and codes* $\mathbf{M}$*. Acceptability (for* $\mathbf{M}$*) is the statement that there exists A such that A witnesses Acceptability.*

In the literature, the term *Acceptability* is usually used for the following, closely related notion, which we will refer to as strong Acceptability (one might want to refer to our above defined notion as weak Acceptability; as it is this notion we are mostly interested in in this article, we will stick to the shorter term Acceptability):

Definition 13. *If A is a bfp, we say that A is strongly acceptable iff for any ordinals $\gamma \geq \delta$, if there is a new subset of δ in $L_{\gamma+1}[A]$, then there is a surjection from δ onto γ in $L_{\gamma+1}[A]$.*

We say that A witnesses Strong Acceptability (for $\mathbf{M}$*) iff A is strongly acceptable and codes* $\mathbf{M}$*. Strong Acceptability (for* $\mathbf{M}$*) is the statement that there exists A such that A witnesses Strong Acceptability.*

If A witnesses Strong Acceptability then A witnesses Acceptability:

Lemma 14. *If there is a surjection from β onto γ in $L_{\gamma+1}[A]$, then*

$$H^{L_{\gamma+1}[A]}(\beta) = L_{\gamma+1}[A].$$

Proof. Let β' be least so that there is a surjection from β' onto γ in $L_{\gamma+1}[A]$. β' is definable in $L_{\gamma+1}[A]$ and we may assume that $\beta = \beta'$. Let $H := H^{L_{\gamma+1}[A]}(\beta)$. $\gamma \in H$ as γ is the largest ordinal of $L_{\gamma+1}[A]$. Let f be a surjection from β onto γ in H, which exists by elementarity. Since $\beta \subseteq H$, it follows that $\gamma \subseteq H$. Thus $H = L_{\gamma+1}[A]$. □

Lemma 15. *$\emptyset$ witnesses Acceptability in* $\mathbf{L}$*.*

Proof. Assume $\delta \leq \gamma$ and there is a new subset x of δ in $L_{\gamma+1}$, but no surjection from δ to γ. Let $N := H^{L_{\gamma+1}}(\delta)$. We may assume that δ is least with the above property and hence an element of N using its definability. By elementarity of N there is an x with the above property in N. Let $\bar{N} = \text{coll}(N)$; as $\delta \subseteq N$, $x \in \bar{N}$. By the Condensation properties of $\mathbf{L}$, $\bar{N}$ is a level of $\mathbf{L}$, hence $\bar{N} = L_{\gamma+1}$ by minimality of γ. But $H^{\bar{N}}(\delta) = \bar{N}$. □

Lemma 16. *Acceptability implies* GCH.

Proof. Assume A witnesses Acceptability and for some κ, $2^\kappa > \kappa^+$. Since $L_{\kappa^+}[A]$ has cardinality κ^+, there is $\gamma \geq \kappa^+$ and a new subset x of κ in $L_{\gamma+1}$. By Acceptability, $H^{L_{\gamma+1}[A]}(\kappa) = L_{\gamma+1}[A]$, which is absurd as γ has cardinality greater than κ. □

Definition 17. *If κ is an infinite cardinal, we say that a bfp A on $[\kappa, \kappa^+)$ is trivially acceptable if for every $\alpha \in (\kappa, \kappa^+)$, there is a surjection from κ to α in $L_{\alpha+2}[A]$.*

Lemma 18. *Assume A is a bfp, κ is a cardinal and $\alpha \in (\kappa, \kappa^+)$ is such that $L_\alpha[A] \models \kappa$ is the largest cardinal. Then if there is a new subset of κ in $L_{\alpha+1}[A]$ then $H^{L_{\alpha+1}[A]}(\kappa) = L_{\alpha+1}[A]$.*

Proof. Assume there is a new subset of κ in $L_{\alpha+1}[A]$ and let x be the $<_{L_{\alpha+1}[A]}$-least such. Let $H := H^{L_{\alpha+1}[A\restriction\alpha]}(\kappa)$, noting that $L_{\alpha+1}[A\restriction\alpha] = L_{\alpha+1}[A]$. κ and α are both in H. As $L_\alpha[A] \models \kappa$ is the largest cardinal, it follows that H is transitive below α. Thus the transitive collapse of H equals $L_{\gamma+1}[A\restriction\gamma]$ for some $\gamma \leq \alpha$. Since $x \in \operatorname{coll} H$, it follows that $\gamma = \alpha$, i.e. $H = L_{\alpha+1}[A\restriction\alpha] = L_{\alpha+1}[A]$. The lemma follows as $H \subseteq H^{L_{\alpha+1}[A]}(\kappa)$. □

Corollary 19. *Assume A is a bfp so that $A\restriction[\kappa, \kappa^+)$ is trivially acceptable and $L_\kappa[A] = H_\kappa^{\mathbf{L}[A]}$ for every infinite cardinal κ. Then A is acceptable.*

Proof. If α is an ordinal of cardinality κ and $\lambda < \kappa$, no new subsets of λ appear in $L_{\alpha+1}[A]$, as $L_\kappa[A] = H_\kappa^{\mathbf{L}[A]}$. The Corollary follows as $A\restriction[\kappa, \kappa^+)$ is trivially acceptable, using Lemma 18. □

Corollary 20. GCH *implies Acceptability.*

Proof. Choose a bfp A such that $L_\kappa[A] = H_\kappa$ and $A\restriction[\kappa, \kappa^+)$ is trivially acceptable for every infinite cardinal κ. The latter is easy to achieve by demanding that whenever $L_{\alpha+1}[A\restriction\alpha] \models \alpha$ is a cardinal, we choose $A(\alpha)$ to code a surjection f from κ to α in the sense that $A(\beta, \alpha) = \gamma$ iff $f(\gamma) = \beta$. Note that we obtain $f \in L_{\alpha+2}[A]$. □

To avoid possible confusion, we now want to clarify the exact meaning of the statement of Theorem 2: In [10], it was shown that Local Club Condensation implies the GCH and that Local Club Condensation is consistent with the existence of an ω-superstrong cardinal. Now by Corollary 20, this would imply Theorem 2 as it is stated. This is somewhat imprecise though, because when we say that Local Club Condensation and Acceptability hold, we actually mean (as was indicated before the statement of Theorem 2) that Local Club Condensation is witnessed by an acceptable bfp or, to put it slightly differently, that Local Club Condensation and Acceptability are witnessed by the same bfp.[4] We will see in Claim 24 below that the bfp constructed in Corollary 20 cannot witness any amount of Condensation.

To be exact, Theorem 2 should be rephrased as follows:

Theorem 21. *It is consistent with an ω-superstrong cardinal to have an acceptable bfp witnessing Local Club Condensation.*

The main work of this paper will be to give a proof of this theorem in Section 3. We close this section with two easy facts about Acceptability which we present in the context of $\mathbf{L}$ for simplicity but which may easily be generalized. They are not needed for the remainder of the paper, so the reader who is not interested in these matters may immediately proceed to Section 2.

Lemma 22. *There is a bfp A s.t. $\forall\kappa\ L_\kappa = L_\kappa[A]$, but A is not acceptable.*

Proof. Choose a countable β such that L_β models ZFC without the power set axiom and thinks that ω_1 exists. Let α be the ω_1 of L_β. Let c be a real in $\mathbf{L}$ which is Cohen-generic over L_β. Now consider the bfp A which is empty except that $A(\alpha)$ codes c in a way that $c \in L_{\alpha+2}[A]$. We don't have

[4]Of course it is also this property that we mean in the statement of Theorem 6.

Acceptability for A because c is a new subset of ω in $L_{\alpha+2}[A]$, but there is no collapse of α to ω in $L_\beta[A]$ (and hence $H^{L_{\alpha+2}[A]}(\omega) \neq L_{\alpha+2}[A]$), since c is Cohen generic over L_β. For infinite cardinals κ, $L_\kappa = L_\kappa[A]$ because we chose c to be constructible (and therefore to belong to L_{ω_1}). $\square$

Corollary 23. *There is a bfp B that is acceptable but not strongly acceptable.*

Proof. Let A be the bfp constructed in Lemma 22, let α be as in the proof of Lemma 22. Modify A to A' by mixing $A(\alpha)$ with a code for a surjection f from ω to α so that $f \in L_{\alpha+2}[A']$. Pass from A' to B by eliminating further instances of failures of Acceptability for A' at larger ordinals in the same way. B is acceptable, but not strongly acceptable, as $L_{\alpha+1}[B] = L_{\alpha+1}[A]$. $\square$

2. Acceptability and Stationary Condensation

Stationary Condensation was introduced in [10] and is a generalized Condensation principle weaker than Local Club Condensation. As we are only interested in the context of models of the form $\mathbf{L}[A]$ here, we will give its definition restricted to that context.

Stationary Condensation for $\mathbf{L}[A]$ is the principle that for each α and infinite cardinal $\kappa \leq \alpha$, any structure $(L_\alpha[A], \in, A, \ldots)$ for a countable language has a condensing substructure $(B, \in, A, \ldots)$ with B of size κ, containing κ as a subset. We say that A witnesses Stationary Condensation (in $\mathbf{V}$) iff A codes $\mathbf{V}$ and Stationary Condensation for $\mathbf{L}[A]$ holds.

Claim 24. *Assume $A \cap [\kappa, \kappa^+)$ is trivially acceptable for some infinite cardinal κ. Then A does not witness Stationary Condensation.*

Proof. Assume A witnesses Stationary Condensation and take some condensing, elementary submodel M of $L_{\kappa^{+++}}[A]$ of size κ. It's collapse will be of the form $L_\gamma[A]$ for some $\gamma \in (\kappa, \kappa^+)$. Then $L_\gamma[A]$ has its version of κ^{++}, contradicting the assumption that $A \cap [\kappa, \kappa^+)$ is trivially acceptable. $\square$

We will now take a first step towards proving Theorem 2 by showing how to extend (by forcing) a ground model $\mathbf{V}$ satisfying GCH to a model of the form $\mathbf{L}[A]$ where A is a bfp witnessing Acceptability and Stationary Condensation, while preserving an ω-superstrong cardinal. This is a strengthening of Theorem 6 of [10].

Definition 25. *We refer to p as a κ^+-Cohen condition iff p is a bfp on $[\kappa, |p|)$, where $|p|$, the* length *of p, is an ordinal of size κ. If G is a bfp on κ, then a κ^+-Cohen condition p is acceptable with respect to G if for every $\eta \in [\kappa, |p|)$, for every $\delta \in (\eta, |p|]$, if there is a new subset of η in $L_{\delta+1}[G \cup p]$, then $H^{L_{\delta+1}[G\cup p]}(\eta) = L_{\delta+1}[G \cup p]$. We say that p is correct with respect to G iff $p = \emptyset$ or $L_{|p|}[G \cup p] \models \kappa$ is the largest cardinal.*

Definition 26. *For an infinite cardinal κ and a bfp G on κ, we define the forcing*

$$\mathrm{AAdd}(\kappa^+, G) :=$$
$$\{p \colon p \text{ is an acceptable, correct } \kappa^+\text{-Cohen condition w.r.t. } G\},$$

where conditions are ordered by inclusion.

Lemma 27. $\mathrm{AAdd}(\kappa^+, G)$ *is* κ^+*-closed.*

Proof. Assume $\langle p_i \colon i < \alpha\rangle$ is a strictly descending sequence of conditions in $\mathrm{AAdd}(\kappa^+, G)$ of limit length $\alpha < \kappa^+$. Let $q := \bigcup_{i<\alpha} p_\alpha$. We show that q is acceptable and correct w.r.t. G. Correctness of q w.r.t. G follows immediately from the correctness of the p_i w.r.t. G. Now assume there is a new subset of κ in $L_{|q|+1}[G \cup q]$ and let x be the $<_{L_{|q|+1}[G\cup q]}$-least such. Let $H = H^{L_{|q|+1}[G\cup q]}(\kappa)$. $\kappa \in H$ and H is transitive below $|q|$. But this means that $\mathrm{coll}(H) = L_{\gamma+1}[G \cup q]$ for some $\gamma \leq |q|$. But as $x \subseteq \kappa$, it follows that $x \in L_{\gamma+1}[G \cup q]$ and hence $\gamma = |q|$. Finally if $\kappa < \eta < |q|$ and there is a new subset of η in $L_{|q|+1}[G \cup q]$, then since there is a bijection from η to κ in $L_{|q|}[G \cup q]$, it follows that there is a new subset of κ in $L_{|q|+1}[G \cup q]$. □

Lemma 28. *For every* $\alpha < \kappa^+$*, any condition in* $\mathrm{AAdd}(\kappa^+, G)$ *can be extended to one of length at least* α*.*

Proof. Assume $p \in \mathrm{AAdd}(\kappa^+, G)$ and $\alpha < \kappa^+$ are given. As p is correct w.r.t. G, we may extend p to a condition q of length α in the same way that we constructed trivially acceptable bfps in Corollary 20. □

Theorem 29. *Assume* GCH*. There is a cofinality-preserving forcing extension of the universe of the form* $\mathbf{L}[A]$ *where* A *is an acceptable bfp witnessing Stationary Condensation. Moreover we may preserve a given large cardinal of any of the following kinds: superstrong, hyperstrong and* n*-superstrong for any* $n \leq \omega$*.*[5]

Proof. Let P be the class-sized reverse Easton iteration with Easton support of $\mathrm{AAdd}(\kappa^+, G_\kappa)$ over all infinite cardinals κ, where for each κ, G_κ denotes the generic bfp on κ obtained by forcing with P_κ, the iteration below κ. Let A be the generic bfp on **Ord** obtained by forcing with P. By an easy density argument, $\mathbf{V}^P = L[A]$. The following is a standard claim, using that we work with a reverse Easton iteration where at stage κ, we apply a κ^+-closed forcing of size κ^+ (see for example [8]):

Fact 30. *If* κ *is regular,* P_κ *has a dense subset of size* κ*. Each* P_κ *preserves cofinalities and hence* $\mathbf{P}$ *preserves cofinalities. After forcing with* P_κ*,* $L_\lambda[G_\kappa] = H_\lambda$ *for all infinite cardinals* $\lambda \leq \kappa$*. Therefore after forcing with* $\mathbf{P}$*,* GCH *holds.*

Claim 31. *A witnesses Acceptability in* $\mathbf{L}[A]$*.*

Proof. Assume $\gamma > \delta$ are ordinals and there is a new subset of δ in $\mathbf{L}_{\gamma+1}[A]$. Since $\mathbf{L}_\lambda[A] = {H_\lambda}^{\mathbf{L}[A]}$ for all cardinals λ, it follows that $\delta \geq \mathrm{card}\,\gamma$. Thus Acceptability follows directly from the fact that A is built up from acceptable Cohen conditions. □

Lemma 32. *A witnesses Stationary Condensation in* $\mathbf{L}[A]$*.*

Proof. Let κ be an infinite cardinal $\leq \alpha$ and $\dot{S} = (L_\alpha[A], \in, A, \ldots)$ a name for a structure in $L[A]$ for a countable language. We may assume that $\dot{S}$ is Skolemized. Work in a P_κ-generic extension with generic bfp G_κ. By the

[5] ω-superstrong cardinals were defined in the introduction of this paper. Definitions of the other large cardinal notions may be found in [9].

closure properties of our iteration, we may assume that $\dot{S}$ has a $P[\kappa, \alpha^+)$-name in that model. We claim that below any condition $p \in P[\kappa, \alpha^+)$, there is q^{**} which forces Condensation for the universe X of some substructure of $\dot{S}$ of size κ which contains κ as a subset, i.e. $q^{**} \Vdash (X, \in, A)$ is isomorphic to some $(L_{\bar{\alpha}}[A], \in, A)$.

Choose some large (w.r.t. α), regular ν. Let $p^0 = p$. Let $M_0 \prec H_\nu$ such that $M_0 \supseteq \kappa + 1$, has size κ and contains p^0 and $\dot{S}$ as elements. Given p^i and M_i, choose $\bar{p}^i \leq p^i$ such that $\bar{p}^i$ hits every dense subet of $P[\kappa, \alpha^+)$ of M_i, which is possible as $P[\kappa, \alpha^+)$ is κ^+-closed. Choose $p^{i+1} \leq \bar{p}^i$ such that p^{i+1} decides both $\alpha \cap (\dot{S}$-closure of $(M_i \cap \alpha))$ and $A \restriction (\alpha \cap (\dot{S}$-closure of $(M_i \cap \alpha)))^2$ and forces that $|p^{i+1}(\kappa)| \geq M_i \cap \kappa^+$. Let M_{i+1} be such that $p^{i+1} \Vdash M_{i+1} \supseteq \alpha \cap (\dot{S}$-closure of $(M_i \cap \alpha))$, M_{i+1} has size κ and contains p^{i+1} and M_i as elements. Continue like this for ω-many steps. Let $q := \bigcup_{i<\omega} p^i$ and let $M := \bigcup_{i<\omega} M_i$. Then the following hold:

- $\operatorname{card} M = \kappa$, $\kappa \subseteq M$.
- $|q(\kappa)| = M \cap \kappa^+$.
- $M \cap \alpha = \alpha \cap \bigcup_{i<\omega} \dot{S}$-closure of $(M_i \cap \alpha) = \alpha \cap (\dot{S}$-closure of $(M \cap \alpha))$.
- q decides $A \restriction (M \cap \alpha)^2$.
- q generates a generic G for $P[\kappa, \alpha^+)$ over M in the sense that for every dense subset D of $P[\kappa, \alpha^+)$ in M, there is $t \geq q$ such that $t \in D \cap M$.

Let π denote the collapsing map of M. Let $\bar{M} = \pi''M$, $\bar{G} = \pi''G$ and $\bar{A} = \pi''A$. Let $q^* \supseteq q$ such that for every $\gamma_0 \in M \cap [\kappa^+, \alpha)$ and all $\gamma_1 \leq \gamma_0$ in M,

- $q^*(\kappa)(\pi(\gamma_0), \pi(\gamma_1)) = \pi(q(\operatorname{card} \gamma_0)(\gamma_0, \gamma_1))$ and
- $q^*(\kappa)(\pi(\gamma_1), \pi(\gamma_0)) = \pi(q(\operatorname{card} \gamma_0)(\gamma_1, \gamma_0))$.

We need to show that we can extend $q^*(\kappa)$ to an acceptable, correct κ^+-Cohen condition. We start by showing that $q^*(\kappa)$ is acceptable: If $\xi < (\kappa^{++})^{\bar{M}[\bar{G}]}$, then $q(\kappa^+) \restriction \pi^{-1}(\xi) \in M$, $M \models \mathbf{1}_{P[\kappa, \kappa^+)} \Vdash q(\kappa^+) \restriction \pi^{-1}(\xi)$ is acceptable w.r.t. G_{κ^+}, so $\bar{M}[\bar{G}] \models q^*(\kappa)[(\kappa^+)^{\bar{M}}, \xi)$ is acceptable w.r.t. $\bar{G} \restriction (\kappa^+)^{\bar{M}}$. Furthermore, $\bar{M}[\bar{G}] \models$ For all infinite cardinals λ, all bounded subsets of λ appear in $L_\lambda[A]$, which implies that $q^*(\kappa) \restriction \xi$ is indeed acceptable w.r.t. G_κ and then by Lemma 27, $q^*(\kappa) \restriction (\kappa^{++})^{\bar{M}}$ is acceptable w.r.t. G_κ. Continuing similarly, we may show that whenever $\lambda \leq \alpha$ is a cardinal in M, $q^*(\kappa) \restriction \pi(\lambda)$ is acceptable w.r.t. G_κ. If $\alpha \notin \mathbf{Card}$, we may assume that $|p^0(\operatorname{card} \alpha)| \geq \alpha$, hence $q(\operatorname{card} \alpha) \restriction \alpha \in M$ and elementarity yields that $q^*(\kappa)$ is acceptable w.r.t. G_κ in this case.

Finally we extend $q^*(\kappa)$ to an acceptable, correct κ^+-Cohen condition: Any bfp on $[\kappa, \pi(\alpha)+1)$ extending $q^*(\kappa)$ is acceptable w.r.t. G_κ using elementarity of $\bar{M}$. We may thus extend $q^*(\kappa)$ to $q^{**}(\kappa)$ as desired by having $q^{**}(\kappa)(\alpha)$ code a surjection from κ to α as described in the proof of Corollary 20. Obviously, q^{**} forces Condensation for the $\dot{S}$-closure of $M \cap \alpha$. □

By exactly the same arguments as for the GCH forcing in [9], we may force with P preserving a given large cardinal of one of the following kinds: superstrong, hyperstrong or n-superstrong for any $n \leq \omega$. In fact, using the closure properties of the iteration, many other large cardinals may be preserved while forcing with P. □

By a slight enhancement of the proof of Lemma 32, it is easily seen that A in fact witnesses a stronger generalized Condensation principle in $\mathbf{L}[A]$:
Fat Stationary Condensation. for $\mathbf{L}[A]$ is the principle that for each α, each infinite cardinal $\kappa < \operatorname{card}\alpha$ and ordinal $\xi < \kappa^+$, any club in $[L_\alpha[A]]^\kappa$ contains a continous chain of length ξ of condensing models. We say that A witnesses Fat Stationary Condensation (in $\mathbf{V}$) iff A codes $\mathbf{V}$ and Fat Stationary Condensation for $\mathbf{L}[A]$ holds.

Instead of providing a proof of the above, we will now turn towards our main object of interest, the stronger principle of Local Club Condensation.

3. Forcing Acceptability and Local Club Condensation

Acceptability and Local Club Condensation were defined in Section 1. We now turn to the proof of Theorem 2: We show how to obtain, starting with a ground model $\mathbf{V} \models \mathrm{GCH}$, a generic extension of the form $\mathbf{L}[A]$ such that A is a bfp witnessing both Local Club Condensation and Acceptability while preserving (very) large cardinals.[6] We will force with a class forcing P which will be the direct limit of P_α for $\alpha \in \mathbf{Ord}$, the P_α will be defined inductively. We will also define $P_\alpha^\oplus$ inductively with the property that P_α is a complete subforcing of $P_\alpha^\oplus$, which in turn is a complete subforcing of $P_{\alpha+1}$. We will show that each P_α preserves cofinalities and the GCH. We also allow for $\alpha = \mathbf{Ord}$ here and later on, where we let $P_{\mathbf{Ord}} = P$. P is not any kind of standard iteration, but similar to a reverse Easton iteration. Conditions p in P_α will be α-sequences and each $p(\beta)$ will be of the form $(p(\beta)(0), p(\beta)(1))$. $p(\beta)(0)$ will be a P_β-name for a condition in some forcing $Q(\beta)(0)$ of V^{P_β}, $p(\beta)(1)$ will be a $P_\beta^\oplus$-name for a condition in some forcing $Q(\beta)(1)$ of $V^{P_\beta^\oplus}$. Elements of $P_\beta^\oplus$ are of the form $p{\restriction}\beta^\frown p(\beta)(0)$ with $p{\restriction}\beta \in P_\beta$. We write $p{\restriction}\beta^\oplus$ for $p{\restriction}\beta^\frown p(\beta)(0)$.

