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Project TOSCA

TASK HOAS_LF

Towards a general framework for metareasoning on HOAS encodings

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How to represent binding operators?

Scenario: we have to represent formally (*encode*) an object language (e.g., π-calculus) in some logical framework for doing formal (meta)reasoning

Problem: how to render binding operators (e.g, ν) efficiently?

- **First-order abstract syntax**

```
nu : Name -> Proc -> Proc
```

  Needs lots of machinery about α-equivalence, substitution, …

- **de Bruijn indexes**

```
nu : Proc -> Proc
```

  Good at α-equivalence, but not immediate to understand and needs even more machinery for capture-avoiding substitution than FOAS

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Higher-order abstract syntax [Harper, Honsell, Plotkin 87]

```
nu : (Name -> Proc) -> Proc
```

♡ it delegates successfully many aspects of names management to the metalanguage (α-conversion, capture-avoiding substitution, generation of fresh names,...) $\Rightarrow$ widely used in most logical frameworks

♠ if `Name` is defined as inductive then *exotic terms* (= not corresponding to any real process of the object language) will arise!

```
weird = nu [x:Nat](Cases x of 0 => P
                             | _ => P|Q   end).
```

♠ usually structural induction over higher-order terms (*contexts*, terms with *holes*) is not provided $\Rightarrow$ metatheoretic analysis is difficult/impossible

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A methodology for HOAS metareasoning

We propose a general methodology for dealing with metatheoretic properties of contexts in HOAS-based encodings.

Let Υ be a framework metalanguage corresponding to a theory of Simple Types/Classical Higher-Order Logic *à la Church*. Types:

$\tau ::=$	o	propositions
$\mid$	υ	names
$\mid$	ι	object language terms (e.g., processes)
$\mid$	$\tau \to \tau$	

Two judgements: type assignments and validity derivations

$$\Gamma \vdash_\Sigma M : \tau \qquad \Gamma \vdash_\Sigma P$$

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The logical framework Υ: Syntax

Two basic logical connectives:

$$\Rightarrow : o \to o \to o \qquad \forall_\tau : (\tau \to o) \to o$$

Other logical connectives and Leibniz equality are defined as abbreviations, as usual

$$\begin{array}{rcl rcl}
\forall x^\tau . p & \stackrel{\text{def}}{=} & \forall_\tau(\lambda x^\tau . p) & p \wedge q & \stackrel{\text{def}}{=} & \neg(p \Rightarrow \neg q) \\
\bot & \stackrel{\text{def}}{=} & \forall r^o . r & p \vee q & \stackrel{\text{def}}{=} & \neg p \Rightarrow q \\
\neg p & \stackrel{\text{def}}{=} & p \Rightarrow \bot & p \Leftrightarrow q & \stackrel{\text{def}}{=} & (p \Rightarrow q) \wedge (q \Rightarrow p) \\
\exists x^\tau . p & \stackrel{\text{def}}{=} & \neg \forall x^\tau . \neg p & M =^\tau N & \stackrel{\text{def}}{=} & \forall R^{\tau \to o} .\, RM \Rightarrow RN
\end{array}$$

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The logical framework Υ: Typing and Logical rules

Typing rules:

$$\frac{-}{\Gamma, x : \tau \vdash_\Sigma x : \tau} \qquad \text{(VAR)}$$

$$\frac{-}{\Gamma \vdash_\Sigma M : \tau}(M : \tau) \in \Sigma \qquad \text{(CONST)}$$

$$\frac{\Gamma \vdash_\Sigma M : \tau' \to \tau \quad \Gamma \vdash_\Sigma N : \tau'}{\Gamma \vdash_\Sigma MN : \tau} \qquad \text{(APP)}$$

$$\frac{\Gamma, x : \tau' \vdash_\Sigma M : \tau}{\Gamma \vdash_\Sigma \lambda x^{\tau'} . M : \tau' \to \tau} \qquad \text{(ABS)}$$

Logical rules (next slide)