Definition 33. *We say that s is an α^+-Cohen condition with collapsing information if s is of the form $s = (c, F)$ where c is an α^+-Cohen condition (in the sense of Definition 25) of length $|s|$, F is of the form $F = \langle f_\gamma \colon \gamma \in [\operatorname{card}\alpha, |s|)\rangle$ and each f_γ is a bijection from $\operatorname{card}\alpha$ to γ. We refer to F as the collapsing information of s.*

For any notion of forcing in some forcing extension, we let $\check{\mathbf{1}}$ denote the standard name for its weakest condition $\mathbf{1}$. If $\beta < \omega$, $p(\beta) = (\check{\mathbf{1}}, \check{\mathbf{1}})$ for any $p \in P$, i.e. P_ω is trivial. If $\operatorname{card}\beta$ is ω or singular, $p(\beta)(1) = \check{\mathbf{1}}$, i.e. $Q(\beta)(1)$ is trivial. Given P_α, the forcing below α, and G_α, a generic for that forcing, we will define $Q(\alpha)(0) := S(G_\alpha)$, where $S(G_\alpha)$ will consist of a collection of α^+-Cohen conditions with collapsing information which we set to be pairwise incompatible in $S(G_\alpha)$, together with a weakest condition $\mathbf{1}$. An $S(G_\alpha)$-generic filter will simply choose one such condition. The exact collection will be defined later on. We write $p(\beta)(0)$ as (p_β, F_β^p), where p_β denotes the

[6] If one doesn't require large cardinal preservation, this is achieved by Jensen Coding (see [1], [8]).

α^+-Cohen condition and F^p_β denotes the collapsing information specified by $p(\beta)(0)$. The underlying set of $P^\oplus_\alpha$ will be a proper suborder of $P_\alpha * Q(\alpha)(0)$ which we will only define later on inductively (see Theorem 49, Clause 4). A careful choice of this underlying set will be of central importance to our proof. Given two conditions $p = (p\restriction\alpha, p(\alpha)(0))$ and $q = (q\restriction\alpha, q(\alpha)(0))$ in $P^\oplus_\alpha$, we let $q \le p$ iff

- $q\restriction\alpha \le p\restriction\alpha$ and
- $p(\alpha)(0) = \check{\mathbf{1}}$ or $q(\alpha)(0) = p(\alpha)(0)$.[7]

Assume we are given a P_α-generic G_α. If $\beta_0 < \beta_1$ both have the same cardinality, are less than α and p is a condition in P_α with $p_{\beta_0} \ne \check{\mathbf{1}} \ne p_{\beta_1}$, ${p_{\beta_1}}^{G_\alpha}$ will properly extend ${p_{\beta_0}}^{G_\alpha}$ and ${F^p_{\beta_1}}^{G_\alpha}$ will properly extend ${F^p_{\beta_0}}^{G_\alpha}$. We may thus define a bfp g_α by letting $g_\alpha(\gamma_0,\gamma_1) = \gamma_2$ iff $\exists p \in G_\alpha\, \exists \delta < \alpha\ p\restriction\delta \Vdash p_\delta(\gamma_0,\gamma_1) = \gamma_2$. Note that we may define $g_{\alpha+1}$ already given a generic $G^\oplus_\alpha$ for $P^\oplus_\alpha$. $|g_\alpha| = \alpha$ if α is a cardinal and $\alpha \le |g_\alpha| < \alpha^+$ otherwise. If $p \in P^\oplus_\alpha$ and $\gamma \in [\operatorname{card}\beta, |p_\beta|)$ for some infinite $\beta \le \alpha$, then $F^p_\beta(\gamma)$ specifies a bijection f^p_γ from $\operatorname{card}\gamma$ to γ in $\mathbf{V}$. Note that if $q \le p$, $f^q_\gamma = f^p_\gamma$. To simplify notation, we usually suppress mention of p and write f_γ instead of f^p_γ. We will be in a similar situation given a generic $G^\oplus_\alpha$ for $P^\oplus_\alpha$ and write f_γ instead of $f^{G^\oplus_\alpha}_\gamma$. It should always be clear from context which condition or which generic gave rise to the particular choice of f_γ. Assuming such a context, we define

$$\pi^\eta(\gamma) = \begin{cases} \gamma & \text{if } \gamma < \eta \\ \operatorname{ot} f_\gamma[\eta] & \text{otherwise.} \end{cases}$$

Given a generic $G^\oplus_\alpha$ for $P^\oplus_\alpha$ which specifies a generic bfp $g = g_{\alpha+1}$ on β for some $\beta < \alpha^+$ and f_γ for $\gamma < \beta$ and α is of regular cardinality, let $C(G^\oplus_\alpha)$ denote the following forcing poset to add a club to $\operatorname{card}\alpha$:

If $\operatorname{card}\alpha = \theta^+$ is a successor cardinal, $p = (p^*, p^{**}) \in C(G^\oplus_\alpha)$ iff

- p^* is a subset of $[\operatorname{card}\alpha, \beta)$ of size less than $\operatorname{card}\alpha$ and
- p^{**} is a closed, bounded subset of $[\theta, \operatorname{card}\alpha)$.

If $\operatorname{card}\alpha$ is inaccessible, $p = (p^*, p^{**})$ is a condition in $C(G^\oplus_\alpha)$ iff

- p^* is a subset of $[\operatorname{card}\alpha, \beta)$ of size less than $\operatorname{card}\alpha$ and
- p^{**} is a closed, bounded set of cardinals below $\operatorname{card}\alpha$.

In both cases, $q = (q^*, q^{**})$ extends $p = (p^*, p^{**})$ in $C(G^\oplus_\alpha)$ iff

- $q^* \supseteq p^*$,
- q^{**} end-extends p^{**} and
- $\forall \gamma_0 \in p^*\, \forall \eta \in q^{**} \setminus p^{**}\, \forall \gamma_1 \le \gamma_0$ in $\eta \cup p^*$
  - $g(\pi^\eta(\gamma_0), \pi^\eta(\gamma_1)) = \pi^\eta(g(\gamma_0,\gamma_1))$ and
  - $g(\pi^\eta(\gamma_1), \pi^\eta(\gamma_0)) = \pi^\eta(g(\gamma_1,\gamma_0))$.

[7] In the second disjunct, the names should be equal, not just forced to be equal. This is one of the reasons that P is not a standard iteration. (Another is the use of thinned-out supports at limits; see below.) Note also that this means that a generic for $Q(\alpha)(0)$ doesn't just pick a Cohen condition (with collapsing information), it does a little more - it actually picks a name for one.

We let $Q(\alpha)(1) = C(G^{\oplus}_{\alpha})$ and $P_{\alpha+1} = P^{\oplus}_{\alpha} * Q(\alpha)(1)$, a standard two step iteration with the standard ordering.[8]

Claim 34. *Assume (c^*, c^{**}),(d^*, d^{**}) are compatible conditions in $C(G^{\oplus}_{\alpha})$. Then $(c^* \cup d^*, c^{**} \cup d^{**})$ is stronger than both.*

Proof. It is immediate that $(c^* \cup d^*, c^{**} \cup d^{**})$ is a condition in $C(G^{\oplus}_{\alpha})$. Let (e^*, e^{**}) witness that (c^*, c^{**}) and (d^*, d^{**}) are compatible. Then $e^* \supseteq c^* \cup d^*$ and e^{**} end-extends $c^{**} \cup d^{**}$. Assume for a contradiction that $(c^* \cup d^*, c^{**} \cup d^{**})$ does not extend (c^*, c^{**}). Then $\exists \gamma_0 \in c^* \, \exists \delta \in d^{**} \setminus c^{**} \, \exists \gamma_1 \leq \gamma_0$ such that $\gamma_1 \in \delta \cup c^*$ and either $g_{\alpha+1}(\pi^{\delta}(\gamma_0), \pi^{\delta}(\gamma_1)) \neq \pi^{\delta}(g_{\alpha+1}(\gamma_0, \gamma_1))$ or $g_{\alpha+1}(\pi^{\delta}(\gamma_1), \pi^{\delta}(\gamma_0)) \neq \pi^{\delta}(g_{\alpha+1}(\gamma_1, \gamma_0))$. But this implies that (e^*, e^{**}) does not extend (c^*, c^{**}), a contradiction. ☐

Assume we have defined P_{ξ} for $\xi < \gamma$, γ a limit ordinal and p is a condition in P^*_{γ}, the inverse limit of $\langle P_{\xi} \colon \xi < \gamma \rangle$. We write $(p^*_{\alpha}, p^{**}_{\alpha})$ instead of $p(\alpha)(1)$. We call $\{\gamma \colon p(\gamma)(0) \neq \check{\mathbf{1}}\}$ the string support of p and denote it by S-supp(p), we call $\{\gamma \colon p(\gamma)(1) \neq \check{\mathbf{1}}\}$ the club support of p and denote it by C-supp(p); we let I-supp$(p) = \bigcup_{\gamma \in \text{C-supp}(p)} p^*_{\gamma}$. [9]

For every limit ordinal γ and $p \in P^*_{\gamma}$, we let $p \in P_{\gamma}$ iff:

(1) If γ is regular, S-supp(p) is bounded below γ.
(2) S-supp$(p) \cap [\operatorname{card} \gamma, \gamma) = [\operatorname{card} \gamma, \xi)$ for some $\xi \leq \gamma$.
(3) If $\operatorname{card} \gamma$ is regular, card(C-supp(p)) $< \operatorname{card} \gamma$.
(4) There is $\zeta < \gamma$ so that for all $\xi \geq \zeta$, $p \restriction \xi^{\oplus}$ forces that $p(\xi)(1)$ has a P_{β}-name for some $\beta < \operatorname{card} \gamma$.

We equip P_{γ} with the natural ordering: $q \leq p$ if for all $\beta < \gamma$, $q \restriction \beta \leq p \restriction \beta$ in P_{β}. The limit stages of our forcing are thus restricted inverse limits of the earlier stages. What is missing to complete the definition of the forcing is to define $P^{\oplus}_{\alpha}$ given P_{α}. This will be done in Clause 4 of Theorem 49 inductively. We will usually assume our conditions p to satisfy (A0): $\forall \beta \, \mathbf{1}_{P^{\oplus}_{\beta}} \Vdash p(\beta)(1) \in C(G^{\oplus}_{\beta})$ and (A1): $\forall \beta \, \mathbf{1}_{P_{\beta}} \Vdash p(\beta)(0) \in S(G_{\beta})$.

Definition 35 (upper part of a condition)**.** *Given a cardinal $\eta < \alpha$ and $p \in P_{\alpha}$, we define $u_{\eta}(p) \in P_{\alpha}$ as follows:*

- $(u_{\eta}(p))(\gamma)(0) = \begin{cases} \check{\mathbf{1}} & \textit{if } \gamma < \eta \\ p(\gamma)(0) & \textit{otherwise} \end{cases}$
- $(u_{\eta}(p))(\gamma)(1) = \begin{cases} \check{\mathbf{1}} & \textit{if } \gamma < \eta^+ \\ p(\gamma)(1) & \textit{otherwise} \end{cases}$

and call $u_{\eta}(p)$ the η^+-strategically closed part of p. Let $u_{\eta}(P_{\alpha}) := \{u_{\eta}(p) \colon p \in P_{\alpha}\}$ with the induced ordering.

Note:

- We may think of $u_{\eta}(p)$ as the condition extracting from p its Cohen and collapsing information in the interval $[\eta, \eta^+)$ and everything at and above η^+.

[8] $q \leq p$ iff $q \restriction \alpha^{\oplus} \leq p \restriction \alpha^{\oplus}$ and $q \restriction \alpha^{\oplus}$ forces $q(\alpha)(1) \leq p(\alpha)(1)$.

[9] Note that S-supp(p) and C-supp(p) are ground model objects while I-supp(p) is not.

- A similar definition of course applies to $p \in P_\alpha^\oplus$. It will usually be the case in the following that definitions, statements and facts about P_α will have natural analogues for $P_\alpha^\oplus$, which we will usually not mention (or prove) explicitly.
- $u_\omega(P_\alpha) = P_\alpha$.
- $u_\eta(p) \in P_\alpha$ uses (A0) and (A1).

The careful reader may observe that at some points later on, we will obtain conditions q that do not satisfy (A0) as lower bounds of decreasing sequences of conditions in our forcing. But it will be the case that we may replace such a q by an η^+-strategically equivalent q' which satisfies (A0), for suitable η, where we call q and q' η^+-strategically equivalent iff $u_\eta(q) \leq u_\eta(q')$ and $u_\eta(q') \leq u_\eta(q)$. We will tacitly assume such replacement a couple of times in the following.

Definition 36 (lower part of a condition)**.**
If $\eta < \alpha$ is a cardinal and $p \in P_\alpha$, we define $l_\eta(p)$ as follows:

- $(l_\eta(p))(\gamma)(0) = \begin{cases} \check{\mathbb{1}} & \text{if } \alpha > \gamma \geq \eta \\ p(\gamma)(0) & \text{otherwise} \end{cases}$
- $(l_\eta(p))(\gamma)(1) = \begin{cases} \check{\mathbb{1}} & \text{if } \alpha > \gamma \geq \eta^+ \\ p(\gamma)(1) & \text{otherwise} \end{cases}$

and call $l_\eta(p)$ the η-sized part of p. Note that $l_\eta(p)$ complements $u_\eta(p)$ in the sense that it carries exactly all information about p not contained in $u_\eta(p)$.

Notation: Assume $\langle p^i : i < \delta\rangle$ is a decreasing sequence of conditions in P_α of limit length δ and $\gamma < \alpha$. Then $\langle p^i_\gamma : i < \delta\rangle$ is eventually constant and we denote its limit by $\bigcup_{i<\delta} p^i_\gamma$. Similar for $\langle F_\gamma^{p^i} : i < \delta\rangle$. We say that r is the componentwise union of $\langle p^i : i < \delta\rangle$ iff for every $\gamma < \alpha$,

$$r_\gamma = \bigcup_{i<\delta} p^i_\gamma, \quad F^r_\gamma = \bigcup_{i<\delta} F^{p^i}_\gamma, \quad r^*_\gamma = \bigcup_{i<\delta} (p^i)^*_\gamma \quad \text{and} \quad r^{**}_\gamma = \bigcup_{i<\delta} (p^i)^{**}_\gamma.$$

r is usually not a condition in P_α, but the supports of r can be calculated as if r were a condition by letting $\text{S-supp}(r) := \{\gamma : r(\gamma)(0) \neq \check{\mathbb{1}}\} = \bigcup_{i<\delta} \text{S-supp}(p^i)$, $\text{C-supp}(r) := \{\gamma : r(\gamma)(1) \neq \check{\mathbb{1}}\} = \bigcup_{i<\delta} \text{C-supp}(p^i)$ and $\text{I-supp}(r) = \bigcup_{\gamma \in \text{C-supp}(r)} r^*_\gamma = \bigcup_{i<\delta} \text{I-supp}(p^i)$.

Definition 37 (stable below η^+)**.** *Assume $\langle p^i : i < \delta\rangle$ is a decreasing sequence of conditions in P_α of limit length $\delta < \eta^+$, $\eta < \alpha$ a cardinal. We say that $\langle p^i : i < \delta\rangle$ is stable below η^+ iff*

- *$\langle l_\eta(p^i) : i < \delta\rangle$ is eventually constant or*
- *η is singular and for every cardinal $\mu < \eta$, $\langle l_\mu(p^i) : i < \delta\rangle$ is eventually constant.*

Fact 38. *If $\langle p^i : i < \delta\rangle$ is a decreasing sequence of conditions in P_α of limit length $\delta < \eta^+$ which is stable below η^+ where $\eta < \alpha$ is a cardinal, then the componentwise union of $\langle p^i \restriction \eta^+ : i < \delta\rangle$ is a greatest lower bound for $\langle p^i \restriction \eta^+ : i < \delta\rangle$.*

Proof. It suffices to observe that $\bigcup_{i<\delta} \text{S-supp}(p^i)$ is bounded below η^+. □

If $p \in P_\alpha$ and $\kappa \leq \alpha$ is a cardinal, let

$$|p|_\kappa := \bigcup_{\gamma \in \text{S-supp}(p) \cap \kappa^+} |p_\gamma|.$$

In case $\text{S-supp}(p) \cap [\kappa, \kappa^+) = \emptyset$, we let $|p|_\kappa = \kappa$. In particular, this is the case if $\alpha = \kappa$. Our iteration will be defined (see Clause 4 of Theorem 49) in such a way that each $|p_\gamma|$ is a ground model object and thus $|p|_\kappa$ is a ground model object. Given any ordinals γ_0 and γ_1 and a condition $p \in P_\alpha$, we want to construct a name $p@(\gamma_0, \gamma_1)$: choose β minimal so that $(\gamma_0, \gamma_1) \in \operatorname{dom} p_\beta$ if possible and let $p@(\gamma_0, \gamma_1) = p_\beta(\gamma_0, \gamma_1)$, let it be undefined otherwise.

Fact 39. *Given a cardinal $\eta < \alpha$ and a decreasing sequence $\langle p^i : i < \delta \rangle$ of conditions in P_α of limit length $\delta < \eta^+$ which is stable below η^+, form their componentwise union r. Observe that* $\text{S-supp}(r)$ *is bounded below every regular cardinal and* $\text{C-supp}(r) \cap \theta^+$ *has size less than θ for every regular θ. We would like to obtain a condition $q \in P_\alpha$ with the following properties for every $\gamma \in \text{C-supp}(r)$, $\gamma \geq \eta^+$, every $\delta_0 \in r^*_\gamma$ and every $\delta_1 < \delta_0$ with $\delta_1 \in \sup r^{**}_\gamma \cup r^*_\gamma$:*

(1) $q \Vdash q@(\pi^{\sup r^{**}_\gamma}(\delta_0), \pi^{\sup r^{**}_\gamma}(\delta_1)) = \pi^{\sup r^{**}_\gamma}(r@(\delta_0, \delta_1))$.
(2) $q \Vdash q@(\pi^{\sup r^{**}_\gamma}(\delta_1), \pi^{\sup r^{**}_\gamma}(\delta_0)) = \pi^{\sup r^{**}_\gamma}(r@(\delta_1, \delta_0))$.
(3) $q^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$.[10]
(4) $q_\xi = r_\xi$ *and* $F^q_\xi = F^r_\xi$ *for every* $\xi \in \text{S-supp}(r)$, $q^*_\xi = r^*_\xi$ *for all* ξ *and* $q^{**}_\xi = r^{**}_\xi$ *for every* $\xi < \eta^+$.

If such q exists as a condition in P_α, q is a lower bound for $\langle p^i : i < \delta \rangle$, i.e. $q \leq p^i$ for each $i < \delta$.

Definition 40 (reducing dense sets)**.** *If D is a dense subset of P_α and $\eta < \alpha$ is a cardinal, we say that q reduces D below η if for every $r \in P_\alpha$ with $u_\eta(r) \leq u_\eta(q)$, there is $s \leq r$ with $u_\eta(s) = u_\eta(r)$ and s meets D in the sense that $\exists d \in D \; s \leq d$.*

Definition 41 (suitable pre-genericity)**.** *Let $p \in P_\alpha$, $\zeta \leq \alpha$, $\theta < \zeta$ regular, M of size less than θ, transitive below θ. Let $\bar\zeta := \min(\zeta, \theta^+)$. We say $q \leq p$ is suitably pre-generic for P_ζ at θ over M if the following hold:*

0. $\sup(\text{S-supp}(q) \cap \theta) \geq \operatorname{card} M$ *and* $\geq M \cap \theta$.
1a. *If $\bar\zeta < \alpha$, then $q \restriction \bar\zeta$ reduces every dense subset of $P_{\bar\zeta}$ in M below* $\operatorname{card} M$.
1b. *If $\zeta = \bar\zeta = \alpha$, then for every $\xi < \alpha$, $q \restriction \xi^\oplus$ reduces every dense subset of $P^\oplus_\xi$ in M below* $\operatorname{card} M$.

Definition 42 (suitable genericity)**.** *Under the assumptions of Def. 41, we say $q \leq p$ is suitably generic for P_ζ at θ over M if $q \leq p$ is suitably pre-generic for P_ζ at θ over M and if $\theta = \operatorname{card} \alpha$ and $\alpha = \zeta = \beta + 1$ is a successor ordinal, $u_{\operatorname{card} M}(q \restriction \beta^\oplus)$ forces that*

2a. $\sup(q^{**}_\beta) \geq M \cap \theta$ *and*
2b. $q^*_\beta \supseteq M \cap [|q \restriction \beta|_\kappa, |q_\beta|)$.

[10]More exactly, we want to set $q^{**}_\gamma = r^{**}_\gamma \cup \{(\sup r^{**}_\gamma, \mathbf{1})\}$ so that $\mathbf{1} \Vdash q^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$.

Remarks:

- If α is a limit ordinal or $\zeta < \alpha$, the notions of suitable genericity and suitable pre-genericity coincide.
- If q is suitably (pre)generic for P_ζ at θ over M and $q' \leq q$, then q' is suitably (pre)generic for P_ζ at θ over M.
- If q is suitably (pre)generic for P_ζ at θ over M, $\theta < \zeta' < \zeta$ and M is closed under the operation which takes any dense subset D of $P_{\zeta'}$ in M to $D^* := \{t \in P_\zeta : t \restriction \zeta' \in D\}$, then q is suitably (pre)generic for $P_{\zeta'}$ at θ over M. This is because if q reduces D^* below $\operatorname{card} M$ then $q \restriction \zeta'$ reduces D below $\operatorname{card} M$.

Definition 43. *If p is a condition in P_α, we say that p is* fully string supported *iff* S-supp$(p) \supseteq [\operatorname{card} \alpha, \alpha)$. *If α is a cardinal, any condition $p \in P_\alpha$ is fully string supported. We say that $p \in P_\alpha$ is a* top string condition *iff p is fully string supported and $p = u_{\operatorname{card} \alpha}(p)$, i.e. iff* S-supp$(p) = [\operatorname{card} \alpha, \alpha)$ *and* C-supp$(p) = \emptyset$. *For any fully string supported condition $p \in P_\alpha$, we define the* top string *of p as* $\operatorname{ts}(p) := u_{\operatorname{card} \alpha}(p)$.