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$$\frac{\Gamma \vdash_\Sigma p : o \quad \Gamma \vdash_\Sigma q : o \quad \Gamma \vdash_\Sigma r : o}{\Gamma \vdash_\Sigma (p \Rightarrow q \Rightarrow r) \Rightarrow (p \Rightarrow q) \Rightarrow p \Rightarrow r} \quad \text{(S)}$$

$$\frac{\Gamma \vdash_\Sigma p : o \quad \Gamma \vdash_\Sigma q : o}{\Gamma \vdash_\Sigma p \Rightarrow q \Rightarrow p} \quad \text{(K)}$$

$$\frac{\Gamma \vdash_\Sigma P : \tau \to o \quad \Gamma \vdash_\Sigma M : \tau}{\Gamma \vdash_\Sigma \forall_\tau(P) \Rightarrow PM} \quad (\forall\text{-E})$$

$$\frac{\Gamma \vdash_\Sigma p : o}{\Gamma \vdash_\Sigma \neg\neg p \Rightarrow p} \quad \text{(DN)}$$

$$\frac{\Gamma, x : \tau \vdash_\Sigma M : \sigma \quad \Gamma \vdash_\Sigma N : \tau}{\Gamma \vdash_\Sigma (\lambda x^\tau.M)N =^\sigma M[N/x]} \quad (\beta)$$

$$\frac{\Gamma \vdash_\Sigma M : \tau \to \sigma}{\Gamma \vdash_\Sigma \lambda x^\tau.Mx =^{\tau\to\sigma} M} x \notin FV(M) \quad (\eta)$$

$$\frac{\Gamma, x : \sigma \vdash_\Sigma M =^\tau N}{\Gamma \vdash_\Sigma \lambda x^\sigma.M =^{\sigma\to\tau} \lambda x^\sigma.N} \quad (\xi)$$

$$\frac{\Gamma \vdash_\Sigma p \Rightarrow q \quad \Gamma \vdash_\Sigma p}{\Gamma \vdash_\Sigma q} \quad \text{(MP)}$$

$$\frac{\Gamma \vdash_\Sigma p : o \quad \Gamma, x : \tau \vdash_\Sigma p \Rightarrow q}{\Gamma \vdash_\Sigma p \Rightarrow \forall x^\tau.q} \quad \text{(Gen)}$$

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An example encoding Σ in the logical framework Υ

Example of object language $\mathcal{L}$:

$$P ::= 0 \mid \tau.P \mid P_1 | P_2 \mid [x \neq y]P \mid (\nu x)P$$

Corresponding signature Σ:

$$\begin{array}{rcl} 0 & : & \iota \\ \tau & : & \iota \to \iota \\ | & : & \iota \to \iota \to \iota \\ [\cdot \neq \cdot]\cdot & : & \upsilon \to \upsilon \to \iota \to \iota \\ \nu & : & (\upsilon \to \iota) \to \iota \end{array} \qquad \begin{array}{rcl} Rec^\iota_\tau & : & \tau \to (\tau \to \tau) \to \\ & & (\tau \to \tau \to \tau) \to \\ & & (\upsilon \to \upsilon \to \tau \to \tau) \to \\ & & ((\upsilon \to \tau) \to \tau) \\ & & \to \iota \to \tau \end{array}$$

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Σ^{++}: A Theory of Contexts on Σ

The theory of contexts Σ^{++} is obtained by adding to Σ the following three axioms:

$$\frac{\Gamma \vdash_\Sigma P : \iota}{\Gamma \vdash_\Sigma \exists x^\upsilon . x \not\in P} \quad (\mathrm{Unsat}^\upsilon_\iota)$$

$$\frac{\Gamma \vdash_\Sigma P : \upsilon^n \to \iota \qquad \Gamma \vdash_\Sigma x : \upsilon}{\Gamma \vdash_\Sigma \exists Q^{\upsilon^{n+1}\to\iota} . x \not\in^{n+1} Q \wedge P =^{\upsilon^n \to \iota} (Q\, x)} \quad (\beta\_exp^{\upsilon^n\to\iota})$$

$$\frac{\Gamma \vdash_\Sigma P : \upsilon^{n+1} \to \iota \qquad \Gamma \vdash_\Sigma Q : \upsilon^{n+1} \to \iota \qquad \Gamma \vdash_\Sigma x : \upsilon}{\Gamma \vdash_\Sigma x \not\in^{n+1} P \Rightarrow x \not\in^{n+1} Q \Rightarrow (P\, x) =^{\upsilon^n\to\iota} (Q\, x) \Rightarrow P =^{\upsilon^{n+1}\to\iota} Q} \quad (\mathrm{Ext}^{\upsilon^{n+1}\to\iota})$$

Questions:

- Expressivity: are these axioms *really* useful? $\Rightarrow$ case studies
- Soundness: are these axioms consistent? $\Rightarrow$ a model for HOAS
- Completeness: in what sense?

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Case studies: π-calculus

Full language, with recursion and mismatch.

- Encoded the full theory (transition system, strong late bisimulation)
- Proved all main results in *A calculus of mobile processes* by Milner, Parrow, Walker (algebraic laws and Lemmata 1–7).

  In particular: For p, q processes, x, y names, $x \not\in p$:

  Lemma 3 if $p \xrightarrow{\alpha} q$ then $p[x/y] \xrightarrow{\alpha[x/y]} q[x/y]$

  Lemma 6 if $p \sim q$ then $p[x/y] \sim q[x/y]$

  Both are instances of the general property (cf. Cardelli)

  If $\Gamma \vdash_\Sigma P$ then for all h injective: $\Gamma[h] \vdash_\Sigma P[h]$

  Both these name replacements are readily encoded by applications of higher-order terms to names.

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Case studies: λ-calculus

Both call-by-name and call-by-value, simply typed:

- full theory: substitution, small step and big step semantics, typing system
- functionality of substitution relation (totality and determinism)
- equivalence of small step and big step semantics
- confluence of big step semantics
- subject reduction

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Substitution of the λ-calculus as a (functional) relation

```
Inductive subst [N:tm] : (var->tm) -> tm -> Prop :=
     subst_var  : (subst N isvar N)
   | subst_void : (y:var)(subst N [_:var]y y)
   | subst_App  : (M1,M2:var->tm)(M1',M2':tm)
           (subst N M1 M1') -> (subst N M2 M2') ->
           (subst N [y:var](App (M1 y) (M2 y)) (App M1' M2'))
   | subst_Lam  : (M:var->var->tm)(M':var->tm)
           ((z:var)(subst N [y:var](M y z) (M' z))) ->
           (subst N [y:var](Lam (M y)) (Lam M')).
```

Axioms used for proving

- Determinism: Ext^{ι}, $\mathrm{Ext}^{\upsilon\to\iota}$, $\mathrm{Unsat}^{\upsilon}_{\iota}$
- Totality: higher-order recursion

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Other case studies (minor/work in progress)

First Order Logic full theory: validity judgement, substitution;
metatheory: functionality of substitution.

spi calculus full theory; metatheory: some algebraic laws

ν**-calculus** theory

$\lambda\sigma$**-calculus** theory; some metatheoretic result

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Towards a categorical model

In order to interpret a HOAS signature in a model based on functor categories, we adopt the following protocol [Hof99]:

- the metalanguage is interpreted in a suitable functor category $\check{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathcal{V}}$ such that
  - if a constructor type contains a negative occurrence of a given type, the latter must have a representable interpretation (e.g. since we have $\nu : (\upsilon \to \iota) \to \iota$, $[\![\upsilon]\!]$ must be representable, i.e., $[\![\upsilon]\!] \cong \check{\mathcal{Y}}(X)$ for some X);
- the structure of functional types will be unraveled by means of the equation $\check{\mathcal{Y}}(X) \Rightarrow A \cong A^X$, where $A^X_Y \stackrel{\text{def}}{=} A_{X \uplus Y}$.

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The model $\mathcal{U}$

The ambient category is $\check{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathcal{V}}$, where $\mathcal{V}$ is defined as follows:

- objects are finite sets of variables;
- morphisms are substitution of variables for variables.