Definition 44. *If Q is any notion of forcing and $D \subseteq Q$ we say that D is an* equivalent dense subset *of Q iff for any $q \in Q$ there is $d \in D$ such that $q \leq d$ and $d \leq q$.*

Definition 45. *We say that $\langle M_i : i < \gamma \rangle$ is an increasing chain iff for all $i < \gamma$ and all $j < i$, $M_j \subseteq M_i$ and $\langle M_k : k \leq i \rangle \in M_{i+1}$.*

Definition 46. *If P is a notion of forcing and η is a cardinal, we say that P is η^+-strategically closed iff Player I has a winning strategy in the following two player game of perfect information: Player I and Player II alternately make moves in which they play a condition in P. Player I has to start and play $\mathbf{1}_P$ in the first move. Player II is allowed to play any condition stronger than the condition just played by Player I in each of his moves. Player I has to play a condition stronger than all previously played conditions in each move, Player I has to make a move at every limit step of the game. We say that Player I wins if he can find conditions to play in any such game of length η^+ (arriving at η^+, the game ends and no further condition has to be played).*

The central technical theorem of our paper at its core will establish that our iteration P is Δ-distributive. Before stating that theorem, we will provide the reader with the definition of Δ-distributivity, which is originally given in [8] and restated here in a less general version, slightly adapted to our iteration P:

Definition 47. *We say P_α is Δ-distributive if whenever $\langle D_i : i < \operatorname{card} \alpha \rangle$ are dense subsets of P_α and $p \in P_\alpha$, there is $q \leq p$ which reduces D_i below i^+ for every i, where we let $i^+ = \omega$ for finite i.*

Now we adapt this definition to the context of class forcing:

Definition 48. *We say that P is Δ-distributive at κ if whenever $\langle D_i : i < \kappa \rangle$ is a definable sequence of dense classes of P and $p \in P$, then there is $q \leq p$ which reduces D_i below i^+ for every i. We say that P is Δ-distributive if P is Δ-distributive at κ for every uncountable cardinal κ.*

Definition and Theorem 49. *Suppose $\omega \leq \eta \leq \kappa = \operatorname{card}\alpha$, $\eta \in$ **Card**. Suppose P_α is defined.*

(1) *[A nice dense subset]*
P_α has a dense subset D_α of conditions p which are fully string supported such that $\text{C-supp}(p) \subseteq \text{S-supp}(p)$ and $p \restriction i^\oplus$ forces (in $P_i^\oplus$) that p_i^* and p_i^{**} have a $P_{\sup \text{S-supp}(p) \cap \operatorname{card} i}$-name for each $i \in \text{C-supp}(p)$.

(2) *[Smallness of the iteration]*
If α is regular, D_α has an equivalent dense subset E_α of size α. Otherwise D_α has an equivalent dense subset E_α of size α^+. If κ is regular, $E_\alpha(p)$, the forcing E_α below p, has size κ for any fully string supported condition $p \in E_\alpha$.

(3) *[Definition of the top string code, $\operatorname{tsc}(p)$]*
Given below.

(4) *[Definition of $P_\alpha^\oplus$]*
Given below.

(5) *[Definability of the Forcing]*
If α is regular, E_α is uniformly definable over H_α.

(6) *[Definability of $E_\alpha(p)$]*
- *Assume α is a limit ordinal and $s \in E_\alpha$ is a top string condition. If $\delta \geq |s|_\kappa$ is such that $|s|_\kappa$ collapses definably over $L_\delta[\operatorname{tsc}(s)]$, then an isomorphic copy of $E_\alpha(s)$ is definable over $L_\delta[\operatorname{tsc}(s)]$. Moreover, a bijection from κ to $E_\alpha(s)$ is definable over $L_\delta[\operatorname{tsc}(s)]$.*
- *Assume $s \in E_\alpha^\oplus$ is a top string condition, let $\chi := |s \restriction \alpha|_\kappa + 1$ and $M \supseteq L_\chi[\operatorname{tsc}(s)]$. If α is a successor ordinal, an isomorphic copy E of $E_\alpha(s \restriction \alpha)$ and a bijection from κ to E are uniformly definable from $\operatorname{tsc}(s) \restriction \chi$ over M.*

(7) *[Definition of String Choice Coding]*
Given below.

(8) *[String Extendibility]*
For $\xi < \alpha^+$, $Q(\alpha)(0)$ is forced to contain a condition $s = (c, F)$ with $|c| \geq \xi$.

(9) *[Chain Condition]*
Assume η is regular. If J is an antichain of P_α such that whenever p and q are in J, $u_\eta(p) \parallel u_\eta(q)$, then $|J| \leq \eta$.

(10) *[Strategic Closure]*
$u_\eta(P_\alpha)$ and $u_\eta(P_\alpha^\oplus)$ are both η^+-strategically closed.

(11) *[Reducing dense sets]*
- *Assume η is regular and $\langle D_i \colon i < \eta \rangle$ is a collection of dense subsets of P_α. Then any condition in P_α can be strengthened to a conditon q with the same η-sized part so that for every $i < \eta$, q reduces D_i below η.*
- *Assume η is singular and $\langle D_i \colon i < \eta \rangle$ is a collection of dense subsets of P_α. Then for any $\zeta < \eta$, any condition in P_α can be strengthened to a condition q with the same ζ-sized part so that for every $i < \eta$, there exists $\eta_i < \eta$ so that q reduces D_i below η_i.*
- *P_α is Δ-distributive.*

(12) *[Club Extendibility]*
If $I \subseteq \alpha$ is such that $\operatorname{card}(I \cap \theta^+) < \theta$ for every regular θ, $I \subseteq \bigcup_{\theta \text{ regular}}[\theta, \theta^+)$ and $\langle \bar{\delta}^i \colon i \in I\rangle$ is s.t. $\bar{\delta}_i < \operatorname{card} i$ for every $i \in I$, then for every $p \in P_\alpha$, there is $q \le p$ s.t. $\forall i \in I\ q \restriction i^\oplus \Vdash \max q_i^{**} \ge \bar{\delta}_i$. Moreover if $\eta < \operatorname{card} \min I$, we can assure that $l_\eta(q) = l_\eta(p)$.

(13) *[Early names]*
- *Assume η is regular and $\dot{f}$ is a P_α-name for an ordinal-valued function with domain η. Then any condition in P_α can be strengthened to a condition q with the same η-sized part forcing that for every $i < \eta$, there is a maximal antichain of size at most η below q deciding $\dot{f}(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot{f}$ has a P_γ-name for some $\gamma < \eta^+$.*
- *Assume η is singular and $\dot{f}$ is a P_α-name for an ordinal-valued function with domain η. Then for any $\zeta < \eta$, any condition in P_α can be strengthened to a condition q with the same ζ-sized part, forcing that for every $i < \eta$, there is a maximal antichain of size less than η below q deciding $\dot{f}(i)$, where for every element a of that antichain, $u_\eta(a) = u_\eta(q)$. In particular, q forces that $\dot{f}$ has a P_η-name.*

(14) *[Coding H_η]*
$H_\eta = L_\eta[g_\eta]$ in the generic extension after forcing with P_α.

(15) *[Preservation of the* GCH*]*
After forcing with P_α, GCH *holds.*

(16) *[Covering, Preservation of Cofinalities]*
For every cardinal θ, for every $p \in P_\alpha$ and every P_α-name $\dot{x}$ for a set of ordinals of size θ there is a set X in $\mathbf{V}$ of size θ and an extension q of p such that $q \Vdash \dot{x} \subseteq X$.
Therefore forcing with P_α preserves all cofinalities.

Proof. We will provide definitions and proofs by induction on α. To start the induction, note that P_ω was defined to be trivial.

Proof of 1 - A nice dense subset: We may assume that α has regular cardinality. 1 is clear at successor ordinal stages using 1, 8, and 13 inductively. If α is a limit ordinal, by the definition of the limit stages of our forcing there exists $\zeta < \alpha$ so that for all $\xi \ge \zeta$, $p \restriction \xi^\oplus$ forces that $p(\xi)(1)$ has a P_β-name for some $\beta < \operatorname{card} \alpha$. So 1 follows using 1 inductively at stage ζ. Note that we use here that if $\beta < \alpha$ and $p \in P_\alpha$, $q \in P_\beta$ and $q \le p \restriction \beta$, then $p' \in P_\alpha$ where $p' \restriction \beta = q$ and $p'[\beta, \alpha) = p[\beta, \alpha)$; also $p' \le p$. This will become clear (inductively) from the definition of P_α.

Proof of 2 - Smallness of the iteration: For any $i < \alpha$, the number of possible choices for $p(i)(0)$ is limited to i^+ by the definition of $P_i^\oplus$ given in 4 below inductively. Assume E_ξ is given inductively for $\xi < \alpha$ as desired. Let E_α be the equivalent dense subset of conditions $p \in D_\alpha$ so that p_i^* and p_i^{**} are nice $E_\xi(\operatorname{ts}(p \restriction \xi))$-names for some $\xi < \sup(\text{S-supp}(p) \cap \operatorname{card} i)$ for every $i \in \text{C-supp}(p)$, in the sense that each of them is represented by $\delta < \operatorname{card} i$-many functions $\langle A_j \colon j < \delta\rangle$ each with domain a maximal antichain of $E_\xi(\operatorname{ts}(p \restriction \xi))$ and range $|p|_{\operatorname{card} i}$. If A_j is such a function, $A_j(a) = \nu$ should be interpreted as "a forces that the j^{th} element of p_i^* (or p_i^{**}) equals ν. Using 9 and 2 inductively, this gives κ-many possibilities for $\{(p_i^*, p_i^{**}) \colon i \in \text{C-supp}(p)\}$. It follows that E_α has size α^+. Note that if $\alpha_0 < \alpha$, then $E_{\alpha_0} = \{p \restriction \alpha_0 \colon p \in E_\alpha\}$. For the last statement of the claim, note that E_α has size α^+ only because there are α^+-many possible top strings of conditions in E_α. Those possibilities are "eliminated" by passing to $E_\alpha(p)$ for a fully string supported condition $p \in E_\alpha$.

3 - Definition of the top string code, $\text{tsc}(p)$*:* If $p \in P_\alpha$ is fully string supported, we say $t = \text{tsc}(p)$ is the *top string code* for p if t is a bfp on $[\kappa, |p|_\kappa)$ and for every $\gamma \in [\kappa, \alpha)$,

- $t \restriction [|p\restriction\gamma|_\kappa, |p_\gamma|)$ codes p_γ in the sense (inductively) of 7 and
- $t(|p\restriction\gamma|_\kappa, |p\restriction\gamma|_\kappa) = \gamma$.

Note that the latter requirement doesn't contradict the former as the code for p_γ does not use the diagonal (see 7).

4 - Definition of $P_\alpha^\oplus$*:* Let G_α be generic for P_α and let g_α be the generic bfp obtained from G_α. Let s denote $g_\alpha \restriction [\kappa, |g_\alpha|)$. Note that s is an acceptable, correct κ^+-Cohen condition by Lemma 27. Let $F_{\text{old}} = \langle f_\gamma \colon \gamma \in [\kappa, |g_\alpha|)\rangle$ be the collapsing information above κ specified by (conditions in) G_α.

If κ is regular and $\alpha = \beta + 1$ is a successor ordinal, let t be a fully string supported condition in G_α and let $Q(\alpha)(0) = S(G_\alpha)$ denote the forcing poset consisting of a weakest condition $\mathbf{1}$ and incompatibly to each other, all κ^+-Cohen conditions q with collapsing information F which obey the following conditions:

- $q \supseteq s$, $F \supseteq F_{\text{old}}$,
- q is acceptable and correct w.r.t. g_κ,
- $q(|s|)$ codes $f_{|s|}$ and f_γ for each $\gamma \in [\kappa, |s|)$ so that $f_\gamma \in L_{|s|+2}[g_\kappa {}^\frown q]$ for $\gamma \leq |s|$,[11]
- $q(|s|+1)$ codes an $E_\beta(\text{ts}(t))$-name $\dot{x} \in L_{|q|}[\text{tsc}(t)]$ for $L_{|g_\beta|}[s]$ so that $\dot{x}$ is an element of any admissible structure containing that code for $\dot{x}$. The existence of $\dot{x}$ follows (by choosing $|q|$ sufficiently large) from 6 inductively and the fact that $s \restriction |g_\beta|$ has an $E_\beta(\text{ts}(t))$-name in $L[\text{tsc}(t)]$.
- $\exists \delta \in (|s|, |q|)$ $L_\delta[s]$ is admissible,
- $q(|s| + \kappa, |s| + \kappa) = \beta$,
- $q \restriction [|s| + |g_\beta|, |s| + |s|))$ codes t_β in the sense of 7 for some $t \in G_\alpha$ with $\beta \in \text{S-supp}(t)$ and [12]
- if $\gamma \neq |s| + \kappa$, $\gamma \geq |s|$, then $q(\gamma, \gamma) = 0$.

If $\alpha > \omega$ is a regular cardinal, let $Q(\alpha)(0) = S(G_\alpha)$ denote the forcing poset consisting of a weakest condition $\mathbf{1}$ and incompatibly to each other, all acceptable, correct (both w.r.t. g_α) α^+-Cohen conditions q with collapsing information F for which

- $q(\alpha + \gamma_0, \gamma_1) = g_\alpha(\gamma_0, \gamma_1)$ for all $\gamma_0, \gamma_1 < \alpha$,

[11] Code $f_{|s|}$ as in Claim 20, but only into $\{q(\xi, |s|) \colon \xi \text{ even}\}$. Now using $f_{|s|}$ and a bijection from κ to $\kappa \times \kappa$, we may code all f_γ for $\gamma \in [\kappa, |s|)$ into $\{q(\xi, |s|) \colon \xi \text{ odd}\}$.

[12] The code we obtain from 7 is actually a bfp on $[|g_\beta|, |s|)$, but can easily be placed within the above, larger area, avoiding the diagonal.

- $|q| \geq \alpha \cdot \alpha$ and
- $\forall \gamma \in [\alpha, |q|)\ q(\gamma, \gamma) = 0$.

If $\alpha > \kappa > \omega$, κ is regular and α is a limit ordinal, let $Q(\alpha)(0) = S(G_\alpha)$ denote the forcing poset consisting of a weakest condition $\mathbf{1}$ and incompatibly to each other, all acceptable, correct (both w.r.t. g_κ) κ^+-Cohen conditions q with collapsing information F which obey the following conditions:

- $q \supsetneq s$, $F \supseteq F_{\text{old}}$,
- if γ is not a multiple of $\kappa \cdot \omega$, $\gamma \geq |s|$, then $q(\gamma, \gamma) = 0$.

If κ is singular or ω, let $Q(\alpha)(0) = S(G_\alpha)$ denote the forcing poset consisting of a weakest condition $\mathbf{1}$ and incompatibly to each other, all acceptable, correct (both w.r.t. g_κ) κ^+-Cohen conditions q with collapsing information F so that $q \supsetneq s$ and $F \supseteq F_{\text{old}}$.

We let $P_\alpha^\oplus$ be the set of all $p_\alpha^\oplus = (p{\upharpoonright}\alpha, p_\alpha, F_\alpha^p)$ such that

- $p{\upharpoonright}\alpha \in P_\alpha$.
- Let $s := \text{ts}(p{\upharpoonright}\alpha)$.
- $p_\alpha = \check{\mathbf{1}}$ or $p_\alpha \supseteq p_\xi$ for all $\xi \in [\kappa, \alpha)$ and p_α is a nice $E_\alpha(s)$-name for a nontrivial condition in $Q(\alpha)(0)$ so that s decides $|p_\alpha|$ and p_α is represented by a collection of functions $\langle A_{\gamma_0,\gamma_1} : (\gamma_0, \gamma_1) \in [\kappa, |p_\alpha|)^\urcorner \rangle$ s.t. each $\operatorname{dom} A_{\gamma_0,\gamma_1}$ is a maximal antichain of $E_\alpha(s)$ and $\operatorname{range} A_{\gamma_0,\gamma_1} = \max(\gamma_0, \gamma_1)$ with the intended meaning that if $A_{\gamma_0,\gamma_1}(t) = \gamma_2$ then t forces $p_\alpha(\gamma_0, \gamma_1) = \gamma_2$.
- If $\mathbf{1}_{E_\alpha(s)}$ forces $p_\alpha(\gamma_0, \gamma_1) = \gamma_2$ for some γ_2, then $\operatorname{dom} A_{\gamma_0,\gamma_1} = \{\mathbf{1}\}$.
- s decides $F \in \mathbf{V}$.
- We may assume that any condition in P_α incompatible to s forces $p(\alpha)(0) = \check{\mathbf{1}}$ and therefore $\mathbf{1}_{P_\alpha} \Vdash p(\alpha)(0) \in Q(\alpha)(0)$.

Proof of 5 - Definability of the Forcing: If $\alpha = \omega_1$, this is immediate from the definition of P_{ω_1}. If α is inaccessible, this is immediate inductively. If $\alpha = \lambda^+$ for some regular λ, E_λ is uniformly definable over H_λ inductively and hence also over H_α. We want to show by induction that for $\beta \in (\lambda, \alpha]$, E_β is uniformly definable over H_α. If $\alpha = \lambda^+$ for some singular cardinal λ, we proceed similarly but start by noting that conditions in E_λ are elements of H_α and that E_λ is uniformly definable from λ over H_α using inductive uniform definability of E_ν for $\nu < \lambda$.

Assume now that E_γ is uniformly definable over H_α for $\gamma < \beta$ of cardinality λ. We want to show that E_β is uniformly definable over H_α. Assume first that $\beta = \gamma + 1$ is a successor ordinal: Elements of E_β are of the form $p{\upharpoonright}\gamma^\frown(p_\gamma, F_\gamma^p)^\frown p(\gamma)(1)$, where $p{\upharpoonright}\gamma \in E_\gamma$, (p_γ, F_γ^p) is as described in 4 and $p(\gamma)(1)$ may be identified with a collection of $< \lambda$-many antichains of $E_\xi(\text{ts}(p{\upharpoonright}\xi))$ for some $\xi < \lambda$, each of size $\leq \lambda$, elementwise paired with ordinals $< \alpha$. Therefore the underlying set of E_β may be identified with a subset of H_α. We have to argue that this set and the extension relation on this set are (uniformly) definable over H_α - we first consider (p_γ, F_γ^p): instead of giving a detailed proof, we just remark that the main point is that all objects possibly relevant for deciding whether or not (p_γ, F_γ^p) is such that $p{\upharpoonright}\gamma^\oplus$ is an element of $P_\gamma^\oplus$ are either ground model objects in H_α or objects in the H_α of an $E_\gamma(\text{ts}(p{\upharpoonright}\gamma))$-generic forcing extension. As

$E_\gamma(\mathrm{ts}(p\restriction\gamma))$ is a forcing in H_α, they have names in the H_α of the ground model and thus whether $p\restriction\gamma^\oplus \in E^\oplus_\gamma$ may be decided within H_α. That the extension relation for $E^\oplus_\gamma$ is definable is obvious, as it is very simple. Using very similar arguments, we may treat $p(\gamma)(1)$ and thus show that E_β and its extension relation are (uniformly) definable over H_α. If β is a limit ordinal, the above is immediate.

Proof of 6 - Definability of $E_\alpha(p)$: We may assume α has regular cardinality. If α is a limit ordinal and $s \in E_\alpha$ is a top string condition, inductively by 6 we know that for every $\xi \in [\kappa, \alpha)$, $E_\xi(s\restriction\xi)$ is definable in $L_{|s|_\kappa}[\mathrm{tsc}(s)]$ using ξ as parameter. Note that conditions in $E_\alpha(s)$ are (basically) sequences of length α with $< \kappa$ nontrivial sequence elements, all of which are in H_κ, thus given a bijection from κ to α, they may be represented using this bijection; the first part of the claim thus follows. This also implies the following:

Corollary 50. *Assume $s \in E^\oplus_\alpha$ is a top string condition, let $\chi := |s\restriction\alpha|_\kappa + 1$ and $M \supseteq L_\chi[\mathrm{tsc}(s)]$. If $\alpha = \gamma+1$ and γ is a limit ordinal, then an isomorphic copy E of $E_\gamma(s\restriction\gamma)$ and a bijection from κ to E are uniformely definable from* $\mathrm{tsc}(s)\restriction\chi$ *over M.*

Assume now $\alpha = \beta + 1$ is a successor ordinal, $s \in E^\oplus_\alpha$ is a top string condition, $\chi = |s\restriction\alpha|_\kappa + 1$ and $L_\chi[\mathrm{tsc}(s)] \subseteq M$. α is (uniformly) definable in M from $\mathrm{tsc}(s)\restriction\chi$ as α is the maximal value on the diagonal of $\mathrm{tsc}(s)\restriction\chi$. Also, $\langle f_\gamma \colon \gamma \le |s\restriction\alpha|_\kappa\rangle$ is (uniformly) definable over M from $\mathrm{tsc}(s)\restriction\chi$. Inductively by 6 or by invoking Corollary 50, $E_\beta(s\restriction\beta)$ is uniformly definable in M and there is a bijection from κ to $E_\beta(s\restriction\beta)$ in M. This allows us to decode s_β from $\mathrm{tsc}(s\restriction\alpha)$ in M. Note that $H_\kappa \subseteq M$ and this implies (using induction) that the underlying set of $E_\alpha(s\restriction\alpha)$ is definable over M.[13] To argue that the extension relation of $E_\alpha(s\restriction\alpha)$ is uniformly definable over M from $\mathrm{tsc}(s)\restriction\chi$, note that (similar to 5) all objects involved in the definition of extension are either s_β or elements of H_κ and thus elements of M; moreover the formulas involved in the definition of the extension relation are sufficiently absolute so that their validity may be recognized within M. Finally, the existence of the desired bijection is easy to observe using the fact that H_κ only has size κ, which can be seen in M.

7 - Definition of String Choice Coding: Let $p \in P^\oplus_\alpha$, $\beta \in \text{S-supp}(p) \cap [\kappa, \alpha]$. Let p_β be represented by a collection of functions $\langle A_{\gamma_0,\gamma_1} \colon (\gamma_0, \gamma_1) \in [\kappa, |p_\beta|)^\urcorner\rangle$ as in 4. We say that c codes p_β if c is a bfp on $[|p\restriction\beta|, |p_\beta|)$ so that for all $\gamma_0 \in [|p\restriction\beta|, |p_\beta|)$, $c(\gamma_0)$ codes $\langle A_{\gamma_0,\gamma_1} \colon \gamma_1 \le \gamma_0\rangle$ and $\langle A_{\gamma_1,\gamma_0} \colon \gamma_1 \le \gamma_0\rangle$ as follows: given γ_0 and γ_1, we code the domain of A_{γ_0,γ_1} as a subset d_{γ_0,γ_1} of κ using the $L[\mathrm{tsc}(p)\restriction(|p\restriction\alpha|_\kappa + 2)]$-least bijection between κ and $E_\beta(p\restriction\beta)$, which exists by 6. Now we code A_{γ_0,γ_1} by a function with domain d_{γ_0,γ_1} and range γ_0 in the obvious way. Using now the L-least bijection between $\kappa \times (\gamma_0 + 1)$ and γ_0 enables us to perform a coding as desired, by mixing the $d_{\gamma_0,\gamma_1} \subseteq \kappa$ for $\gamma_1 \le \gamma_0$ into γ_0 and similarly for the d_{γ_1,γ_0}. This also ensures that we may keep the diagonal free from information.

[13] $p \in E_\alpha(s\restriction\alpha)$ iff $p\restriction\beta \in E_\beta(s\restriction\beta)$, $p_\beta = s_\beta$, $F^p_\beta = s^p_\beta$, $\exists\xi < \kappa\ \sup \text{S-supp}(p) \cap \xi^+ \ge \xi$ and p^*_β, p^{**}_β are nice $E_\xi(\mathrm{ts}(p\restriction\xi))$-names (note that the latter are elements of H_κ).

Proof of 8 - String Extendibility: Use Lemma 28 and easy observations showing that all requirements from 4 can be satisfied (simultaneously).