The model $\mathcal{U}$ of Υ is defined by means of the following protocol:

- types and contexts are interpreted as covariant presheaves: $[\![\tau]\!] \in Obj(\check{\mathcal{V}})$ and $[\![\Gamma]\!] \in Obj(\check{\mathcal{V}})$;
- terms are interpreted as natural transformations: $[\![\Gamma \vdash_\Sigma M : \tau]\!] \in \check{\mathcal{V}}([\![\Gamma]\!], [\![\tau]\!])$;

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Interpreting basic datatypes

- $[\![\upsilon]\!] \stackrel{\text{def}}{=} Var : \mathcal{V} \longrightarrow \mathcal{S}et$ defined as follows:

$$Var_X \stackrel{\text{def}}{=} X \qquad Var_h(x) \stackrel{\text{def}}{=} h(x), \quad \text{for } x \in X, h \in \mathcal{V}(X,Y)$$

  Hence, it is isomorphic to the representable functor $\check{\mathcal{Y}}(\{\star\})$.

- $[\![\iota]\!] \stackrel{\text{def}}{=} Proc : \mathcal{V} \longrightarrow \mathcal{S}et$ defined as follows:

$$Proc_X \stackrel{\text{def}}{=} \{P \mid FV(P) \subseteq X\}$$

$$Proc_h(P) \stackrel{\text{def}}{=} P[h], \quad \text{for } P \in Proc_X, h \in \mathcal{V}(X,Y)$$

  $Proc$ is not representable

Prop.: For all n, $Var^n \Rightarrow Proc$ is an initial algebra for a suitable functor.

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Toposes are not enough

Being $\check{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathcal{V}}$ a topos, we could use the canonical interpretation for the propositions type:

$$\begin{aligned} [\![o]\!]_X &\stackrel{\text{def}}{=} Sub(\check{\mathcal{Y}}(X)) = Sub(\mathcal{V}(X, \_)) \\ [\![o]\!]_f(S) &\stackrel{\text{def}}{=} \{g \in Arr(\mathcal{V}) \mid dom(g) = Y \text{ and } g \circ f \in S\} \end{aligned}$$

(where $f : X \longrightarrow Y$ and $S \in [\![o]\!]_X$)

However, this does not work because the axiom of unique choice would be validated:

$$\begin{aligned} AC!_{\sigma,\tau} \quad : \quad & (\forall a^\sigma. \exists ! b^\tau. \, R(a, b)) \Rightarrow \\ & \exists ! f^{\sigma \to \tau}. \, \forall a^\sigma. \, R(a, f(a)) \end{aligned}$$

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Toposes are not enough (cont.)

whence:

- $AC!$ allows to derive the characteristic function of the equality over names $eq : \upsilon \to \upsilon \to nat$ (defined by $\forall x, y : \upsilon.\, x = y \Leftrightarrow eq(x, y) = 1$, where $=$ is Leibniz equality);
- $Q \stackrel{\text{def}}{=} \lambda x^\upsilon.\, if\ eq(x, y)\ then\ p\ else\ q$ (where $y : \upsilon$ e $p, q : \iota$);
- using $Ext^{\upsilon \to \iota}$ one can prove that $Q =^{\upsilon \to \iota} \lambda x^\upsilon.\, q$;
- hence it is possible to show that all processes are syntactically equal (absurd).

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Interpreting o in $\check{\mathcal{V}}$

Given $F \in \check{\mathcal{V}}$, predicates over F (**Pred**(F)) are $\mathcal{V}$-indexed familes of sets $\{P_X\}_{X\in\mathcal{V}}$ such that:

1. $P_X \subseteq F_X$ where $X \in \mathcal{V}$;
2. for all $h \in \mathcal{I}(X, Y)$, if $f \in P_X$ then $F_h(f) \in P_Y$;
3. if $f \in F_X$ and $F_h(f) \in P_Y$ for some $h \in \mathcal{I}(X, Y)$, then $f \in P_X$.