Proof of 9 - Chain Condition: Assume J is an antichain of P_α such that whenever p and q are in J, $u_\eta(p) \parallel u_\eta(q)$. We may assume that all conditions in J are from E_α. Assume for a contradiction that J has size at least η^+. Then $p{\restriction}\eta$ is the same for η^+-many conditions in J and thus we may assume it is the same for all conditions in J. By GCH and a Δ-system argument, there is $W \subseteq J$ of size η^+ and a size less than η subset A of η^+ such that $\text{C-supp}(p) \cap \text{C-supp}(q) \cap [\eta, \eta^+) = A$ whenever $p \neq q$ are both in W. But using that GCH holds after forcing with P_η by 15 inductively, it follows that for η^+-many conditions p in W, $\langle p(i)(1) \colon i \in A\rangle$ is the same (modulo equivalence). But - using the assumption that $u_\eta(p) \parallel u_\eta(q)$ - any two such conditions are compatible, thus W (and hence also J) is not an antichain.

10 (Strategic Closure) implies 11 (Reducing dense sets):

Claim 51. *Assume $p \in P_\alpha$, D is a dense subset of P_α and $\nu < \alpha$ is regular. Then there is $q \leq p$ such that $l_\nu(q) = l_\nu(p)$ and q reduces D below ν.*

Proof. Build a decreasing sequence of conditions in P_α below p as follows: Let $p^0 = p$. Choose q^0 so that $q^0 \leq p^0$ and $q^0 \in D$. By possibly passing to an equivalent condition, we may also ensure that $u_\nu(q^0) \leq u_\nu(p^0)$. At stage $j+1$, let $p^{j+1} \leq p^0$ be any condition incompatible to all q^k, $k \leq j$, such that $u_\nu(p^{j+1}) = u_\nu(q^j)$ if such exists and choose q^{j+1} such that:

- $q^{j+1} \leq p^{j+1}$,
- $q^{j+1} \in D$ and
- $u_\nu(q^{j+1})$ is chosen according to the strategy for ν^+-strategic closure below $\langle u_\nu(q^k) \colon k \leq j\rangle$.

At limit stages $j < \nu^+$, let $p^j \leq p^0$ be a condition which is incompatible to all q^k, $k < j$ so that for all $k < j$, $u_\nu(p^j) \leq u_\nu(q^k)$ if such exists. Note that a p^j satisfying the latter condition can always be found by the strategic choice of the $u_\nu(q^k)$. Choose $q^j \leq p^j$ so that $q^j \in D$ and $u_\nu(q^j) \leq u_\nu(p^j)$. Proceed until at some stage j no condition p^j as above can be chosen. By 9, this will be the case for some $j < \nu^+$. We can then find $q \in P_\alpha$ so that $u_\nu(q) \leq u_\nu(q^k)$ for every $k < j$ and $l_\nu(q) = l_\nu(p)$. By our construction, q reduces D below ν. □

Using the claim for $\nu = \eta$, the case of regular η follows immediately, applying 10 once more. For the case of $\eta \leq \alpha$ singular, choose a continuous, cofinal in η, increasing sequence $\langle \eta_i \colon i < \operatorname{cof} \eta\rangle$ of cardinals where each η_{i+1} is regular. Build a sequence of conditions $\langle q^i \colon i < \operatorname{cof} \eta\rangle$ so that $q^{i+1} = q^i$ for limit ordinals i and otherwise q^{i+1} reduces the first η_i-many given dense sets below η_i, $l_{\eta_i}(q^{i+1}) = l_{\eta_i}(q^i)$ and $u_{\eta_i}(q^{i+1})$ is chosen according to the strategy for $(\eta_i)^+$-strategic closure of $u_{\eta_i}(P_\alpha)$ for each $i < \operatorname{cof} \eta$. At limit stages $i \leq \operatorname{cof} \eta$, we may take lower bounds of the conditions obtained so far using stability of the obtained sequence of conditions below η_i together with $(\eta_i)^+$-strategic closure of $u_{\eta_i}(P_\alpha)$ provided by 10. One needs to observe that strategies for strategic closure cohere nicely between different cardinals, i.e.

if $\theta_0 < \theta_1$ are cardinals and $\langle u_{\theta_0}(p^j)\colon j < i\rangle$ follows the strategy for ${\theta_0}^+$-strategic closure of $u_{\theta_0}(P_\alpha)$, then $\langle u_{\theta_1}(p^j)\colon j < i\rangle$ follows the strategy for ${\theta_1}^+$-strategic closure of $u_{\theta_1}(P_\alpha)$.

Δ-distributivity is easily inferred using the above.

Proof of 10 (Strategic Closure), 11 (Reducing dense sets) and 12 (Club Extendibility): We distinguish several cases according to α:

	α	line of argument
Case 1	the trivial cases	trivial
Case 2	succ. ord. of reg. card.	prove 12, then 10
Case 3	sing. card. or inacc.	prove 10 similar to Case 2
Case 4	succ. of a reg. card.	prove 11, then 10
Case 5	lim. ord. of inacc. card.	prove 10, building on Case 2
Case 6	lim. ord. of succ. card.	prove 10, building on Case 2
Finally	lim. ord.	prove 12, using 10

Case 1: $\kappa = \omega$, κ is singular and $\alpha > \kappa$, $\alpha = \omega_1$ or $\alpha = \lambda^+$ for some singular cardinal λ: If $\alpha = \omega$, P_α denotes the trivial forcing and therefore 10 is trivially valid. If κ is singular or ω and α is not a cardinal, 10 is clear inductively. If $\alpha = \omega_1$, 10 is clear. If $\alpha = \lambda^+$ for some singular cardinal λ, 10 is clear using 10 inductively (at stage λ). 12 is clear inductively in all cases.

Case 2: $\alpha = \beta + 1$ for some β, κ is regular: This case already introduces many of the ideas that will turn up in the proofs of 10, 11 and 12. The main issue in the proofs to follow is reducing dense sets related to $\mathrm{ts}(p)$ for some fully string supported $p \in P_\alpha$. In Case 2, $\mathrm{ts}(p)$ is a P_β-name and we will make use of the fact that the properties of P_β that were already established inductively allow us to treat issues at the top cardinal κ exactly like issues at smaller cardinals $\theta < \kappa$. We will show how to handle those $\theta < \kappa$ first (exactly the same treatment needs to be done in all subsequent cases at all $\theta < \kappa$) and remark in the end why κ may be treated in the same way. What makes Case 3 simpler is the fact that $\mathrm{ts}(p) = \mathbf{1}$ or to put it differently, that we only have to treat cardinals $\theta < \kappa$ exactly as in Case 2; nothing has to be done at κ here. In the hardest Cases 5 and 6, we will have to draw upon the fact that $\mathrm{ts}(p)$ is a union of P_β-names for $\beta < \alpha$ when we want to treat issues at the top cardinal κ. Now we turn to the actual proof of Case 2:

The proofs of 10 and 12 will be very similar to each other in this case. We proceed by first giving the proof of 10 assuming 12 and then give a sketch of how to prove 12 in a similar way. The basic difference is that while we use the notion of suitable genericity in the proof of 10 and need to use 12 there to find suitably generic conditions, we will use the weaker notion of suitable pre-genericity to prove 12, making use of the fact that suitably pre-generic conditions are easier to find than suitably generic ones (in particular without using 12 itself).

The proof of 10 proceeds in two steps: First we show that if we build a decreasing sequence of conditions alongside an increasing sequence of models and ensure enough suitable genericity of our conditions at successor stages over the current model in our sequence of models, we can ensure that at limit stages lower bounds for our decreasing sequence of conditions do exist as long as this is not ruled out by obviously blowing the support restraints. The second step will be to show that we can always choose suitably generic successive conditions. This basic scheme of proof will also be employed in Cases 5 and 6.

Claim 52. *[Lower Bounds at Successor Stages]*
Assume $\langle p^i : i < \gamma\rangle$ is a decreasing sequence of conditions in P_α of limit length $\gamma < \eta^+$ which is stable below η^+. Assume that for every $\theta \in [\eta^+, \alpha)$: if $j < \gamma$ is least such that $\text{C-supp}(p^j \cap [\theta, \theta^+)) \neq \emptyset$, then $\langle M^i_\theta : j \leq i < \gamma\rangle$ is an increasing chain of domains of elementary submodels of H_ν for some large (w.r.t. α), regular ν with union $M_\theta = \bigcup_{j\leq i<\gamma} M^i_\theta$, each M^i_θ has cardinality $< \theta$, is transitive below θ, contains p^i and θ as elements and for each $i \in [j, \gamma)$, p^{i+1} is suitably generic for $P_{\min\{\theta^+,\alpha\}}$ at θ over M^i_θ.
Then the sequence $\langle p^i : i < \gamma\rangle$ has a lower bound.

Proof. Assume $\langle p^i : i < \gamma\rangle$ is as in the statement of the claim, using models M^i_θ. We want to show by induction on δ that $\langle p^i\restriction\delta : i < \gamma\rangle$ has a lower bound. Fix some $\delta \in (\eta^+, \alpha]$ and assume that q^ξ denotes an inductively obtained lower bound of $\langle p^i\restriction\xi : i < \gamma\rangle$, $(q^\xi)^\oplus$ denotes an inductively obtained lower bound of $\langle p^i\restriction\xi^\oplus : i < \gamma\rangle$ for $\xi < \delta$. It is easy to see that the existence of q^ξ implies the existence of $(q^\xi)^\oplus$. Let r be the componentwise union of the $p^i\restriction\delta$. Let $|r|_\theta := \bigcup_{i<\gamma} |p^i\restriction\delta|_\theta$. We will only treat the hardest case when $\delta = \alpha = \beta + 1$ is a successor ordinal - the case that $\delta < \alpha$ is similar and easier. We will next establish a notion of coherence that will be central in what follows:

Definition 53. *Assume $\langle p^i : i < \gamma\rangle$ is a decreasing sequence of conditions in P_α of limit length $\gamma < \eta^+$ which is stable below η^+ and let r be the componentwise union of $\langle p^i : i < \gamma\rangle$. Assume $\mathcal{M} = \langle M_\theta : \text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset\rangle$ is a sequence where each M_θ has cardinality $< \theta$, is transitive below θ and contains θ as element. We say that r matches $\mathcal{M}$ iff for every θ with $\text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset$,*

1. *If θ is inaccessible,* $\operatorname{card} M_\theta = \sup(\text{S-supp}(r) \cap \theta) = M_\theta \cap \theta$.
2. *If θ is a successor cardinal, $\theta = \lambda^+$,* $|r|_\lambda = \sup(\text{S-supp}(r) \cap \theta) = M_\theta \cap \theta$.
3. *If $\theta < \operatorname{card}\alpha$,* $\text{C-supp}(r) \cap [\theta, \theta^+) = M_\theta \cap [\theta, \theta^+)$. *If $\theta = \operatorname{card}\alpha$,* $\text{C-supp}(r) \cap [\theta, \alpha) = M_\theta \cap [\theta, \alpha)$.
4. *If $\xi \in \text{C-supp}(r)$ has cardinality θ and $\langle p^i\restriction\xi^\oplus : i < \gamma\rangle$ has a lower bound, then this lower bound forces that $r^*_\xi = M_\theta \cap [\theta, |r_\xi|)$.*
5. *If $\theta < \operatorname{card}\alpha$ and $\langle p^i\restriction\theta^+ : i < \gamma\rangle$ has a lower bound, then this lower bound forces that* $\text{I-supp}(r) \cap [\theta, \theta^+) = M_\theta \cap [\theta, \theta^+)$. *If $\theta = \operatorname{card}\alpha$ and $\langle p^i : i < \gamma\rangle$ has a lower bound, then this lower bound forces that* $\text{I-supp}(r) \cap [\theta, \theta^+) = M_\theta \cap [\theta, |r|_\theta)$.
6. *If D is a dense subset of $E_{\min\{\theta^+,\zeta\}}$ for some $\zeta \in [\operatorname{card}\alpha, \alpha)$ in M_θ, then there is $i < \gamma$ so that p^i reduces D below λ for some $\lambda < \theta$.*
7. *If $\xi \in \text{C-supp}(r)$, $\xi \geq \eta^+$ and $\langle p^i\restriction\xi^\oplus : i < \gamma\rangle$ has a lower bound, then this lower bound forces that* $\sup r^{**}_\xi = \sup(\text{S-supp}(r) \cap \theta)$.

As might be expected, it is the case that if we construct a decreasing sequence of conditions alongside an increasing sequence of models as in the statement of Claim 52, the above-defined coherence notion applies to the componentwise union of our conditions and the unions of our models, which is shown in the following claim:

Claim 54. *r matches $\langle M_\theta \colon \text{C-supp}(r) \cap [\theta, \theta^+) \neq \emptyset \rangle$.*

Proof. Proof of 1: By Clause 0 of suitable genericity, $\sup(\text{S-supp}(p^{i+1}) \cap \theta) \geq \operatorname{card} M^i_\theta$ and $\geq M^i_\theta \cap \theta$, implying that $\sup(\text{S-supp}(r) \cap \theta) \geq \operatorname{card} M_\theta$ and $\geq M_\theta \cap \theta$. As $p^i \in M^i_\theta$, $\theta \in M^i_\theta$ and M^i_θ is transitive below θ, it follows that $M_\theta \cap \theta$ and $\operatorname{card} M_\theta$ are both $\geq \sup(\text{S-supp}(r) \cap \theta)$.

Proof of 2: Since $\lambda \in M^0_\theta$, $|p^i|_\lambda \in M^i_\theta$, hence $|r|_\lambda \leq M_\theta \cap \theta$. For the other direction, $|p^i|_\lambda \geq \sup(\text{S-supp}(p^i) \cap \theta)$, hence $|r|_\lambda \geq \sup(\text{S-supp}(r) \cap \theta)$.

Proof of 3: Since $\theta \in M^0_\theta$, $\operatorname{card}(\text{C-supp}(p^i) \cap [\theta, \theta^+)) \in M^i_\theta$ and hence $\text{C-supp}(p^i) \cap [\theta, \theta^+) \subseteq M^i_\theta$, hence $\text{C-supp}(r) \cap [\theta, \theta^+) \subseteq M_\theta$. For the other direction, assume $\xi \in M^i_\theta \cap [\theta, \min(\theta^+, \alpha))$. If $\alpha = \xi + 1$, by clause 2 of suitable genericity we obtain that $\xi \in \text{C-supp}(p^1)$. If $\alpha \neq \xi + 1$, then $D = \{t \in P_{\xi+1} \colon \xi \in \text{C-supp}(t)\}$ is dense in $P_{\xi+1}$ and definable in M^i_θ and hence reduced below $\operatorname{card} M^i_\theta$ by p^{i+1}. But this means that $\xi \in \text{C-supp}(p^{i+1})$. Hence $M_\theta \cap [\theta, \theta^+) \subseteq \text{C-supp}(r) \cap [\theta, \theta^+)$.

Proof of 4: Choose $i < \gamma$ so that $\xi \in \text{C-supp}(p^i)$. By suitable genericity, p^{i+1} decides $|p^i_\xi| = |r_\xi|$. Let ζ be an element of $M_\theta \cap [\theta, |r_\xi|)$ and choose $j < \gamma$ greater than i so that $\zeta \in \text{C-supp}(p^j)$. Then $\{t \in P_{\xi+1} \colon t \restriction \xi^\oplus \Vdash \zeta \in t^*_\xi\}$ is dense and definable in M^j_θ, hence by suitable genericity, $p^{j+1} \restriction \xi^\oplus$ forces that $\zeta \in (p^{j+1})^*_\xi$. For the other direction, by suitable genericity and clause 16 of theorem 49 inductively, there exists $x \in \mathbf{V}$ of size $< \theta$ such that $p^{i+1} \restriction \xi^\oplus \Vdash r^*_\xi \subseteq x$ and hence $M^{i+1}_\theta \supseteq x$ by elementarity of M^{i+1}_θ.

5 is immediate from 4. 6 is immediate using suitable genericity of the p^i.

Proof of 7: If $\alpha = \beta + 1$ is a successor ordinal and $\xi = \beta$, $\sup r^{**}_\xi \geq \sup(\text{S-supp}(r) \cap \theta)$ follows using clause 2 of suitable genericity. Otherwise, $\sup r^{**}_\xi \geq \sup(\text{S-supp}(r) \cap \theta)$ follows by easy density arguments and clause 1 of suitable genericity. $\sup(\text{S-supp}(r) \cap \theta) \geq \sup r^{**}_\xi$ also follows by easy density arguments and clause 1 of suitable genericity. □

We now turn back to the proof of Claim 52. We want to define a condition q constituting a lower bound of $\langle p^i \colon i < \gamma \rangle$. It will be immediate from the definition of q that it extends each p^i as soon as we know that q actually is a valid condition. To prove the latter will then finish the proof of Claim 52. We will make constant and usually tacit use of the coherence properties obtained in Claim 54.

Assume $\eta^+ \leq \xi < \alpha$, $\xi \in \text{C-supp}(r)$, $\operatorname{card} \xi = \theta$ and $(q^\xi)^\oplus$ forces $\rho \in r^*_\xi$. Then f_ρ is a bijection between θ and ρ, by elementarity of M_θ thus $f_\rho \restriction (M_\theta \cap \theta)$ is a bijection between $M_\theta \cap \theta$ and $M_\theta \cap \rho$. Thus if we let π_θ denote the collapsing map of M_θ, it follows that $(q^\xi)^\oplus$ forces $\pi_\theta(\rho) = \operatorname{ot}(f_\rho[\sup r^{**}_\xi])$. If θ is inaccessible, $\pi_\theta(\rho) \geq M_\theta \cap \theta = \sup(\text{S-supp}(r) \cap \theta) = \operatorname{card} M_\theta$; thus for any $\rho_0 \neq \rho_1$ in I-supp(r), $\pi_{\mathrm{card}\,\rho_0}(\rho_0) \neq \pi_{\mathrm{card}\,\rho_1}(\rho_1)$. Let $\bar\theta := \pi_\theta(\theta)$ and let $\bar M_\theta = \pi_\theta{}'' M_\theta$. We want to build q out of r as follows:

- for every regular $\theta \geq \eta^+$ with C-supp$(r) \cap [\theta, \theta^+) \neq \emptyset$, construct a $P_{\bar\theta}$-name s_θ for a $\bar\theta^+$-Cohen condition and let $q_{\bar\theta} = s_\theta$ as follows:
  - Let $|s_\theta|^- := \sup \pi_\theta{}''(M_\theta \cap |r|_\theta)$. If θ is inaccessible, let $|s_\theta| = |s_\theta|^-$; if θ is a successor cardinal, let $|s_\theta| = |s_\theta|^- + 2$.
  - Choose $F^q_{\bar\theta}$ to be $\langle f_\gamma \colon \gamma \in [\mathrm{card}\,\bar\theta, |s_\theta|)\rangle$ with f_γ chosen freely as a bijection from card $\bar\theta$ to γ for $\gamma \geq \bar\theta$ and equal to the f_γ picked by r otherwise.
  - If $(\rho_0, \rho_1) \in [\mathrm{card}\,\bar\theta, \bar\theta)^\urcorner$, then $s_\theta(\rho_0, \rho_1) = r@(\rho_0, \rho_1)$.
  - For all $(\rho_0, \rho_1) \in ([\theta, \theta^+)^\urcorner \cap {M_\theta}^2)$, $s_\theta(\pi_\theta(\rho_0), \pi_\theta(\rho_1))$ is such that it is forced by q^ξ to be equal to $\pi_\theta(r@(\rho_0, \rho_1))$ whenever ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\theta$.
  - If θ is a successor cardinal, $q^{\bar\theta}$ forces that $s_\theta(|s_\theta|^-)$ codes $f_{|s_\theta|^-}$.
- for all $\xi \in$ C-supp(r), $\xi \geq \eta^+$, $q^{**}_\xi = r^{**}_\xi \cup \{\sup r^{**}_\xi\}$,
- $q_\xi = r_\xi$ and $F^q_\xi = F^r_\xi$ for all $\xi \in$ S-supp(r), $q^*_\xi = r^*_\xi$ for all ξ and $q^{**}_\xi = r^{**}_\xi$ for all $\xi < \eta^+$.

This will be possible once we know that

(a) Whenever $\theta \geq \eta^+$, $(\rho_0, \rho_1) \in ([\theta, \theta^+)^\urcorner \cap {M_\theta}^2)$ and ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\theta$, then q^ξ forces that $r@(\rho_0, \rho_1)$ has a $P_{\bar\theta}$-name.

(b) If ρ is not a multiple of θ and q^ξ forces $\rho \in r^*_\xi$, then q^ξ forces $r@(\rho, \rho) = 0$. This implies that if $\theta = \lambda^+$ is a successor cardinal and $\bar\rho \geq |r|_\lambda$ is not a multiple of $\omega \cdot \lambda$, $q^{\bar\theta}$ forces $s_\theta(\bar\rho, \bar\rho) = 0$.

(c) $q^{\bar\theta}$ forces that s_θ is an acceptable, correct $\bar\theta^+$-Cohen condition.

Proof of (a): Choose $i < \gamma$ and ξ s.t. $p^i \restriction \xi^\oplus$ forces $(\rho_0, \rho_1) \in [\theta, \theta^+)^\urcorner \cap (\bar\theta \cup (p^i)^*_\xi)$. The set of conditions deciding $p^i@(\rho_0, \rho_1)$ is dense in P_ξ and an element of M^i_θ. It follows by suitable genericity of p^{i+1} for P_ξ at θ over M^i_θ that $p^{i+1} \Vdash p^i@(\rho_0, \rho_1)$ has a $P_{\sup(\text{S-supp}(p^{i+1}) \cap \theta)}$-name and (a) follows as $\sup(\text{S-supp}(p^{i+1}) \cap \theta) < \bar\theta$.

Proof of (b): The first statement is obvious from the definition of the forcing. The second statement follows from the first one, noting that multiples of θ will be collapsed to multiples of $\bar\theta$ under π_θ and that $\bar\theta$ is a multiple of $\omega \cdot \lambda$.

Proof of (c): As we may show this separately for every relevant θ, fix some such θ. Assume first that $\theta < \mathrm{card}\,\alpha$. If $\xi \in$ C-supp$(r) \cap [\theta, \theta^+)$, $r_\xi \in M_\theta$ is a P_ξ-name for an acceptable, correct θ^+-Cohen condition. Let $s = \bigcup_{\xi \in \text{C-supp}(r) \cap [\theta, \theta^+)} r_\xi$. Let $\bar r_\xi := \pi_\theta(r_\xi)$ and let $\bar s = \bigcup_{\xi \in \text{C-supp}(r) \cap [\theta, \theta^+)} \bar r_\xi$. Let $s^\xi_\theta := s_\theta \restriction \pi_\theta(|r \restriction \xi|_\theta)$.