Then we can define $[\![o]\!] \stackrel{\text{def}}{=} Prop$, where $Prop_X \stackrel{\text{def}}{=} \textbf{Pred}(\check{\mathcal{Y}}(X))$

For each X, $Prop_X$ is a Boolean algebra, where order is given by (pointwise) inclusion.

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Interpreting o in $\check{\mathcal{V}}$: the formal justification

This approach can be explained by the existence of the adjunction $(\cdot)^r \dashv (\cdot)^*$, where $(\cdot)^r : \check{\mathcal{V}} \longrightarrow \check{\mathcal{I}}$ and $\check{\mathcal{I}} \stackrel{\text{def}}{=} \mathcal{S}et^{\mathcal{I}}$:

- objects of $\mathcal{I}$ are finite sets of variables;
- morphisms of $\mathcal{I}$ are *injective* substitution of variables for variables.

Indeed we have the following:

$$\textbf{Pred}(F) \stackrel{\text{def}}{=} \textbf{Pred}_{\check{\mathcal{I}}}(F^r) \cong \check{\mathcal{I}}(F^r, \Omega) \cong \check{\mathcal{V}}(F, \Omega^*)$$

Hence, choosing $F = \check{\mathcal{Y}}(X)$, we have

$$\textbf{Pred}(\check{\mathcal{Y}}(X)) \cong \check{\mathcal{V}}(\check{\mathcal{Y}}(X), \Omega^*) \cong \Omega^*_X$$

This suggests to take $[\![o]\!] \stackrel{\text{def}}{=} Prop$, where $Prop_X \stackrel{\text{def}}{=} \textbf{Pred}(\check{\mathcal{Y}}(X))$.

Interpreting the truth judgment

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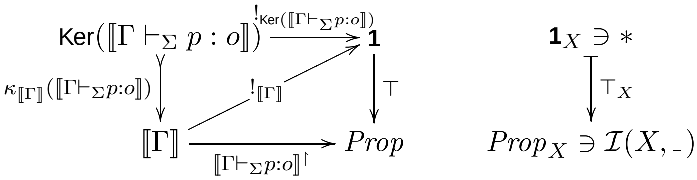
$\Gamma \vdash_\Sigma p$ holds iff for all $X \in \mathcal{V}$ and $\eta \in [\![\Gamma]\!]_X$ we have

$$[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta) \geq \mathcal{I}(X, \_).$$

Intuitive meaning: proposition p holds on (a tuple of) terms η if it is preserved at least by all injective substitutions ($\mathcal{I}(X, \_)$).



$$[\![\Gamma]\!]_X \ni \eta \longmapsto [\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta) \wedge \mathcal{I}(X, \_)$$

Forcing

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Given $X \in \mathcal{V}$, Γ, $\eta \in [\![\Gamma]\!]_X$, and p such that $\Gamma \vdash_\Gamma p : o$ the *forcing judgment*

$$X \Vdash_{\Gamma,\eta} p$$

stands for $\eta \in \kappa_{[\![\Gamma]\!]}([\![\Gamma \vdash_\Sigma p : o]\!])_X$ (i.e., $[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta) \geq \mathcal{I}(X, \_)$).

The forcing judgment is a powerful tool allowing to streamline the computation of the truth value of propositions:

$\Gamma \vdash_\Sigma p$ is valid iff for all $X \in \mathcal{V}$ and $\eta \in [\![\Gamma]\!]_X$ we have $X \Vdash_{\Gamma,\eta} p$.