Claim 55. *If $\bar G \ni q^{\bar\theta}$ is generic for $P_{\bar\theta}$, then $\langle p^i \restriction \theta^+ \colon i < \gamma\rangle$ together with $\bar G$ generates a generic G^{**} for P_{θ^+} over M_θ. Let $G^* = \pi_\theta$"G^{**}. Then $(s_\theta \restriction |s_\theta|^-)^{\bar G} = \bar s^{G^*}$.*

Proof. Each p^i is compatible with $\bar{G}$ and thus $\langle p^i \restriction \theta^+ : i < \gamma\rangle$ together with $\bar{G}$ generates a filter $G^{**} \subseteq P_{\theta^+}$. Assume $D \in M_\theta$ is an open dense subset of P_{θ^+}. As D is an element of some M^i_θ, $i < \gamma$, $p^{i+1}\restriction\theta^+$ reduces D below $\operatorname{card} M^i_\theta$ by suitable genericity, which implies $D^* = \{q \in P_{\bar{\theta}} \colon q^\frown p^{i+1}[\bar{\theta}, \theta^+) \in D\}$ is dense below $p^{i+1}\restriction\bar{\theta}$ and thus hit by some condition $q \in \bar{G}$. Let $\bar{p}^i$ denote $\pi_\theta(p^i)$. For the second statement, if $\xi_0, \xi_1 < |s_\theta|^-$,

$$
\begin{aligned}
& {s_\theta}^{\bar{G}}(\xi_0, \xi_1) = \xi_2 \text{ iff}\\
& \exists p \in \bar{G}\ p \Vdash s_\theta(\xi_0, \xi_1) = \xi_2 \text{ iff}\\
& \exists p \in \bar{G}\, \exists i < \gamma\ p^\frown p^i[\bar{\theta}, \theta^+) \Vdash p^i @({\pi_\theta}^{-1}(\xi_0), {\pi_\theta}^{-1}(\xi_1)) = {\pi_\theta}^{-1}(\xi_2) \text{ iff}\\
& \exists p^* \in G^*\ p^* \Vdash \bar{p}^i @(\xi_0, \xi_1) = \xi_2 \text{ iff}\\
& \bar{s}^{G^*}(\xi_0, \xi_1) = \xi_2.
\end{aligned}
$$

□

Let $\bar{P}_\xi = \pi_\theta(P_\xi)$. If θ is inaccessible, $\bar{\theta}$ is a (singular) cardinal and thus by elementarity, each $\bar{r}_\xi$ is a $\bar{P}_\xi$-name for an acceptable, correct $\bar{\theta}^+$-Cohen condition w.r.t. $g_{\operatorname{card}\bar{\theta}}$. By the above claim, we thus know that $q^{\bar{\theta}}$ forces that each s^ξ_θ is an acceptable, correct $\bar{\theta}^+$-Cohen condition w.r.t. $g_{\operatorname{card}\bar{\theta}}$. By Lemma 27, this implies that $q^{\bar{\theta}}$ forces that s_θ is an acceptable, correct $\bar{\theta}^+$-Cohen condition w.r.t. $g_{\operatorname{card}\bar{\theta}}$ and hence we are done in that case.

If θ is a successor cardinal, $\theta = \lambda^+$, $\bar{\theta} \in (\lambda, \theta)$ is not a cardinal (of $\mathbf{V}$).

Claim 56. *$q^{\bar{\theta}}$ forces that s_θ is correct w.r.t. g_λ.*

Proof. $q^{\bar{\theta}}$ forces that $f_{|s_\theta|^-} \in L_{|s_\theta|}[g_\lambda^\frown s_\theta]$ by the definition of s_θ. □

It remains to show that s_θ is an acceptable θ-Cohen condition:

Claim 57. *$q^{\bar{\theta}}$ forces that all subsets of λ in $L_{|s_\theta|^-+1}[g_\lambda{}^\frown s_\theta]$ appear before $\bar{\theta}$.*

Proof. Let $\bar{G} \ni q^{\bar{\theta}}$ be generic for $P_{\bar{\theta}}$. $\langle p^i \colon i < \gamma\rangle$ together with $\bar{G}$ generates a generic G^{**} for P_{θ^+} over M_θ. Therefore $G^* := \pi_\theta{}''[G^{**}]$ is generic for $\bar{P}_{\theta^+}$ over $\bar{M}_\theta$ and we get an elementary embedding j_θ from $\bar{M}_\theta[G^*]$ to $M_\theta[G^{**}]$ in $V[\bar{G}]$. $M_\theta \models \mathbf{1}_{P_{\theta^+}} \Vdash$ "$L[g_{\theta^+}] \models$ all subsets of λ appear before θ" inductively, hence $M_\theta[G^{**}] \models$ "$L[g_{\theta^+}] \models$ all subsets of λ appear before θ". Applying elementarity, $\bar{M}_\theta[G^*] \models$ "$L[{j_\theta}^{-1}(g_{\theta^+})] \models$ all subsets of λ appear before $\bar{\theta}$", but ${j_\theta}^{-1}(g_{\theta^+}) = g_\lambda{}^\frown(s_\theta\restriction|s_\theta|^-)^{\bar{G}}$ and since $\bar{M}_\theta$ is transitive and satisfies a large fragment of ZFC, absoluteness yields that $L_{\mathbf{Ord}(\bar{M}_\theta)}[g_\lambda^\frown(s_\theta\restriction|s_\theta|^-)^{\bar{G}}] \models$ all subsets of λ appear before $\bar{\theta}$, which proves the claim. □

Claim 58. *$q^{\bar{\theta}}$ forces that s^ξ_θ is an acceptable θ-Cohen condition w.r.t. g_λ and $L_{|s^\xi_\theta|}[g_\lambda{}^\frown s^\xi_\theta] \models \bar{\theta}$ is the largest cardinal. We abbreviate the latter by saying that s^ξ_θ is correct relative to $\bar{\theta}$.*

Proof. By elementarity, using the properties of $\bar{r}_\xi$ and Claim 55. □

We will finish showing that s_θ is acceptable w.r.t. g_λ by the following:

Claim 59. *$q^{\bar{\theta}}$ forces that s_θ is acceptable w.r.t. g_λ for subsets of $\bar{\theta}$, in the sense that $q^{\bar{\theta}}$ forces that whenever $|s_\theta| \geq \nu > \rho \geq \bar{\theta}$ and there is a new subset of ρ in $L_{\nu+1}[g_\lambda{}^\frown s_\theta]$, then $H^{L_{\nu+1}[g_\lambda{}^\frown s_\theta]}(\rho) = L_{\nu+1}[g_\lambda{}^\frown s_\theta]$.*

Proof. Using claim 58, it follows as in the proof of Lemma 27 that $q^{\bar{\theta}}$ forces that $s_\theta\restriction|s_\theta|^-$ is an acceptable w.r.t. g_λ for subsets of $\bar{\theta}$, correct relative to $\bar{\theta}$, θ-Cohen condition. The claim follows as $q^{\bar{\theta}} \Vdash f_{|s_\theta|^-} \in L_{|s_\theta|}[g_\lambda^\frown s_\theta]$. □

To finish the proof of (c) and thus of Claim 52, we finally have to consider the case $\theta = \operatorname{card}\alpha$. We omit a detailed proof as this is very similar to (but easier than) the case $\theta < \operatorname{card}\alpha$, letting $s = r_\beta \in M_\theta$, $\bar{s} = \pi_\theta(s)$ and using the facts that if $\bar{G}$ is generic for $P_{\bar{\theta}}$, then $\langle p^i : i < \gamma\rangle$ together with $\bar{G}$ generates a generic G^{**} for $P_\beta^\oplus$ over M_θ and that g_α can already be read off from the $P_\beta^\oplus$-generic. □

We now come to what we described as the second part of proof in the paragraph before the statement of Claim 52 - the next claim, together with Claim 52, finally establishes 10 in Case 2 (except that we still have to prove 12 in this case without using 10):

Claim 60. *If $p \in P_\alpha$ and for every $\theta \in [\eta^+, \alpha)$ with* C-supp$(p) \cap [\theta, \theta^+) \neq \emptyset$, *$M_\theta$ is of cardinality $< \theta$ and transitive below θ, then there is $q \leq p$ which is suitably generic for P_α at θ over M_θ for all such θ.*

Proof. By 11 inductively and by 12, which we need for suitable genericity at κ. Note that we may successively extend p to q, at each step ensuring suitable genericity of q at θ for one particular cardinal θ. We may do this at every particular θ without changing our condition below $\operatorname{card} M_\theta$ or at or above θ^+, thus no problems arise to obtain q as a lower bound of those successive extensions in the end. □

Now we turn to the proof of 12: Using 12 inductively, we may assume that $I = \{\beta\}$ and it thus suffices to show that for any given $\delta < \kappa$, we can extend any given $p \in P_\alpha$ to $q \leq p$ so that $q \Vdash \sup(q_\beta^{**}) \geq \delta$. Let such δ and p be given. We want to build a decreasing sequence of conditions $\langle p^i : i < \omega\rangle$ as in Claim 52. At κ though, we only demand that each p^{i+1} is suitably pre-generic for P_α at κ over M_κ^i. Choose M_κ^0 so that $M_\kappa^0 \cap \kappa \geq \delta$. Choose each p^{i+1} so that $(p^{i+1})_\beta^{**} = (p^i)_\beta^{**}$. Let r be the componentwise union of the p^i. $\langle p^i\restriction\beta : i < \omega\rangle$ is as desired for Claim 52 to go through at stage β and hence we may inductively obtain a lower bound $\bar{q}$ of $\langle p^i\restriction\beta : i < \omega\rangle$. Let $\mathcal{M} = \langle M_\nu : \text{C-supp}(r) \cap [\nu, \nu^+) \neq \emptyset\rangle$. Similar to Claim 54, r matches $\mathcal{M}$, except that item 7 of definition 53 does not hold for $\xi = \beta$. It is not hard to see that we may still, additional to obtaining the lower bound $\bar{q}$, add $M_\kappa \cap \kappa \geq \delta$ to r_β^{**} (as we do for all other r_ξ^{**} when ξ is of cardinality κ) and obtain a lower bound q of $\langle p^i : i < \omega\rangle$ as we would have done in Claim 52.

Case 3: α is a singular cardinal or inaccessible: The analogue of Claim 60 is immediate inductively, the analogue of Claim 52 then follows as in Case 2 (but easier, as we only have to consider $\theta < \kappa$), yielding 10.

Case 4: α is a successor of a regular cardinal: Let λ be so that $\alpha = \lambda^+$. It is immediate that $u_\lambda(P_\alpha)$ is α-closed. But this allows us to prove 11 in the case $\eta = \lambda$. We use this to prove 11 in the general case: Given a collection $\langle D_i : i < \eta\rangle$ of dense subsets of P_α and $\eta < \lambda$, we can reduce each D_i below λ by a single condition $p \in P_\alpha$ by the above. But then $p \in P_\beta$ for some

$\beta < \alpha$ and we may use 11 at stage β inductively to extend p to q (with p and q equal at and above β) so that q reduces each D_i below η. Now 11 suffices to prove 10 as it allows us to find suitably generic conditions, i.e. prove the analogue to Claim 60. The analogue to Claim 52 then follows as in Case 2 above.

Case 5: α is a singular limit ordinal, κ is inaccessible
We proceed similar to Case 2 here. First we show that we may take lower bounds at limit stages of decreasing sequences of conditions in our forcing as long as we ensure enough suitable genericity along the sequence. In the second step, we show that we may always ensure enough suitable genericity along the sequence. With the methods of Case 2 available, this case will be quite easy:

Claim 61. *[Lower Bounds at Singular Limit Stages of Inaccessible Cardinality]*
Assume $\langle p^i \colon i < \gamma\rangle$ is a decreasing sequence of conditions in P_α of limit length $\gamma < \eta^+$ which is stable below η^+. Assume that for every $\theta \in [\eta^+, \alpha)$: if $j < \gamma$ is least such that C-supp$(p^j \cap [\theta, \theta^+)) \neq \emptyset$, *then $\langle M^i_\theta \colon j \leq i < \gamma\rangle$ is an increasing chain of domains of elementary submodels of H_ν for some large (w.r.t. α), regular ν with union $M_\theta = \bigcup_{j \leq i < \gamma} M^i_\theta$, each M^i_θ has cardinality $< \theta$, is transitive below θ, contains p^i and θ as elements and for each $i \in [j, \gamma)$, p^{i+1} is suitably generic for $P_{\min\{\theta^+, \alpha\}}$ at θ over M^i_θ. Then the sequence $\langle p^i \colon i < \gamma\rangle$ has a lower bound.*

Proof. Assume $\langle p^i \colon i < \gamma\rangle$ is as in the statement of the claim, using models M^i_θ. Let r be the componentwise union of the p^i. We want to show by induction on ξ that $\langle p^i \restriction \xi \colon i < \gamma\rangle$ has a lower bound. Let $\xi \leq \alpha$ and assume that q^ζ denotes the inductively obtained lower bound of $\langle p^i \restriction \zeta \colon i < \gamma\rangle$, $(q^\zeta)^\oplus$ denotes the inductively obtained lower bound of $\langle p^i \restriction \zeta^\oplus \colon i < \gamma\rangle$ for $\zeta < \xi$. Cardinals $\theta < \kappa$ are handled as in Case 2. We will thus basically ignore them for the rest of this argument and focus solely on handling κ. We will only treat the hardest case when $\xi = \alpha$ - other cases are handled as in Case 2. Similar to Claim 54 in Case 2, we obtain that r matches $\langle M_\theta \colon$ C-supp$(r) \cap [\theta, \theta^+) \neq \emptyset\rangle$. Assume $\nu \in [\kappa, \alpha)$ and $(q^\nu)^\oplus$ forces $\rho \in r^*_\nu$. Let π_κ denote the collapsing map of M_κ. Then $(q^\nu)^\oplus$ forces that $\pi_\kappa(\rho) = \mathrm{ot}(f_\rho[\sup r^{**}_\nu])$ as in Case 2 above. Let $\bar\kappa = \pi_\kappa(\kappa)$ and let $\bar M_\kappa = \pi_\kappa{}'' M_\kappa$.

We want to build q out of r similar than we did in Case 2, in particular, we will do the same at cardinals less than κ, which we will ignore during the rest of this argument. At κ, we do the following:

- Construct a $P_{\bar\kappa}$-name s_κ for a $\bar\kappa^+$-Cohen condition and let $q_{\bar\kappa} = s_\kappa$ as follows:
  - Let $|s_\kappa| = \sup \pi_\kappa{}''(M_\kappa \cap |r|_\kappa)$.
  - For all $(\rho_0, \rho_1) \in ([\kappa, \kappa^+)^\urcorner \cap {M_\kappa}^2)$, $s_\kappa(\pi_\kappa(\rho_0), \pi_\kappa(\rho_1))$ is such that it is forced by q^ξ to be equal to $\pi_\kappa(r@(\rho_0, \rho_1))$ whenever ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\kappa$.
- Choose arbitrary collapsing information $F^q_{\bar\kappa}$.
- For all $\nu \in$ C-supp(r), $\nu \geq \kappa$, $q^{**}_\nu = r^{**}_\nu \cup \{\sup r^{**}_\nu\}$.

- $q_\nu = r_\nu$ and $F^q_\nu = F^r_\nu$ for all $\nu \geq \kappa$, $q^*_\nu = r^*_\nu$ for all $\nu \geq \kappa$.

This will be possible once we know that

(a) Whenever $(\rho_0, \rho_1) \in ([\kappa, \kappa^+)^\urcorner \cap M_\kappa{}^2)$ and ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\kappa$, then q^ξ forces that $r@(\rho_0, \rho_1)$ has a $P_{\bar\kappa}$-name.
(b) $q^{\bar\kappa}$ forces that s_κ is an acceptable, correct $\bar\kappa^+$-Cohen condition.

(a) is shown exactly as in Case 2. For (b), it follows as in case 2 that $q^{\bar\kappa}$ forces that every restriction of s_κ is an acceptable, correct $\bar\kappa^+$-Cohen condition w.r.t. $g_{\bar\kappa}$ by elementarity (the key point here is that $\bar\kappa$ is a cardinal, this saves us from all the hard work that has to be done in Case 6 below). Thus by Lemma 27, $q^{\bar\kappa}$ forces that s_κ is an acceptable, correct $\bar\kappa^+$-Cohen condition w.r.t. $g_{\bar\kappa}$, as desired. □

We are now done with the first step of our proof in Case 5. A new ingredient for the second step will be a slightly refined version of our usual claim about obtaining lower bounds of sequences of conditions (Claim 61 in this case), which we will need to invoke to construct suitably generic conditions in Claim 63 below in case α has "small cofinality":

Claim 62. *Assume α has cofinality $\gamma < \kappa$ and $\langle p^i : i < \gamma \rangle$ is a decreasing sequence of conditions in E_α of limit length $\gamma < \eta^+$ which is stable below η^+. Assume that for every $\theta \in [\eta^+, \kappa)$: if $j < \gamma$ is least such that* C-supp$(p^j \cap [\theta, \theta^+)) \neq \emptyset$*, then $\langle M^i_\theta : j \leq i < \gamma \rangle$ is an increasing chain of domains of elementary submodels of H_ν for some large (w.r.t. α), regular ν with union $M_\theta = \bigcup_{j \leq i < \gamma} M^i_\theta$, each M^i_θ has cardinality $< \theta$, is transitive below θ, contains p^i and θ as elements and for each $i \in [j, \gamma)$, p^{i+1} is suitably generic for $P_{\theta+}$ at θ over M^i_θ. Assume further that $\langle \alpha_i : i < \operatorname{cof} \alpha \rangle$ is continuous, increasing and cofinal in α with $\alpha_0 > \kappa$. Assume that $\langle M^i_\kappa : i < \gamma \rangle$ is an increasing chain of domains of elementary submodels of H_ν with union M_κ, each M^i_κ has cardinality $< \kappa$, is transitive below κ, contains p^i and κ as elements and for each $i < \gamma$, p^{i+1} is suitably generic for P_{α_i} at κ over M^i_κ and $p^{i+1} \upharpoonright [\alpha_i, \alpha) = p^i \upharpoonright [\alpha_i, \alpha)$.*
Then the sequence $\langle p^i : i < \gamma \rangle$ has a lower bound.

Proof. Our current assumptions suffice to perform exactly the same proof that we gave for Claim 61. □

We finish the proof in Case 5 by the following:

Claim 63. *If $p \in P_\alpha$ and M_θ is of cardinality $< \theta$ for every $\theta \in [\eta^+, \alpha)$ with* C-supp$(p) \cap [\theta, \theta^+) \neq \emptyset$*, then there is $q \leq p$ which is suitably generic for P_α at θ over M_θ for all such θ.*

Proof. First use inductive distributivity to obtain a condition q^* which is suitably generic for P_α at θ over M_θ for all $\theta < \kappa$. If α has cofinality $\geq \kappa$, then M_κ is bounded in α and therefore we may use inductive distributivity from stage $\sup(M_\kappa \cap \alpha)$ to obtain suitable genericity. If α has cofinality $< \kappa$, construct a decreasing sequence of conditions $\langle p^i : i < \operatorname{cof} \alpha \rangle$ with $p^0 = q^*$ as in Claim 62 while additionally making sure that for each $i < \gamma$, p^{i+1} reduces all dense subsets of P_{α_i} in M_κ below $\operatorname{card} M_\kappa$ (which is possible by

11 inductively) and note that the resulting lower bound is suitably generic for P_α at κ over M_κ. □

Case 6: α is a singular limit ordinal, $\kappa = \lambda^+$ is a successor cardinal: Assume p^0 is a condition in E_α. Work in some P_α-generic extension with generic G_α and let $s = g_\alpha \restriction [\kappa, |g_\alpha|)$.

Case 6 is the hardest case. It is only for the proof in this case to go through that the somewhat intricate definition of $P_\alpha^\oplus$ in 4 is required. Although the proof structure in this case is basically the same two-step construction we used in Case 2 and Case 5, what actually follows is a long chain of preliminaries culminating in the central preliminary result (Claim 75) that no new subset of λ appears at the top of s (i.e. in $L_{|s|+1}[s]$). This was easily seen in all other cases, but requires a careful proof in the present case, which mainly involves establishing "definable versions" of Step 1 and Step 2. With Claim 75 established, we may then proceed with the usual two-step proof as in Case 2 and Case 5. Instead of giving the detailed argument for Steps 1 and 2 by mostly repeating once more what we did in Case 2 and Case 5 (we will also go through a somewhat similar line of argument once more in the "definable versions" below), we decided to avoid too much repetition and just hint at necessary changes to those cases in the proof of Claim 76 below, which will then finish the proof in Case 6. This presentation should be more agreeable to the reader.

Claim 64. *For $\beta \in [\kappa, \alpha)$, $\mathrm{tsc}(p^0 \restriction \beta)$ is uniformly definable in $L_{|s|}[s]$ from β.*

Proof. For every $\gamma \in [\kappa, \beta)$, p^0_γ is coded (in the sense of 7) by s in the interval $(|p^0_{\gamma+1}| + |p^0_\gamma|, |p^0_{\gamma+1}| \cdot 2)$. Let ν be the unique successor multiple of κ s.t. $s(\nu, \nu) = \gamma$. If γ is a limit ordinal, $|p^0_\gamma|$ equals the least ξ s.t. for no successor multiple ζ of κ in $[\xi, \nu)$, $s(\zeta, \zeta) \neq 0$ and equals the least ξ s.t. $s(\xi + \kappa, \xi + \kappa) = \zeta$ if $\gamma = \zeta + 1$ is a successor ordinal. $|p^0_\gamma|$ is definable in $L_{|s|}[s]$ and hence our claim follows, as $\mathrm{tsc}(p^0 \restriction \beta)$ is obtained by putting together, one after another, the codes (in the sense of 7) of the p^0_γ, $\gamma < \beta$, additionally letting $\mathrm{tsc}(p^0 \restriction \beta)(|p^0 \restriction \gamma|_\kappa, |p^0 \restriction \gamma|_\kappa) = \gamma$ for all $\gamma < \beta$. □

Corollary 65. *$L_{|s|}[\mathrm{tsc}(p^0)]$ is a definable class of $L_{|s|}[s]$, therefore*

$$L_{|s|+1}[\mathrm{tsc}(p^0)] \subseteq L_{|s|+1}[s].$$

Claim 66. *$L_{|s|}[s]$ and $L_{|s|}[\mathrm{tsc}(p^0)]$ both think that κ is the largest cardinal, i.e. s and $\mathrm{tsc}(p^0)$ are both correct w.r.t. $\emptyset$.*

Proof. By 4, because s and $\mathrm{tsc}(p^0)$ both code f_γ for $\gamma < |s|$ in a way that each such f_γ is an element of both $L_{|s|}[s]$ and $L_{|s|}[\mathrm{tsc}(p^0)]$. □

If α has definable cofinality $< \kappa$, note that the definable cofinality of $|s|$ over $L_{|s|}[\mathrm{tsc}(p^0)]$ equals that of α and as $H_\kappa \subseteq L_{|s|}[\mathrm{tsc}(p^0)]$ it is equal to the actual cofinality of α in $\mathbf{V}$. An isomorphic copy of $E_\alpha(\mathrm{ts}(p^0))$ is definable over $L_{|s|}[\mathrm{tsc}(p^0)]$ by 6. For a condition $t \in E_\alpha(\mathrm{ts}(p^0))$, let $c(t)$ denote its counterpart in that isomorphic copy. We say that a dense $D \subseteq E_\beta(\mathrm{ts}(p^0))$ is coded in M^0 if it is a dense subset of the version of $E_\beta(\mathrm{ts}(p^0))$ in that isomorphic copy and definable over M^0. We have been somewhat vague in 6 about the exact way to obtain the isomorphic copy of $E_\alpha(\mathrm{ts}(p^0))$, but we will need a rigorous definition of $c(t)$ in the following, which we will give now: Note that by 5, $t\restriction\kappa \in L_{\kappa\cdot 2+1}[\mathrm{tsc}(p^0)]$, so we will take $c(t)$ to be of the form $(c(t)_\kappa, c(t)^\kappa)$ where $c(t)_\kappa$ simply equals $t\restriction\kappa$. Let f be the $L[\mathrm{tsc}(p^0)]$-least bijection from κ to α, note that $f \in L_{|s|+1}[\mathrm{tsc}(p^0)]$ and apply f^{-1} pointwise to the indices of $t\restriction[\kappa,\alpha)$ to obtain $t' = \langle (f^{-1}(\delta), t(\delta)(1)) \colon t(\delta)(1) \neq \check{\mathbf{1}}, \delta \in [\kappa,\alpha)\rangle$. Now obtain t'' from t' by replacing each $t(\delta)(1)$ by a code for $t(\delta)(1)$ in H_κ, obtained as follows: By 2, $t(\delta)(1)$ is represented by $\rho < \kappa$-many functions $\langle A_j \colon j < \rho\rangle$ each with domain a maximal antichain of E_κ of size $< \kappa$ and range $|s|$. Let g be the $L[\mathrm{tsc}(p^0)]$-least bijection from κ to $|s|$. We modify each A_j to A'_j by letting $A'_j(x) = g^{-1}(A_j(x))$. Now each A'_j is an element of H_κ and we let $\langle A'_j \colon j < \rho\rangle \in H_\kappa$ be our desired code for $t(\delta)(1)$. Let $c(t)^\kappa = t''$.