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Some properties derived by means of forcing

- $X \Vdash_{\Gamma,\eta} \forall x^\tau.p$ iff for all Y, $h \in \mathcal{I}(X,Y)$, $a \in [\![\tau]\!]_Y$ we have $Y \Vdash_{(\Gamma,x:\tau),\ \langle[\![\Gamma]\!]_h(\eta),a\rangle}\ p$;
- $X \Vdash_{\Gamma,\eta} p \Rightarrow q$ iff $X \Vdash_{\Gamma,\eta} p$ implies $X \Vdash_{\Gamma,\eta} q$;
- it is never the case that $X \Vdash_{\Gamma,\eta} \bot$.
- $X \Vdash_{\Gamma,\eta} \neg p$ iff it is never the case that $X \Vdash_{\Gamma,\eta} p$;
- $X \Vdash_{\Gamma,\eta} p \wedge q$ iff $X \Vdash_{\Gamma,\eta} p$ and $X \Vdash_{\Gamma,\eta} q$;
- $X \Vdash_{\Gamma,\eta} p \vee q$ iff $X \Vdash_{\Gamma,\eta} p$ or $X \Vdash_{\Gamma,\eta} q$;
- $X \Vdash_{\Gamma,\eta} \exists x^\tau.p$ iff there exist Y, $h \in \mathcal{I}(X,Y)$, $a \in [\![\tau]\!]_Y$ s.t. $Y \Vdash_{(\Gamma,x:\tau),\ \langle[\![\Gamma]\!]_h(\eta),a\rangle}\ p$.
- For all Γ, M, N, X and $\eta \in [\![\Gamma]\!]_X$:

$$X \Vdash_{\Gamma,\eta} M =^\tau N \quad\Longleftrightarrow\quad [\![\Gamma \vdash_\Sigma M : \tau]\!]_X(\eta) = [\![\Gamma \vdash_\Sigma N : \tau]\!]_X(\eta)$$

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The model validates the theory of contexts

Using forcing, all HOAS axioms have been verified.

Unsat$^\upsilon_\iota$: if $\Gamma \vdash_\Sigma P : \iota$, then for all $X, \eta \in [\![\Gamma]\!]_X$: $X \Vdash_{\Gamma,\eta} \exists x^\upsilon.x \notin P$.

Ext$^{\upsilon\to\iota}$: if $\Gamma \vdash_\Sigma P : \upsilon \to \iota$, $\Gamma \vdash_\Sigma Q : \upsilon \to \iota$ and $\Gamma \vdash_\Sigma x : \upsilon$, then for all X, $\eta \in [\![\Gamma]\!]_X$: $X \Vdash_{\Gamma,\eta} x \notin^1 P \Rightarrow x \notin^1 Q \Rightarrow (P\,x) =^\iota (Q\,x) \Rightarrow P = Q$.

β_exp^ι: if $\Gamma \vdash_\Sigma P : \iota$ and $\Gamma \vdash_\Sigma x : \upsilon$, then for all $X, \eta \in [\![\Gamma]\!]_X$: $X \Vdash_{\Gamma,\eta} \exists Q^{\upsilon\to\iota}.x \notin^1 Q \wedge P =^\iota (Q\,x)$.

Closure under injective substitutions

If $\Gamma \vdash_\Sigma P$ then for all h injective: $\Gamma[h] \vdash_\Sigma P[h]$

corresponds to monotonicity of forcing:

if $X \Vdash_{\Gamma,\eta} P$ then for all $Y, h \in \mathcal{I}(X,Y)$: $Y \Vdash_{\Gamma,\eta[h]} P$.

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AC! is not validated by $\mathcal{U}$

Suppose AC! true in the model $\mathcal{U}$. Since

$$y : \upsilon \vdash \forall x^{\upsilon} \exists ! n^{nat}. x =^{\upsilon} y \iff n =^{nat} 1$$

holds, by AC! we have

$$y : \upsilon \vdash \exists f^{\upsilon \to nat}. x =^{\upsilon} y \iff (f\ x) =^{nat} 1$$

that is: for all Y and $y' \in Y$, there exists Z, $h \in \mathcal{I}(Y, Z)$ and $g \in (\mathit{Var} \Rightarrow nat)_Z$ such that for all X, $h' \in \mathcal{I}(Z, X)$, $x' \in X$:

$$Y \Vdash_{y,f,x;y'[h;h'],g[h'],x'} x =^{\upsilon} y \iff (f\ x) = 1$$

But this condition means that g is not a natural transformation $\Rightarrow$ contradiction.