The next preliminary result will be the existence of what might be called "definably suitably generic" conditions (those will actually be called "suitably generic for codes" in Definition 70 below):

Claim 67. *If $p^0 \in P_\alpha$ is fully string supported and $M^0 \supseteq \lambda + 1$ is definable over $L_{|s|}[\mathrm{tsc}(p^0)]$ and has size λ, then there is $q \leq p^0$ so that for all $\beta < \alpha$, q reduces all dense subsets of P_β which are coded in M^0 below λ.*

Proof. Let $\langle \alpha_i \colon i < \operatorname{cof}\alpha\rangle$ be an increasing sequence with limit α, Σ_n-definable over $L_{|s|}[\mathrm{tsc}(p^0)]$. We may assume that n is s.t. M^0 is Σ_n-definable over $L_{|s|}[\mathrm{tsc}(p^0)]$. Also choose n sufficiently large for the argument to come. We construct a decreasing sequence of conditions $\langle p^i \colon i < \operatorname{cof}\alpha\rangle$ so that p^{i+1} reduces all dense subsets of P_{α_i} which are coded in M^0 below λ and $\langle p^i \colon i < \operatorname{cof}\alpha\rangle$ has a lower bound q which will be as desired. We do this by constructing an increasing, continuous sequence of models $\langle M^i \colon i \leq \operatorname{cof}\alpha\rangle$, each of size λ so that each M^{i+1} is the Σ_n-Skolem Hull in $L_{|s|}[\mathrm{tsc}(p^0)]$ of $(M^i \cap \kappa) \cup \{c(p^i), M^i \cap \kappa\}$ and choosing each p^{i+1} to have the least code in the canonical well-ordering of $L[\mathrm{tsc}(p^0)]$ so that it reduces all dense subsets of P_{α_i} which are coded in M^i below λ and to agree with p^i at and above α_i. It remains to show that for each limit ordinal $\gamma \leq \operatorname{cof}\alpha$, we obtain a lower bound p^γ of the conditions chosen so far. Choose p^γ to have the least-possible code in the canonical well-ordering of $L[\mathrm{tsc}(p^0)]$ in that case.

We will only treat the hardest case when $\gamma = \operatorname{cof}\alpha$. Let r be the componentwise union of the p^i. We want to show by induction on $\xi \leq \alpha$ that $\langle p^i\restriction\xi \colon i < \gamma\rangle$ has a lower bound. Let $\xi \leq \alpha$ and assume that q^ζ denotes the inductively obtained lower bound of $\langle p^i\restriction\zeta \colon i < \gamma\rangle$, $(q^\zeta)^\oplus$ denotes the inductively obtained lower bound of $\langle p^i\restriction\zeta^\oplus \colon i < \gamma\rangle$ for $\zeta < \xi$. We will only treat the hardest case when $\xi = \alpha$. Let $N := M^\gamma$. Similar to Claim 54 in Case 2, we obtain that r matches N in the sense of the following claim, completely ignoring issues at cardinals $< \kappa$ which are handled as in Case 2:

Claim 68. (1) *If D is a dense subset of E_ζ for some $\zeta \in [\kappa,\alpha)$ that is coded in N, then there is $i < \gamma$ so that p^i reduces D below λ.*
(2) $|r|_\lambda = \sup(\text{S-supp}(r) \cap \kappa) = N \cap \kappa$.
(3) $\text{C-supp}(r) \cap [\kappa,\alpha) = N \cap [\kappa,\alpha)$.

(4) *If* $\rho \in$ C-supp(r) *has cardinality* κ*,* $(q^\rho)^\oplus$ *forces* $r^*_\rho = N \cap [\kappa, |r_\rho|)$ *and* $\sup r^{**}_\rho = \sup(\text{S-supp}(r) \cap \kappa)$*.*

Proof. Proof of 1: Immediate from our construction.

Proof of 2: Since $\lambda \in M^0$, $|p^i|_\lambda \in M^i$ and hence $|r|_\lambda \leq N \cap \kappa$. For the other direction, note that $\{t \in E_\kappa \colon |t|_\lambda > M^i \cap \kappa\}$ is dense in E_κ and coded in M^{i+1}, hence $|p^{i+2}|_\lambda \geq M^i \cap \kappa$. The proof for $\sup(\text{S-supp}(r) \cap \kappa)$ is similar.

Proof of 3: If for some $i < \gamma$, $\rho \in$ C-supp$(p^i) \cap [\kappa, \alpha)$, then as $\operatorname{tcl}(\{c(p^i)^\kappa\}) \subseteq M^{i+1}$, $\rho = f(j)$ for some $j < \kappa$ in M^{i+1} and therefore $\rho \in M^{i+1}$ by Σ_n-elementarity of M^{i+1}. For the other direction, assume $\rho \in M^i \cap [\kappa, \alpha_i)$ for some $i < \gamma$. $\{t \in E_{\alpha_i} \colon \rho \in \text{C-supp}(t)\}$ is dense in E_{α_i} and coded in M^i, hence $\rho \in$ C-supp(p^{i+1}).

Proof of 4: Similar to the above and the corresponding items in Claim 54. □

Assume $\nu \in [\kappa, \alpha)$ and $(q^\nu)^\oplus$ forces $\rho \in r^*_\nu$. Let π denote the collapsing map of N. Then $(q^\nu)^\oplus$ forces that $\pi(\rho) = \operatorname{ot}(f_\rho[\sup r^{**}_\nu])$ as in Case 2. Let $\bar\kappa = \pi(\kappa)$ and let $\bar N = \pi'' N$. We want to build q out of r similar as in Case 2 - in particular, we will do the same at cardinals less than κ, which we will ignore during the rest of this argument. At κ, we do the following:

- Construct a $P_{\bar\kappa}$-name s_κ for a κ-Cohen condition and set $q_{\bar\kappa} = s_\kappa$ as follows:
  - Let $|s_\kappa|^- = \sup \pi_\kappa''(N \cap |r|_\kappa)$, let $|s_\kappa| = |s_\kappa|^- + 2$.
  - Choose $F^q_{\bar\kappa}$ to be $\langle f_\gamma \colon \gamma \in [\lambda, |s_\kappa|)\rangle$ with f_γ chosen freely as a bijection from λ to γ for $\gamma \geq \bar\kappa$ and equal to the f_γ picked by r otherwise.
  - If $(\rho_0, \rho_1) \in [\lambda, \bar\kappa)^\urcorner$, then $s_\theta(\rho_0, \rho_1) = r@(\rho_0, \rho_1)$.
  - For all $(\rho_0, \rho_1) \in ([\kappa, \kappa^+)^\urcorner \cap N^2)$, $s_\kappa(\pi_\kappa(\rho_0), \pi_\kappa(\rho_1))$ is such that it is forced by q^ξ to be equal to $\pi_\kappa(r@(\rho_0, \rho_1))$ whenever ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\kappa$.
  - $q^{\bar\kappa}$ forces that $s_\kappa(|s_\kappa|^-)$ codes $f_{|s_\kappa|^-}$.
- For all $\nu \geq \kappa$ in C-supp(r), $q^{**}_\nu = r^{**}_\nu \cup \{\sup r^{**}_\nu\}$.
- $q_\nu = r_\nu$, $F^q_\nu = F^r_\nu$ and $q^*_\nu = r^*_\nu$ for all $\nu \geq \kappa$.

This will be possible once we know that

(a) Whenever $(\rho_0, \rho_1) \in ([\kappa, \kappa^+)^\urcorner \cap N^2)$ and ξ is so that both ρ_0 and ρ_1 are either forced by $(q^\xi)^\oplus$ to belong to r^*_ξ or are smaller than $\bar\kappa$, then q^ξ forces that $r@(\rho_0, \rho_1)$ has a $P_{\bar\kappa}$-name.

(b) If ρ is not a multiple of κ and q^ξ forces $\rho \in r^*_\xi$, then q^ξ forces $r@(\rho, \rho) = 0$. This implies that if $\bar\rho \geq |r|_\lambda$ is not a multiple of $\omega \cdot \lambda$, $q^{\bar\kappa}$ forces $s_\kappa(\bar\rho, \bar\rho) = 0$.

(c) $q^{\bar\kappa}$ forces that s_κ is an acceptable, correct κ-Cohen condition.

Proof of (a): Choose $i < \gamma$ and ξ s.t. $p^i \restriction \xi^\oplus$ forces $(\rho_0, \rho_1) \in [\kappa, \kappa^+)^\urcorner \cap (\bar\kappa \cup (p^i)^*_\xi)$. The set of conditions deciding $p^i@(\rho_0, \rho_1)$ is dense in P_ξ and coded in M^i. Therefore p^{i+1} reduces this dense set below λ and hence forces that $p^i@(\rho_0, \rho_1)$ has a $P_{\sup(\text{S-supp}(p^{i+1})\cap\kappa)}$-name and (a) follows as $\sup(\text{S-supp}(p^{i+1}) \cap \kappa) < \bar{\kappa}$.

Proof of (b): As in Case 2.

Proof of (c): We proceed similar to Case 2, additionally using the following claim, which ensures Acceptability at $|s_\kappa|^-$:

Claim 69. *$\bar{\kappa}$ is collapsed to λ definably over $L_{|s_\kappa|^-}[s_\kappa]$.*

Proof. Note that our whole construction has been done Σ_n-definably over $L_{|s|}[\text{tsc}(p^0)]$. Now as N is Σ_n-elementary in $L_{|s|}[\text{tsc}(p^0)]$, it is easy to observe that the same construction can be done Σ_n-definably over N and hence the collapsed version of that construction can be done Σ_n-definably over $\bar{N}$. $\bar{N}$ is of the form $L_{|\bar{s}|}[\text{tsc}(\bar{p}^0)]$ for some $\bar{p}^0$. Similar to Claim 55, if $\bar{G} \ni q^{\bar{\kappa}}$ is generic for $P_{\bar{\kappa}}$, then $\langle c(p^i) \colon i < \gamma\rangle$ together with $\bar{G}$ generates generics G^{**}_ξ for $E_\xi(p^0)$ over N whenever $\xi < \alpha$. Let $G^*_\xi = \pi'' G^{**}_\xi$. Then $(\bar{p}^0_{\pi(\xi)})^{G^*_\xi} = s_\kappa \restriction |p^0_{\pi(\xi)}|$ and $\bigcup_{\xi<\alpha}(\bar{p}^0_{\pi(\xi)})^{G^*_\xi} = s_\kappa \restriction |s_\kappa|^-$. Moreover $\text{tsc}(\bar{p}^0)$ is definable over $L_{|s_\kappa|^-}[s_\kappa]$ in a way that $L_{|s_\kappa|^-}[\text{tsc}(\bar{p}^0)]$ is a definable class of $L_{|s_\kappa|^-}[s_\kappa]$, similar to Claim 65. But this makes the collapsed version of our construction definable over $L_{|s_\kappa|^-}[s_\kappa]$. From this construction we can read off the sequence $\langle \sup(M^i \cap \bar{\kappa}) \colon i < \gamma\rangle$ which is cofinal in $\bar{\kappa}$ and hence we obtain a surjection from λ to $\bar{\kappa}$ definably over $L_{|s_\kappa|^-}[s_\kappa]$. □

Finally, if α has definable cofinality $\geq \kappa$ over $L_{|s|}[\text{tsc}(p^0)]$, Claim 67 is immediate by induction, as in this case M is bounded in α. □

Definition 70. *Let $p \in E_\alpha$ and M be of size λ, transitive below κ. We say $q \leq p$ is suitably generic for P_α at κ for codes in M if $q\restriction\xi^\oplus$ reduces every dense subset of $E^\oplus_\xi(\text{ts}(p))$ that is coded in M below λ for every $\xi < \alpha$.*

The following is now immediate by handling cardinals $\theta < \kappa$ as in Case 2 and handling κ as in Claim 67:

Corollary 71. *If $p \in E_\alpha$ and for every $\theta \in [\eta^+, \kappa)$ with* C-supp$(p)\cap[\theta, \theta^+) \neq \emptyset$*, M_θ is of cardinality $< \theta$ and M_κ is definable over $L_{|s|}[\text{tsc}(p^0)]$ and of cardinality λ, then there is $q \leq p$ which is suitably generic for P_α at θ over M_θ for every $\theta \in [\eta^+, \kappa)$ with* C-supp$(p) \cap [\theta, \theta^+) \neq \emptyset$ *and suitably generic for P_α at κ for codes in M_κ.*

Next we establish what may be called "definable reduction" for E_α:

Claim 72. *If $p \in E_\alpha$ and D is a dense subset of $E_\alpha(p)$ which has a code definable over $L_{|s|}[\text{tsc}(p)]$, then there is $q \leq p$ which reduces D below λ.*

Proof. We will again completely ignore issues at cardinals less than κ, which are handled as usual. Build a decreasing sequence of conditions in P_α below p and an increasing sequence of models as follows: Fix $n < \omega$ sufficiently large. Let $p^0 = p$. Choose q^0 to have the least code (i.e. $c(q^0)$ least in the canonical well-ordering of $L[\text{tsc}(p^0)]$) so that $q^0 \leq p^0$ and $q^0 \in D$. Choose M^0 to be the Σ_n-Skolem Hull of $(\lambda + 1) \cup \{c(q^0)\}$ in $L_{|s|}[\text{tsc}(p^0)]$. At stage $j + 1$, let $p^{j+1} \leq p^0$ be the condition with least code which is incompatible to all q^k, $k \leq j$, such that $u_\lambda(p^{j+1}) = u_\lambda(q^j)$ if such exists and choose q^{j+1} to have the least code such that:

- $q^{j+1} \leq p^{j+1}$,
- $q^{j+1} \in D$ and
- $u_\lambda(q^{j+1}) \leq u_\lambda(p^{j+1})$ is suitably generic for P_α at κ for codes in M^j.

Choose M^{j+1} to be the Σ_n-Skolem Hull of $(M^j \cap \kappa) \cup \{c(q^{j+1}), M^j \cap \kappa\}$ in $L_{|s|}[\mathrm{tsc}(p^0)]$. At limit stages $j < \kappa$, let $p^j \leq p^0$ be the condition with least code which is incompatible to all q^k, $k < j$ so that for all $k < j$, $u_\lambda(p^j) \leq u_\lambda(q^k)$ if such exists. Choose q^j w.r.t. p^j as q^{j+1} was chosen w.r.t. p^{j+1} above. Choose M^j to be the Σ_n-Skolem Hull of $((\bigcup_{k<j} M^k \cap \kappa) + 1) \cup \{c(q^j)\}$.

Proceed until at some stage j no condition p^j as above can be chosen. By 9, this will be the case for some $\gamma < \kappa$. We finish the proof by showing that for every limit ordinal $j \leq \gamma$, we may obtain a lower bound for $\langle u_\lambda(q^k) \colon k < j\rangle$. For $j = \gamma$, this lower bound will then give rise to our desired q as in Claim 51. Let $N = \bigcup_{k<j} M^k$. Let π be the collapsing map of N, let $\bar\kappa = \pi(\kappa)$. As $j \leq \bar\kappa$, it follows that $j < \bar\kappa$ by Σ_n-elementarity of N.[14] Let r be the componentwise union of $\langle p^i \colon i < j\rangle$. We want to form q out of r similar to the proof of Claim 67. In particular, we define s_κ as in that claim. Similar to the proof of Claim 69, we obtain that the collapsed version of this construction (over the collapse of N) is Σ_n-definable over $L_{|s_\kappa|^-}[s_\kappa]$, giving rise to a collapse of $\bar\kappa$ to j definably over $L_{|s_\kappa|^-}[s_\kappa]$ and hence to λ as $L_{\bar\kappa}[s_\kappa] \models \lambda$ is the largest cardinal. □

Claim 73. *If $p \in E_\alpha$ and $D = \langle D_i \colon i < \lambda\rangle$ is a sequence of codes of dense sets which is definable over $L_{|s|}[\mathrm{tsc}(p^0)]$, then there is $q \leq p$ which reduces each D_i below λ.*

Proof. Let $n < \omega$ be so that D is Σ_n-definable over $L_{|s|}[\mathrm{tsc}(p^0)]$. Build a decreasing sequence of conditions in E_α below p and an increasing sequence of Σ_n-elementary submodels of $L_{|s|}[\mathrm{tsc}(p^0)]$ both of length λ, reducing a single D_i in each step (using Claim 72) and using Claim 67 in each step to reduce dense subsets of E_β for all $\beta < \alpha$ which are coded in the relevant model. Note that this can be done so that the decreasing sequence of conditions will be Σ_n-definable over $L_{|s|}[\mathrm{tsc}(p^0)]$. By the same arguments as for Claim 67, we obtain a lower bound of our sequence of conditions. This lower bound will be as desired. □

We're almost ready to give a proof of what we called the central preliminary result above. Before that, we need one more technical claim the validity of which was basically ensured by the definition of our forcing in 4:

Claim 74.
Every element of $L_{|s|+1}[s]$ has an $E_\alpha(\mathrm{ts}(p^0))$-name in $L_{|s|+1}[\mathrm{tsc}(p^0)]$.

Proof. It is easy to see that s has an $E_\alpha(\mathrm{ts}(p^0))$-name in $L_{|s|+1}[\mathrm{tsc}(p^0)]$. By the definition of conditions in 4, for every $\beta < \alpha$, $L_{|g_\beta|}[s]$ has an $E_\beta(\mathrm{ts}(p^0))$-name in $L_{|s|}[\mathrm{tsc}(p^0)]$ and this name is uniformely definable from β. It follows that we can obtain as the union of those names definably over $L_{|s|}[\mathrm{tsc}(p^0)]$ an $E_\alpha(\mathrm{ts}(p^0))$-name for $L_{|s|}[s]$. We will finish by showing that every definable subset x of $L_{|s|}[s]$ has an $E_\alpha(\mathrm{ts}(p^0))$-name in $L_{|s|+1}[\mathrm{tsc}(p^0)]$: Let $x = \{y \in$

[14] j is Σ_n-definable over N. $\bar\kappa$ cannot be Σ_n-definable over N.

$L_{|s|}[s]\colon L_{|s|}[s] \models \varphi(y,p)\}$ for some formula φ which may also refer to the predicate s and some $p \in L_{|s|}[s]$. As $L_{|s|}[s]$ has a name in $L_{|s|+1}[\mathrm{tsc}(p^0)]$, we know that p has a name $\dot{p}$ in $L_{|s|+1}[\mathrm{tsc}(p^0)]$ by transitivity of that structure. Let $\dot{x} = \{(t,y)\colon t \Vdash_{E_\alpha[\mathrm{ts}(p^0)]} L_\gamma[\dot{s}] \models \varphi(y,\dot{p})\}$. $\dot{x}$ is a name for x and definable over $L_{|s|}[\mathrm{tsc}(p^0)]$. □

Claim 75. *There is no new subset of λ in $L_{|s|+1}[s]$.*

Proof. Assume to the contrary that there is a new subset x of λ in $L_{|s|+1}[s]$. By Claim 74, x has an $E_\alpha(\mathrm{ts}(p^0))$-name $\dot{x}$ in $L_{|s|+1}[\mathrm{tsc}(p^0)]$. But then by Claim 73, it is dense to force that x has a P_ξ-name of size λ for some $\xi < \kappa$, implying that $x \in L_{\kappa\cdot 2}[s]$, contradicting our assumption. □

Claim 76. *$u_\eta(P_\alpha)$ is η^+-strategically closed.*

Proof. To show that decreasing sequences of suitably generic conditions give rise to lower bounds at limit stages, perform an argument very similar to that of Claim 52, using Claim 75 and elementarity to verify the instance of Claim 57 about new subsets of λ in $L_{|s_\kappa|^-+1}[g_\lambda^\frown s_\kappa]$. The existence of suitably generic conditions is seen as in Case 5, Claim 62 and Claim 63. □

Finally: We want to show that if α is a limit ordinal, 12 follows from 10. That we can extend **-components below the top cardinal follows inductively. At the top cardinal κ, let $\nu := \mathrm{card}(I \cap [\kappa,\kappa^+)) < \kappa$, let $I = \{x_i\colon i < \nu\}$. Assume $\eta \in [\nu,\kappa)$. Let $D_i := \{t \in u_\eta(P_\alpha)\colon t{\restriction}x_i^\oplus \Vdash \sup t^{**}_{x_i} \geq \bar{\delta}^{x_i}\}$ for $i < \nu$. Each D_i is dense in $u_\eta(P_\alpha)$ by 12 inductively. Use 10 to build a decreasing sequence of conditions $\langle p^i\colon i < \nu\rangle$ in P_α with $p^0 = p$ and lower bound q so that each $u_\eta(p^{i+1})$ hits D_i, i.e. forces that $\sup p^{i+1} \geq \bar{\delta}^{x_i}$, and $\langle l_\eta(p^i)\colon i < \nu\rangle$ is constant.

We are now finished with proving 10, 11 and 12.

Proof of 13 - Early Names: Apply 11 to reduce the dense sets D_i of conditions which decide $\dot{f}(i)$, $i < \eta$.

Proof of 14 (Coding H_η), 15 (Preservation of the GCH*) and 16 (Covering, Preservation of Cofinalities):* 14 is an easy density argument using 13. 15 and 16 follow from Δ-distributivity (see [8], Lemma 2.10 and Lemma 2.13). □

Corollary 77. *P preserves* ZFC*, cofinalities and the* GCH*.*

Proof. By the same proof as for Clause 11 of Theorem 49, P is easily seen to be Δ-distributive. By Lemma 2.23 of [8], this implies that P is tame and hence preserves ZFC and cofinalities. GCH preservation is immediate from Clause 14 of Theorem 49. □

To verify that A witnesses Local Club Condensation in $\mathbf{V}[G]$, we will use the following two observations from [10], the former reformulated in the context of models of the form $\mathbf{L}[A]$:

Lemma 78. [10] *The statement that A witnesses Local Club Condensation is equivalent to the following, seemingly weaker statement: If α has uncountable cardinality κ, then the structure $\mathcal{A} = (L_\alpha[A], \in, A, F)$ has a continuous chain $\langle \mathcal{B}_\gamma\colon \gamma \in C\rangle$ of condensing substructures with domains B_γ,*

$\bigcup_{\gamma\in C} B_\gamma = L_\alpha[A]$, $C \subseteq \kappa$ *is club,* C *consists only of cardinals if* κ *is a limit cardinal, each* B_γ *has cardinality* $\operatorname{card}\gamma$ *and contains* γ *as subset, where* F *denotes the function* $(f, x) \mapsto f(x)$ *whenever* $f \in L_\alpha[A]$ *is a function and* $x \in \operatorname{dom}(f) \cap L_\alpha[A]$.

Fact 79. [10] *Assume* $f_\beta\colon \operatorname{card}\beta \to \beta$ *is a bijection from the cardinality of* β*, a regular uncountable cardinal, to* β*. There is a club of* $\delta < \operatorname{card}\beta$ *such that* $f_\alpha[\delta] = f_\beta[\delta] \cap \alpha$ *for all* $\alpha \in f_\beta[\delta] \setminus \operatorname{card}\beta$.

Claim 80. *Let* G *be* P*-generic. Let* A *be the generic predicate obtained from* G*, i.e.* $\alpha \in A \leftrightarrow \exists p \in G\ p \Vdash p@\alpha = 1$*. Then* A *witnesses Local Club Condensation and Acceptability in* $V[G]$.

Proof. That $L[A] = V[G]$ follows from Clause 14 of Theorem 49. That A witnesses Acceptability can be seen as in Lemma 31. It remains to show that A witnesses Local Club Condensation.

Assume first that α has regular uncountable cardinality κ. Note that for all $\beta_0, \beta_1 \in \alpha$ we have $A(\pi^\delta(\beta_0), \pi^\delta(\beta_1)) = \pi^\delta(A(\beta_0, \beta_1))$ for all sufficiently large δ in the club $\bigcup_{p\in G} p^{**}_\gamma \subseteq \kappa$ where γ is such that for some $p \in G$, $p \Vdash \{\beta_0, \beta_1\} \setminus \kappa \subseteq p^*_\gamma$. Moreover $f_\beta[\delta] = f_\alpha[\delta] \cap \beta$ for all $\beta \in f_\alpha[\delta] \setminus \kappa$ for a club of $\delta < \kappa$ by Fact 79. Let C denote the intersection of those clubs. Let $M^*_\alpha = (L_\alpha[A], \in, A, F, \dots)$ be a Skolemized structure for a countable language and for any $X \subseteq \alpha$ let $M^*_\alpha(X)$ be the least substructure of M^*_α containing X as a subset. Consider the continuous chain $\langle M^*_\alpha(f_\alpha[\delta])\colon \delta \in D\rangle$, where D consists of all elements δ of C s.t. $\delta = f_\alpha[\delta] \cap \kappa$ and $f_\alpha[\delta] = M^*_\alpha(f_\alpha[\delta]) \cap \mathbf{Ord}$. Then $M^*_\alpha(f_\alpha[\delta])$ condenses for each $\delta \in D$.

The remaining case is to verify Local Club Condensation for α when α has singular cardinality κ. Let F denote the function $(f, x) \mapsto f(x)$ whenever $f \in L_\alpha[A]$ is a function with $x \in \operatorname{dom}(f)$. Suppose that $\dot S \in \mathbf{V}$ is a P_β-name for a structure $(L_\alpha[A], \in, A, F, \dots)$ for a countable language in $\mathbf{L}[A]$ such that the $\dot S$-closure of κ is all of $L_\alpha[A]$. We may assume that β is a limit ordinal. We show that any condition p has an extension q which forces that there is a continuous chain $\langle Y_\gamma\colon \gamma \in C\rangle$ of condensing substructures of $\dot S$ whose domains $\langle y_\gamma\colon \gamma \in C\rangle$ have union $L_\alpha[A]$ such that $\langle y_\gamma \cap \mathbf{Ord}\colon \gamma \in C\rangle$ belongs to the ground model, where C is a closed unbounded subset of $\mathbf{Card} \cap \kappa$, each y_γ has cardinality γ and contains γ as subset. Choose C to be any club subset of $\mathbf{Card} \cap \kappa$ of ordertype $\operatorname{cof}\kappa$ whose minimum is either ω or a singular cardinal and is at least $\operatorname{cof}\kappa$ so that C is bounded below every inaccessible cardinal. Write C in increasing order as $\langle \gamma_i\colon i < \operatorname{cof}\kappa\rangle$. Choose some large (w.r.t. β), regular ν.

Let $p^0 = p$. Let $\langle M^0_\theta\colon \exists i < \operatorname{cof}\kappa\ \theta = \gamma_i\rangle$ be a sequence of elementary submodels of H_ν such that each M^0_θ has size less than θ, is transitive below θ and contains p^0 and $\dot S$ as elements. Moreover make sure that whenever $\theta_0 < \theta_1$, $M^0_{\theta_0} \subseteq M^0_{\theta_1}$.

Given p^i, choose $p^{i+1} \le p^i$ such that p^{i+1} reduces every dense subset of P_β in M^i_θ below $\operatorname{card} M^i_\theta$ and such that $\sup(\text{S-supp}(p^{i+1}) \cap \theta) \ge \operatorname{card}(M^i_\theta)$ and $\ge M^i_\theta \cap \theta$ for all θ. Choose $\langle M^{i+1}_\theta\colon\ \text{C-supp}(p^{i+1}) \cap [\theta, \theta^+) \neq \emptyset\rangle$ such that each $M^{i+1}_\theta \supseteq M^i_\theta$ has size less than θ, is transitive below θ, contains p^{i+1}

and M^i_θ as elements and is elementary in H_ν. Also make sure that whenever $\theta_0 < \theta_1$ are as above, $M^{i+1}_{\theta_0} \subseteq M^{i+1}_{\theta_1}$ and that whenever γ_j is a limit point of C, $M^{i+1}_{\gamma_j{}^+} = \bigcup_{k<j} M^{i+1}_{\gamma_k{}^+}$. The latter is possible as $\gamma_0 \geq \operatorname{cof}\kappa$, we may thus sufficiently enlarge the $M^{i+1}_{\gamma_k{}^+}$ after choosing $M^{i+1}_{\gamma_j{}^+} \supseteq \bigcup_{k<j} M^{i+1}_{\gamma_k{}^+}$ in the first place.

Finally, let r be the componentwise union of $\langle p^i : i < \omega\rangle$. We will construct a lower bound q for $\langle p^i : i < \omega\rangle$ which will be as desired. Let $y_\gamma := \bigcup_{i<\omega} M^i_{\gamma^+}$ for every $\gamma \in C$. We have obtained the following properties for every $\gamma \in C$:

(1) y_γ is transitive below γ^+,
(2) $y_\gamma \cap [\gamma, \gamma^+) = \text{S-supp}(r) \cap [\gamma, \gamma^+) = |r|_\gamma$,
(3) $y_\gamma \cap [\gamma^+, \gamma^{++}) = \text{I-supp}(r) \cap [\gamma^+, \gamma^{++})$,
(4) any lower bound of $\langle p^i : i < \operatorname{cof}\kappa\rangle$ forces that the $\dot{S}$-closure of y_γ intersected with **Ord** equals y_γ,
(5) any lower bound of $\langle p^i : i < \operatorname{cof}\kappa\rangle$ forces that $A \cap {y_\gamma}^2$ has a $P_{y_\gamma \cap \gamma^+}$-name and
(6) $\langle y_\gamma : \gamma \in C\rangle$ is continuous and increasing.

(1) is immediate as each of the $M^i_{\gamma^+}$ is transitive below γ^+, (2) and (3) follow similar to Claim 54. (4) now follows as $p^{i+2} \in M^{i+2}_{\gamma^+}$: using elementarity, p^{i+2} forces that we can cover the $\dot{S}$-closure of $M^i_{\gamma^+}$ by a set in $M^{i+2}_{\gamma^+}$ of size γ; as $\gamma \subseteq M^{i+2}_{\gamma^+}$, this covering set will be contained (as a subset) in $M^{i+2}_{\gamma^+}$. (5) follows similar to (4), using easy density arguments. (6) is immediate by our requirements on the M^i_θ.

Let π_γ be the collapsing map of y_γ, let $\bar{\gamma}^+ = \pi_\gamma(\gamma^+)$. We obtain q by choosing $q_{\bar{\gamma}^+}$ of length $\sup(\pi_\gamma'' y_\gamma) + 2$ for every $\gamma \in C$ (at cardinals not in C but in C-supp(r), we do the usual construction necessary to obtain a lower bound): If $\xi \in y_\gamma$, f_ξ is a bijection from $\operatorname{card}\xi$ to ξ, hence $f_\xi \restriction (y_\gamma \cap \operatorname{card}\xi)$ is a bijection from $y_\gamma \cap \operatorname{card}\xi$ to $y_\gamma \cap \xi$ by elementarity, i.e. $\pi_\gamma(\xi) = \operatorname{ot}(f_\xi[y_\gamma \cap \operatorname{card}\xi])$. If $(\xi_0, \xi_1) \in [\gamma, \mathbf{Ord})^\urcorner \cap {y_\gamma}^2$, let $q_{\bar{\gamma}^+}(\pi_\gamma(\xi_0), \pi_\gamma(\xi_1)) = \pi_\gamma(r@(\xi_0, \xi_1))$. Let $q_{\bar{\gamma}^+}(\sup(\pi_\gamma'' y_\gamma))$ code a bijection from γ to $\sup(\pi_\gamma'' y_\gamma)$ and let $q_{\bar{\gamma}^+}(\sup(\pi_\gamma'' y_\gamma) + 1) = 0$. $q_{\bar{\gamma}^+}$ is obviously correct. That $q_{\bar{\gamma}^+}$ is acceptable follows by elementarity of y_γ similar to Claims 57 and 59. Our construction made sure that q forces y_γ to condense for every $\gamma \in C$. □

Theorem 81. *Local Club Condensation and Acceptability are simultaneously consistent with the existence of an ω-superstrong cardinal.*

Proof. Assume κ is ω-superstrong, witnessed by the embedding $j\colon \mathbf{V} \to \mathbf{M}$. Let P be the Local Club Condensation and Acceptability forcing as defined at the beginning of this section. We want to show that forcing with P may preserve the ω-superstrength of κ. Let P^* denote the $\mathbf{M}$-version of P (using the definition of P in $\mathbf{M}$). Note that for every $n < \omega$, $P_{j^n(\kappa)} = P^*_{j^n(\kappa)}$. We want to find a $\mathbf{V}$-generic $G \subseteq P$ and an $\mathbf{M}$-generic $G^* \subseteq P^*$ such that $j''G \subseteq G^*$ and $V[G]_{j^\omega(\kappa)} \subseteq M[G^*]$. After finding a suitable $P_{j^\omega(\kappa)}$-generic $G_{j^\omega(\kappa)}$, we will let $G^*_{j^\omega(\kappa)}$ be $G_{j^\omega(\kappa)} \cap P^*_{j^\omega(\kappa)}$. We will let G^* be the filter generated by $G^*_{j^\omega(\kappa)}$ together with the image of G under j. We have to show the following:

(1) $G^*_{j^\omega(\kappa)}$ is $P^*_{j^\omega(\kappa)}$-generic over $\mathbf{M}$.
(2) G^* is P^*-generic over $\mathbf{M}$.
(3) We can choose $G_{j^\omega(\kappa)}$ in such a way that $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$.

We will assume 3 for the moment and proof 1 and 2 using 3. We will then proof 3 without using either 1 or 2. Assume that j is given by an ultrapower embedding, which means that every element of $\mathbf{M}$ is of the form $j(f)(a)$ where f has domain $H_{j^\omega(\kappa)}$ and a belongs to $H_{j^\omega(\kappa)}$.

Proof of 1: Suppose $D \in \mathbf{M}$ is dense on $P^*_{j^\omega(\kappa)}$ and write D as $j(f)(a)$ where $\mathrm{dom}(f) = V_{j^\omega(\kappa)}$ and $a \in V_{j^{n+1}(\kappa)}$ for some $n \in \omega$. Choose $p \in G_{j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^n(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is dense on $P_{j^\omega(\kappa)}$. The existence of p follows from Clause 11 of Theorem 49, using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$. Then $j(p)$ belongs to $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$ by 3 and reduces D below $j^{n+1}(\kappa)$.

Hence $E := \{q \in P_{j^{n+2}(\kappa)} \colon q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D\}$ is dense below $j(p)\restriction j^{n+2}(\kappa)$ in $P_{j^{n+2}(\kappa)}$. Since $G_{j^{n+2}(\kappa)}$ contains $j(p)\restriction j^{n+2}(\kappa)$ and is $P_{j^{n+2}(\kappa)}$-generic over $\mathbf{M}$, $G_{j^{n+2}(\kappa)} \cap E \neq \emptyset$. Choose q in that intersection. Then $q^\frown j(p)[j^{n+2}(\kappa), j^\omega(\kappa)) \in D \cap G^*_{j^\omega(\kappa)}$.

Proof of 2: Like 1, using that $j''G \subseteq G^*$ as an immediate consequence of 3.

Proof of 3: We will specify a master condition $q \in P_{j^\omega(\kappa)}$ so that $q \in G_{j^\omega(\kappa)}$ ensures $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$. Let $\dot{G}$ be the canonical name in $\mathbf{V}$ for the $P_{j^\omega(\kappa)}$-generic. We define r by letting, for all $\gamma \geq j(\kappa)$:

$$r_\gamma = \bigcup_{p \in \dot{G}} j(p)_\gamma,\ F^r_\gamma = \bigcup_{p \in \dot{G}} F^{j(p)}_\gamma,\ r^*_\gamma = \bigcup_{p \in \dot{G}} j(p)^*_\gamma,\ r^{**}_\gamma = \bigcup_{p \in \dot{G}} j(p)^{**}_\gamma.$$

As we did earlier, we write S-supp(r) for $\{\gamma \colon r(\gamma)(0) \neq \check{\mathbf{1}}\}$ and C-supp(r) for $\{\gamma \colon r(\gamma)(1) \neq \check{\mathbf{1}}\}$. It is easily observed as in the proof of Theorem 25 of [10] that S-supp(r) is bounded below every regular cardinal and that $\mathrm{card}(\text{C-supp}(r) \cap \theta^+) < \theta$ for every regular cardinal θ. We want to form q out of r by setting, for every $\gamma \in$ C-supp(r) of cardinality θ and every $(\delta_0, \delta_1) \in [\mathrm{card}\sup r^{**}_\gamma, \mathbf{Ord})^\urcorner \cap (\sup r^{**}_\gamma \cup r^*_\gamma)^2$:

- $q^{**}_\gamma = r^{**}_\gamma \cup \{\sup r^{**}_\gamma\}$,
- If $\gamma \geq j(\kappa)^+$, let

$$q_{\sup r^{**}_\theta}(\pi^{\sup r^{**}_\theta}(\delta_0), \pi^{\sup r^{**}_\theta}(\delta_1)) = \pi^{\sup r^{**}_\theta}(r@(\delta_0, \delta_1)),$$

- Let $t_\theta := \sup_{\zeta \in \text{I-supp}(r) \cap [\theta, \theta^+)} \pi^{\sup r^{**}_\theta}(\zeta)$ and let $q_{\sup r^{**}_\theta}(t_\theta)$ code a bijection from $\mathrm{card}\sup r^{**}_\theta$ to t_θ, let $q_{\sup r^{**}_\theta}(t_\theta + 1) = 0$ and $|q_{\sup r^{**}_\theta}| = t_\theta + 2$.
- Choose arbitrary collapsing information $F^q_{\sup r^{**}_\theta}$.

We also set $q_\gamma = r_\gamma$ and $F^q_\gamma = F^r_\gamma$ for γ in S-supp(r), $q^*_\gamma = r^*_\gamma$ for all γ and let components other than the above have value $\check{\mathbf{1}}$. The following Claim will finish the proof of Theorem 81:

Claim 82. (1) $q \in P_{j^\omega(\kappa)}$.
(2) *q extends $j(p)$ whenever $p\restriction\kappa = \mathbf{1}$.*

(3) *Whenever* $p \le q$, $p \in G$, *then* $p \le j(p)$; *hence if* $p \in G_{j^\omega(\kappa)}$, *then* $j(p) \in G_{j^\omega(\kappa)}$, *i.e.* $j''G_{j^\omega(\kappa)} \subseteq G^*_{j^\omega(\kappa)}$.

Proof. Proof of 1: We want to define, for every cardinal $\theta \ge j(\kappa)^+$ with $\text{C-supp}(r) \cap [\theta, \theta^+) \ne \emptyset$ a model M_θ: Choose some large (w.r.t. $j^\omega(\kappa)$), regular (in $\mathbf{M}$) $\nu \in \text{range}(j)$, choose a wellorder of $H_\nu{}^{\mathbf{M}}$ in $\text{range}(j)$ and let M_θ be the Skolem Hull of $\sup(\text{S-supp}(r) \cap \theta) \cup (\text{C-supp}(r) \cap [\theta, \theta^+))$ in $H_\nu{}^{\mathbf{M}}$ w.r.t. that wellorder.

Claim 83. *For all* θ *with* $\text{C-supp}(r) \cap [\theta, \theta^+) \ne \emptyset$,

- $M_\theta \cap \theta = \sup(\text{S-supp}(r) \cap \theta) = \sup r^{**}_\theta$.
- $M_\theta \cap [\theta, \theta^+) = \text{C-supp}(r) \cap [\theta, \theta^+) = \text{I-supp}(r) \cap [\theta, \theta^+)$.

Proof. For the first statement, assume $\xi \in M_\theta$, $\xi < \theta$. Then ξ can be defined using finite sets of parameters $S_0 \subseteq \sup(\text{S-supp}(r) \cap \theta)$ and $S_1 \subseteq \text{C-supp}(r) \cap [\theta, \theta^+)$. Choose $p \in G$ so that $S_0 \subseteq \text{S-supp}(j(p) \cap \theta)$ and $S_1 \subseteq \text{C-supp}(j(p)) \cap [\theta, \theta^+)$. Let $t \le p$ in G be such that whenever $\text{C-supp}(p) \cap [\rho, \rho^+) \ne \emptyset$, $\sup(\text{S-supp}(t) \cap \rho) \ge \sup(H^{H_{j^{-1}(\nu)}}(\sup(\text{S-supp}(p) \cap \rho) \cup (\text{C-supp}(p) \cap [\rho, \rho^+))) \cap \rho)$.[15] It follows that $\xi < \sup(\text{S-supp}(j(t)) \cap \theta < \sup(\text{S-supp}(r) \cap \theta)$, which is equal to $\sup r^{**}_\theta$ by the usual arguments. The proof of the second statement is similar. □

Let π_θ denote the collapsing map of M_θ and note that $\pi_\theta \restriction \theta^+ = \pi^{\sup r^{**}_\theta}$. By the usual arguments, it follows that our above definition of q has no conflicting requirements and q has appropriate supports in order to be a condition in $P_{j^\omega(\kappa)}$. It is immediate that $q \restriction j(\kappa)^+ \in P_{j(\kappa)^+}$. It remains to show that each $q_{\sup r^{**}_\theta}$ is an acceptable, correct $(\sup r^{**}_\theta)^+$-Cohen condition.

Claim 84. *For all* $\theta \ge j(\kappa)^+$ *with* $\text{C-supp}(r) \cap [\theta, \theta^+) \ne \emptyset$, *the following hold:*

(1) $q \restriction (M_\theta \cap \theta) \in P_{M_\theta \cap \theta}$ *and extends* $j(p) \restriction (M_\theta \cap \theta)$ *for every* $p \in G$ *with* $p \restriction \kappa = \mathbf{1}$.
(2) *If* $q \restriction (M_\theta \cap \theta) \in G_{M_\theta \cap \theta}$ *and* G *is* P*-generic over* $\mathbf{V}$, *then* $G_{M_\theta \cap \theta}$ *together with* $j''G$ *generates a* $P_{M_{\theta^+} \cap \theta^+}$*-generic filter* H *over* M_{θ^+}.
(3) *Let* $\bar{H} = \pi_\theta'' H$.

$$\left(\pi_\theta'' g[\text{card}(M_\theta \cap \theta), M_{\theta^+} \cap \theta^+)\right)^{\bar{H}} = (q_{M_\theta \cap \theta} \restriction t_\theta)^{G_{M_\theta \cap \theta}}.$$

(4) $q \restriction (M_\theta \cap \theta) \Vdash q_{M_\theta \cap \theta}$ *is an acceptable, correct* $(M_\theta \cap \theta)^+$*-Cohen condition.*
(5) $q \restriction (M_{\theta^+} \cap \theta^+) \in P_{M_{\theta^+} \cap \theta^+}$ *and extends* $j(p) \restriction (M_{\theta^+} \cap \theta^+)$ *for every* $p \in G$ *with* $p \restriction \kappa = \mathbf{1}$.

Proof. By induction on θ.

Proof of 1: If $\theta > j(\kappa)^+$, 1 is immediate from 5 inductively. If $\theta = j(\kappa)^+$, observe that

- $\sup r^{**}_{j(\kappa)} = \kappa$,
- $\forall \gamma \in [\kappa, \kappa^+)$ $\text{ot}\, j(f_\gamma)[\kappa] = \gamma$ and

[15] The Skolem Hull in $H_{j^{-1}(\nu)}$ is taken with respect to the j-preimage of the wellorder of $H_\nu{}^{\mathbf{M}}$ chosen above.

- $\text{C-supp}(r) \cap [j(\kappa), j(\kappa)^+) = \text{I-supp}(r) \cap [j(\kappa), j(\kappa)^+) = j''[\kappa, \kappa^+)$.

For every $\xi \in \text{C-supp}(p) \cap [\kappa, \kappa^+)$, $\delta \in q^{**}_{j(\xi)} \setminus j(p)^{**}_{j(\xi)}$ and $(\gamma_0, \gamma_1) \in [j(\kappa), j(\kappa)^+)^\urcorner \cap (\sup(j(p)^{**}_{j(\xi)}) \cup j(p)^{*}_{j(\xi)})^2$, we have to verify that $q@(\pi^\delta(\gamma_0), \pi^\delta(\gamma_1)) = \pi^\delta(j(p)@(\gamma_0, \gamma_1))$. If $\delta < \kappa$, we use that by a density argument, there exists $t \leq j(p)$ in $j''G$ with $\delta \in t^{**}_{j(\xi)}$. If $\delta = \kappa$, 1 follows from our above observations.