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Recursion and induction principles over $\upsilon^n \to \iota$ are validated

$$\frac{\Gamma \vdash_\Sigma R : (\upsilon^n \to \iota) \to o}{\begin{array}{l}\Gamma \vdash_\Sigma (R\, \lambda \vec{x}^{\upsilon}.0) \Rightarrow (\forall P^{\upsilon^n \to \iota}.(R\, P) \Rightarrow (R\, \lambda \vec{x}^{\upsilon}.(\tau.(P\, \vec{x})))) \Rightarrow \\ \quad (\forall P^{\upsilon^n \to \iota}.(R\, P) \Rightarrow \forall Q^{\upsilon^n \to \iota}.(R\, Q) \Rightarrow (R\, \lambda \vec{x}^{\upsilon}.(P\, \vec{x}) | (Q\, \vec{x}))) \Rightarrow \\ \quad (\forall y^{\upsilon}.\forall z^{\upsilon}.\forall P^{\upsilon^n \to \iota}.(R\, P) \Rightarrow \\ \quad\ (R\, \lambda \vec{x}^{\upsilon}.[x_1 \neq x_1](P\, \vec{x})) \wedge \cdots \wedge (R\, \lambda \vec{x}^{\upsilon}.[x_i \neq x_j](P\, \vec{x})) \wedge \cdots \\ \quad\ \cdots \wedge (R\, \lambda \vec{x}^{\upsilon}.[x_n \neq x_n](P\, \vec{x})) \wedge \\ \quad\ (R\, \lambda \vec{x}^{\upsilon}.[y \neq x_1](P\, \vec{x})) \wedge \cdots \wedge (R\, \lambda \vec{x}^{\upsilon}.[y \neq x_n](P\, \vec{x})) \wedge \\ \quad\ (R\, \lambda \vec{x}^{\upsilon}.[x_1 \neq z](P\, \vec{x})) \wedge \cdots \wedge (R\, \lambda \vec{x}^{\upsilon}.[x_n \neq z](P\, \vec{x})) \wedge \\ \quad\ (R\, \lambda \vec{x}^{\upsilon}.[y \neq z](P\, \vec{x}))) \Rightarrow \\ \quad (\forall P^{\upsilon^{n+1} \to \iota}.(\forall y^{\upsilon}.(R\, \lambda \vec{x}^{\upsilon}.(P\, \vec{x}\, y))) \Rightarrow (R\, \lambda \vec{x}^{\upsilon}.\nu(P\, \vec{x}))) \Rightarrow \\ \quad \forall P^{\upsilon^n \to \iota}.(R\, P)\end{array}} \ (Ind^{\upsilon^n \to \iota})$$

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Related work

- $FO\lambda^{\Delta N}$ by McDowell-Miller (LICS'97) is a metalogic where induction principles are derived from induction over natural numbers.
- Gabbay and Pitts (LICS'99) introduced a language of contexts based on permutative renaming. "New" quantifier, similar to both $\forall$ and $\exists$

$$\frac{\Gamma, y\#\vec{x} \vdash \phi}{\Gamma \vdash \text{И}y.\phi} \qquad \frac{\Gamma \vdash \text{И}y.\phi \quad \Gamma, \phi, y\#\vec{x} \vdash \psi}{\Gamma \vdash \psi}$$

In the theory of contexts, $\text{И}y.\phi$ is definable as

$$\text{И}y.\phi \quad \equiv \quad \forall y^{\upsilon}.y \not\in^1 (\lambda y^{\upsilon}.\phi) \Rightarrow \phi \quad \equiv \quad \exists y^{\upsilon}.y \not\in^1 (\lambda y^{\upsilon}.\phi) \wedge \phi$$

and the rules above are derivable.

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Conclusions and future work

The proposed theory of context is quite expressive, sound and modular.

The model is the basis for future extensions

Future work:

- Expressivity: more case studies (ambient calculus)
- Extending the model to dependent types (useful for dealing with higher-order proof objects, e.g., natural deduction derivations)
- Extending the model to capture-avoiding substitutions of terms for variables
- Realizability semantics (constructive logic)