Proof of 2: By 1, $G_{M_\theta \cap \theta}$ and $j''G$ are compatible and thus generate a filter. If D is a dense subset of $P_{M_{\theta^+} \cap \theta^+}$ in M_{θ^+}, we can define it in $H_\nu{}^{\mathbf{M}}$ using finite sets of parameters $S_0 \subseteq \sup(\text{S-supp}(r) \cap \theta^+)$ and $S_1 \subseteq \text{C-supp}(r) \cap [\theta^+, \theta^{++})$. Choose $p \in G$ so that $S_0 \subseteq \sup(\text{S-supp}(j(p)) \cap \theta^+)$ and $S_1 \subseteq \text{C-supp}(j(p))$. By a density argument, there is $q \leq p$ in G s.t. for all $\theta > \kappa$ with $\text{C-supp}(p) \cap [\theta, \theta^+) \neq \emptyset$, q reduces all dense subsets of P which are definable in $H_{j^{-1}(\nu)}$ using parameters in $\sup(\text{S-supp}(p) \cap \theta) \cup (\text{C-supp}(p) \cap [\theta, \theta^+))$ strictly below θ in the sense that q reduces them below λ for some $\lambda < \theta$. But then $j(q)$ reduces D strictly below θ and we may hit D by further extending $j(q)$ only below $M_\theta \cap \theta$, yielding 2.

Proof of 3: Let $\dot{g}$ be the canonical name (in $\mathbf{V}$) for the generic predicate obtained from $\dot{G}$. If $(\xi_0, \xi_1) \in [\text{card}\, M_\theta \cap \theta, t_\theta)^\urcorner$, then $(\pi_\theta{}''\dot{g})^{\bar{H}}(\xi_0, \xi_1) = \xi_2$ iff $\exists p \in G_{M_\theta \cap \theta}$ and $s \in j''G$ so that $p \wedge s$ forces $s@(\pi_\theta^{-1}(\xi_0), \pi_\theta^{-1}(\xi_1)) = \pi_\theta^{-1}(\xi_2)$ iff $(q_{M_\theta \cap \theta})^{G_{M_\theta \cap \theta}}@(\xi_0, \xi_1) = \xi_2$.

Proof of 4: Follows from 2 and 3 using elementarity, similar to Claims 57 and 59.

Proof of 5: The first statement is immediate from 4. The second statement is immediate from the definition of q. □

Proof of 2 (of Claim 82): Immediate from Claim 84.

Proof of 3: Assume $p \leq q$. Then $p \leq j(p)$ as $p{\restriction}\kappa = j(p){\restriction}\kappa$ and $p[\kappa, j^\omega(\kappa)) \leq q \leq j(p)[\kappa, j^\omega(\kappa))$. □

□

4. Preserving smaller large cardinals

Many smaller large cardinals may be preserved while forcing Local Club Condensation and Acceptability. We will show how to preserve subcompact cardinals, as this will become relevant for our application in Section 6:

Definition 85. *κ is subcompact iff for every $A \subseteq \kappa^+$ there is a cardinal $\lambda < \kappa$, some $\bar{A} \subseteq \lambda^+$ and an elementary embedding $\pi\colon (H_{\lambda^+}, \bar{A}) \to (H_{\kappa^+}, A)$ with $\pi{\restriction}\lambda = \text{id}$.*

Theorem 86. *We may force Local Club Condensation and Acceptability preserving all instances of subcompactness.*

Proof. We will use the following equivalent characterization of subcompactness of κ in the ground model: Given any κ^+-closed transitive T of size κ^+ and $A \in T$ there is a cardinal $\lambda < \kappa$, a λ^+-closed $\bar{T}$ of size λ^+, $\bar{A} \in \bar{T}$ and an elementary embedding π from $(\bar{T}, \bar{A})$ to (T, A) such that π has critical point λ and sends λ to κ.

Let P denote the Local Club Condensation and Acceptability forcing as described in Section 3. Let $\dot{A}$ be a name for a subset of κ^+ in some P-generic extension. As every subset of κ^+ is added by P_ξ for some $\xi < \kappa^{++}$ we may assume that $\dot{A}$ is a P_ξ-name for some $\xi < \kappa^{++}$. Note that below any fully string supported $p \in P_\xi$, P_ξ has a dense subset of size κ^+ and hence an isomorphic copy $E \in H_{\kappa^{++}}$. Let $\dot{B}$ be some E-name in $H_{\kappa^{++}}$ translating $\dot{A}$. We can choose a κ^+-closed transitive T of size κ^+ containing $\dot{B}$ and apply the above form of subcompactness to $(T, \dot{B})$ to obtain a cardinal $\lambda < \kappa$, a λ^+-closed $\bar{T}$ of size λ^+, some $\bar{\dot{B}} \in \bar{T}$ and an elementary embedding π from $(\bar{T}, \bar{\dot{B}})$ to $(T, \dot{B})$.

To show that κ is subcompact after forcing with P, we want to find $q \in E$ forcing that π lifts to $\pi^G \colon (\bar{T}[G], \bar{\dot{B}}^G) \to (T[G], \dot{B}^G)$. This then gives an elementary π_0^G from $(H_{\lambda^+}[G], \bar{\dot{B}}^G)$ to $(H_{\kappa^+}[G], \dot{B}^G)$, establishing subcompactness for $\dot{A}^G$. But to lift π, we just have to make sure that given any P_κ-generic G_κ, we choose $G_{\kappa^{++}}$ extending G_κ so that $\pi'' G_{\lambda^{++}} \subseteq G_{\kappa^{++}}$. But this is easy to ensure choosing q to be a master condition similar to the proof of Theorem 81. $\square$

5. Strong Condensation for ω_2

Apart from its intrinsic interest, the material of this section will also be useful for the application we will give in the next section. In this section, we do not want to restrict the context to models of the form $\mathbf{L}[A]$, but work in the more general context of models with a hierarchy of levels:

Definition 87. *We say $(\mathbf{M}, \langle M_\alpha \colon \alpha \in \mathbf{Ord}\rangle) \models \mathrm{ZFC}$ is a model with a hierarchy of levels if $\langle M_\alpha \colon \alpha \in \mathbf{Ord}\rangle$ is so that $\mathbf{M} = \bigcup_{\alpha \in \mathbf{Ord}} M_\alpha$, each M_α is transitive, $\mathbf{Ord}(M_\alpha) = \alpha$, if $\alpha < \beta$ then $M_\alpha \in M_\beta$ and if γ is a limit ordinal, then $M_\gamma = \bigcup_{\alpha<\gamma} M_\alpha$. We will often use M_α to also denote the structure $(M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle)$, where context will clarify the intended meaning. If $\mathcal{B}$ has domain B and is a substructure of some structure on M_α, we say that $\mathcal{B}$* condenses *or that $\mathcal{B}$* has Condensation *iff $(B, \in, \langle M_\beta \colon \beta \in B\rangle)$ is isomorphic to some $(M_{\bar{\alpha}}, \in, \langle M_\beta \colon \beta < \bar{\alpha}\rangle)$. We also say that B condenses or that B has Condensation in this case.*

Local Club Condensation for a model $\mathbf{M}$ with a hierarchy of levels is the statement that if α has uncountable cardinality κ and $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \leq \gamma < \kappa\rangle$ of condensing substructures of $\mathcal{A}_\alpha$ whose domains have union M_α, where each $\mathcal{B}_\gamma = (B_\gamma, \in, \langle M_\beta \colon \beta \in B_\gamma\rangle, \ldots)$ is such that B_γ has cardinality $\operatorname{card} \gamma$ and contains γ as a subset.

Theorem 88. [16] *If $(\mathbf{M}, \langle M_\alpha \colon \alpha \in \mathbf{Ord}\rangle)$ is a model of Local Club Condensation, τ is an $\mathbf{M}$-cardinal of uncountable cofinality, $\kappa = (\tau^+)^{\mathbf{M}}$, $F =$*

[16]We would like to thank Liu-Zhen Wu for providing significant help on improving an older, slightly weaker version of this theorem.

$\langle f_\alpha \colon \alpha \in [\tau,\kappa)\rangle$ where each f_α is a bijection from τ to α in $\mathbf{M}$, S is a set of Skolem functions for M_κ,

$$X \prec (M_\kappa, \in, \langle M_\alpha \colon \alpha < \kappa\rangle, F, S)$$

and X is transitive below τ, then X condenses. In fact, X need not be an element of $\mathbf{M}$ for the above to hold.

Proof. Let X be as above, let $\bar\tau$ be the image of τ under the transitive collapse of X. For $\alpha \in X$,

$$X \cap M_\alpha \prec (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, f_\alpha, S).$$

Now $X = \bigcup_{\alpha\in X} X \cap M_\alpha$. We want to show that for $\alpha < \sup X$ in X, $X \cap M_\alpha$ condenses and therefore X condenses. Let, for all $\beta \le \tau < \alpha < \kappa$, $H^*_\alpha(\beta)$ be the Skolem Hull (using Skolem functions in S) of β in $(M_\alpha, \in, \langle M_\gamma \colon \gamma < \alpha\rangle, f_\alpha)$. $H^*_\alpha(\tau) = M_\alpha$ for every $\alpha < \kappa$. Using Local Club Condensation, for each $\alpha < \kappa$, there is a club subset of τ in M_κ such that $H^*_\alpha(\theta)$ condenses for every θ in that club. But by elementarity of X in M_κ, there is such a club C_α in X which is club in $\bar\tau$, i.e. $\bar\tau \in C_\alpha$ and thus $H^*_\alpha(\bar\tau) = M_\alpha \cap X$ condenses. □

Strong Condensation for a model $\mathbf{M}$ with a hierarchy of levels is the statement that for every ordinal α, there is a structure $\mathcal{A}_\alpha = (M_\alpha, \in, \langle M_\beta \colon \beta < \alpha\rangle, \ldots)$ for a countable language such that each of its substructures condenses.[17]

Strong Condensation up to β ($\beta \in \mathbf{Card}$, $\mathbf{M}$ as above) is the statement of Strong Condensation restricted to ordinals $\alpha \le \beta$ together with the assumption that $M_\beta = H_\beta$.

Strong Condensation for α ($\alpha \in \mathbf{Card}$, $\mathbf{M}$ as above) is the statement of Strong Condensation for a single cardinal α together with the assumption that $M_\alpha = H_\alpha$.

Local Club Condensation up to β for a cardinal β is the statement of Local Club Condensation restricted to ordinals $\alpha \le \beta$ together with the assumption that $M_\beta = H_\beta$.

Note: For every cardinal α, Strong Condensation implies Strong Condensation up to α implies Strong Condensation for α. For $\alpha = \omega_1$, Strong Condensation up to α is a theorem of ZFC. It is known, using collapsing functions, that Strong Condensation for ω_3 implies that there is no precipitous ideal on ω_1 (see [15]), which makes Strong Condensation for ω_3 already a most interesting property. It is not known whether it is possible to force Strong Condensation for ω_3 with a small forcing. It was shown by Liu-Zhen Wu that there is a small forcing to obtain Strong Condensation for ω_2 in the generic extension and it was observed independently by the first author that Strong Condensation for ω_2 also holds in the model for Local Club Condensation from [10]. But more is true - Local Club Condensation up to ω_2 implies Strong Condensation for ω_2 (using the same hierarchy of levels):

Corollary 89 (Liu-Zhen Wu)**.** *Local Club Condensation up to ω_2 implies Strong Condensation for ω_2.*

[17] Strong Condensation was originally defined by Hugh Woodin in [18].

Proof. Follows directly from theorem 88, letting $\tau = \omega_1$. □

Note: It is easy to see that Strong Condensation for ω_2 implies Strong Condensation up to ω_2, which of course implies Local Club Condensation up to ω_2, hence Local Club Condensation up to ω_2 and Strong Condensation for ω_2 are equivalent. In light of Theorem 88, we think of Local Club Condensation as a global version of Strong Condensation for ω_2.

6. An application to the consistency strength of PFA

Let $\mathbf{c}$ denote $2^{\aleph_0}$. Σ^2_1-indescribable gaps $[\kappa, \kappa^+)$ were introduced in Definition 4. In [13] and [14], Neeman and Schimmerling show the following:

Theorem 90 (Neeman, Schimmerling)**.** [13] *Suppose* $[\kappa, \kappa^+)$ *is* Σ^2_1*-indescribable in a model satisfying* GCH. *Then* PFA($\mathbf{c}^+$*-linked*) *holds in a proper forcing extension in which* $\mathbf{c} = \omega_2 = \kappa$.

Theorem 91 (Neeman)**.** [14] *Suppose* $\mathbf{V}$ *is a proper forcing extension of a fine structural model* $\mathbf{M}$ *and* PFA($\mathbf{c}^+$*-linked*) *holds in* $\mathbf{V}$. *Then* $[\kappa, \kappa^+)$ *is* Σ^2_1*-indescribable in* $\mathbf{M}$ *where* $\kappa = (\omega_2)^{\mathbf{V}}$.

By the methods developed in our paper, we are able to provide a sufficiently fine structural (or rather "**L**-like") model for the arguments of the proof of the latter theorem to go through. But this **L**-like model may, in contrast to current fine structural inner models, also contain large cardinals in the range of a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$. Also we do not need $\mathbf{V}$ to be a proper forcing extension - it suffices if $\mathbf{V}$ is a proper extension of $\mathbf{M}$ (which is also true for Neeman's theorem above):

Theorem 92. *If* $\mathbf{V}$ *is a proper extension of a model* $\mathbf{M}$ *satisfying Local Club Condensation, Acceptability,* $\square$ *on the singular cardinals and* $\square_\lambda$ *for every singular* λ *and* PFA($\mathbf{c}^+$*-linked*) *holds in* $\mathbf{V}$, *then there is a* Σ^2_1*-indescribable gap* $[\kappa, \kappa^+)$ *in* $\mathbf{M}$.

Proof. The properties of $\mathbf{M}$ may be compared with those used by Neeman in his [14], listed at the beginning of its §2: Neeman assumes what we called strong Acceptability (but it is easy to see that he only uses what we call Acceptability in the present paper) and uses a Condensation property different from ours. Also he works in an extender model of the form $\mathcal{J}[\vec{E}]$ while our model is of the form $\mathbf{L}[A]$ for a bfp A. We sketch the argument that Neeman's proof, which may be found in §2 of [14], can be adapted to our present situation. We assume that the reader is familiar with [14] for the rest of this section. We define (this may be compared with the definition in [14] following the proof of its Lemma 2.1):

Definition 93. *β is a point on level α if*

1. $L_\beta[A] \models \alpha$ *is the largest cardinal,*
2. $\beta \in \mathbf{Card}^{L_{\beta+1}[A]}$ *and*
3. $\beta \notin \mathbf{Card}^{\mathbf{L}[A]}$.

The following definitions and claims are carried out as in [14], noting that we are always in a situation similar to condition 1 of Lemma 2.1 of [14] and it is easy to observe that that the proofs of all those claims (which aren't numbered in [14], but generalize Claims 1.2 through 1.9 of §1 of [14]) can easily be adapted to our current setting. The same is true for Lemma 2.3, Claim 2.9, Definition 2.10 and Claim 2.11 of [14]. The same is also basically true for Theorem 2.12 of [14], the main theorem yielding our desired Theorem 92. There's just one point in the argument where additional care is needed: When Neeman applies PFA($\mathbf{c}^+$-linked) in [14], he defines amongst other objects a function $\bar{f}$ from ω_1 to $L_\kappa[A]$, where $\kappa = \tau^+$ is a successor cardinal with τ inaccessible, and lets $\bar{N}$ be the transitive collapse of range($\bar{f}$). We have to show that Local Club Condensation is strong enough to yield that $\bar{N}$ is a level of $\mathbf{L}[A]$, as this is what Neeman uses further on in [14] and what we want to use in our adaption of his proof. As $\bar{N}$ is transitive below τ by the properties of $\bar{f}$ (see [14]) and elementarity w.r.t. the structure defined in the statement of Theorem 88 is easy to arrange, this follows from Theorem 88. □

Corollary 94. *Assuming the consistency of a proper class of subcompact cardinals, it is consistent that there is a proper class of subcompact cardinals, but* PFA *(even restricted to posets which are* $\mathbf{c}^+$*-linked) holds in no proper extension of the universe.*

Proof. We start with a model containing a proper class of subcompact cardinals. Force to obtain the GCH using Theorem 2 of [9], preserving all subcompacts (this may be done using the technique of Theorem 86). Now force to obtain $\square_\lambda$ for every singular λ, preserving all subcompacts (using the technique of Theorem 86) and the GCH, using a reverse Easton iteration of the standard forcing to obtain square (conditions are initial segments of the desired square sequence, ordered by end-extension - see [4], Section 6). Force again to obtain square on the singular cardinals as in [5], preserving all subcompacts (using the technique of Theorem 86), $\square_\lambda$ for every singular λ (as the iteration of [5] is cofinality-preserving) and the GCH. Force once more to obtain a bfp A coding the generic extension, witnessing Local Club Condensation and Acceptability while preserving all subcompacts, using Theorem 86. Note that the forcing used is cofinality-preserving and hence preserves any square principles. If this final model does not contain a Σ^2_1-indescribable gap $[\kappa, \kappa^+)$, let it be our model $\mathbf{M}$, otherwise let $\mathbf{M}$ be the V_κ of this model, where κ is the least Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ of that model. It is easy to verify that below any Σ^2_1-indescribable gap $[\kappa, \kappa^+)$, there is a stationary set of subcompacts, hence V_κ has a proper class of subcompacts in the latter case. Applying Theorem 92, it follows that PFA (even restricted to posets which are $\mathbf{c}^+$-linked) cannot hold in any proper extension of $\mathbf{M}$. □

7. Generalized Condensation Principles and Diamond

Stationary Condensation implies $\diamondsuit$; this was noted without proof already in [10]. The proof of this fact is most similar to the proof that $\diamondsuit$ holds in $\mathbf{L}$, which may be found in [6]. For any regular cardinal κ, Fat Stationary Condensation implies $\diamondsuit_{\kappa^+}(\mathrm{Cof}\ \kappa)$, which again follows by a proof analogous to the respective proof for the same principle in $\mathbf{L}$, which may again be found in [6]. Moreover, the following hold, proofs of which are omitted for similar reasons ([6] might be consulted again):

Lemma 95. *The following are implied by Local Club Condensation:*

- $\diamondsuit_\kappa(E)$ *whenever* κ *is regular and* $E \subseteq \kappa$ *is stationary.*
- $\diamondsuit^*_\kappa$ *for all successor cardinals* κ.
- $\diamondsuit^+_\kappa$ *for all successor cardinals* κ.

8. Separating Local Club Condensation and Stationary Condensation

As we were making a huge effort to obtain Local Club Condensation and Acceptability while preserving large cardinals above and obtain Stationary Condensation and Acceptability quite easily while preserving large cardinals in Theorem 29, we want to end this paper by showing that those principles are actually different. The following separates Stationary Condensation and Local Club Condensation in a strong sense, using Lemma 95:

Theorem 96. *There is a model of* ZFC *in which Stationary Condensation and Acceptability hold but* $\diamondsuit^*_{\kappa^+}$ *fails for every infinite cardinal* κ. *Moreover, assuming the consistency of an* ω*-superstrong cardinal, there is such a model with an* ω*-superstrong cardinal.*

Proof. By [7], there is a σ-closed forcing which destroys $\diamondsuit^*$, preserves cofinalities and the GCH. Observe that this proof can be easily modified (by replacing ω by any infinite cardinal κ throughout the proof) to yield that for every infinite cardinal κ, there is a κ^+-closed forcing[18] which destroys $\diamondsuit^*_{\kappa^+}$, preserves cofinalities and the GCH. Now force Stationary Condensation and Acceptability by first adding an acceptable ω_1-Cohen to give rise to the predicate A below ω_1, then destroy $\diamondsuit^*$ by σ-closed forcing. Now add an acceptable ω_2-Cohen to give rise to the predicate A in the interval $[\omega_1, \omega_2)$, then destroy $\diamondsuit^*_{\omega_2}$ by ω_2-closed forcing. Continue this iteration with reverse Easton support and note that it preserves cofinalities, forces Stationary Condensation and Acceptability (this is seen as in Theorem 29 above, the important point is that since the forcings to destroy $\diamondsuit^*_{\kappa^+}$ are sufficiently closed, A will code the final generic extension) and preserves many large cardinals, in particular preserves ω-superstrong cardinals (this is again seen as in Theorem 29). The iteration starting from κ^+ is κ^{++}-closed, and thus $\diamondsuit^*_{\kappa^+}$ will fail in the final generic extension for every infinite cardinal κ. □

9. Open Questions

Neeman and Schimmerling [13] not only obtain the consistency of PFA for $\mathbf{c}^+$-linked forcings from a Σ^2_1 indescribable "1-gap" $[\kappa, \kappa^+)$, but they obtain a similar result for higher gaps: The consistency of a Σ^2_1 indescribable "n-gap" $[\kappa, \kappa^{+n})$ is sufficient for the consistency of PFA for $\mathbf{c}^{+n}$-linked forcings. Are there analogues of Theorems 6 and 1 for higher gaps? For cardinals $\kappa < \alpha$, the definition of "κ is α-subcompact" can be found in [3].

Question 97. *Is it consistent that there exists a proper class of* κ *which are* κ^{++}*-subcompact but PFA for* $\mathbf{c}^{++}$*-linked forcings fails in all proper extensions?*

[18] the forcing to add κ^{++}-many Cohen subsets of κ^+

This is probably related to the question whether forcing a “global version of Strong Condensation for ω_3” (see the note at the end of Section 5) is possible while preserving very large cardinals, which at its core has the following, seemingly very hard question:

Question 98. *Over an arbitrary ground model is it possible to obtain Strong Condensation for ω_3 by set forcing?*

We showed that Local Club Condensation up to ω_2 and Strong Condensation for ω_2 are equivalent in Section 5. It is easily observed that there is a bfp A on $\omega_{\omega+1}$ witnessing Local Club Condensation up to $\omega_{\omega+1}$, but not Strong Condensation for $\omega_{\omega+1}$. Large cardinals allow us to separate Strong Condensation and Local Club Condensation in a stronger sense, as in any model of Local Club Condensation that has an ω_1-Erdős cardinal, no predicate can witness Strong Condensation (see [10]). An obvious question which should have a negative answer is the following:

Question 99. *Does Local Club Condensation up to ω_3 imply Strong Condensation for ω_3 - in the sense that in any model of the former there is a hierarchy witnessing the latter?*

It would also be very interesting to find quasi-lower bounds on the consistency strength of set theoretic principles other than (fragments of) PFA, for example one could ask the following:

Question 100. *Is a proper class of subcompacts a quasi lower bound on the consistency strength of the failure of $\square_\lambda$, λ singular?*

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