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About permutation algebras, (pre)sheaves and named sets*

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Abstract In this paper we survey some well-known approaches proposed as general models for calculi dealing with names (like for example process calculi with name-passing). We focus on *(pre)sheaf categories*, *nominal sets*, *permutation algebras* and *named sets*, studying the relationships among these models, thus allowing techniques and constructions to be transferred from one model to the other.

Keywords Nominal calculi · Permutation algebras · Presheaf categories · Named sets

1. Introduction

Since the introduction of π-calculus, the notion of *name* has been recognized as central in models for concurrency, mobility, staged computation, metaprogramming, memory region allocation, etc. In recent years, several approaches have been proposed as general frameworks for modeling languages featuring name passing and/or allocation. These approaches are based on category theory, non-standard set theory, automata theory, algebraic specifications, etc. It comes as no surprise that there are so many approaches: although all ultimately cope with the same issues, they are inspired by different aims and perspective, leading to different solutions and design choices.

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The main aim of this paper is to survey some of the most widely used models for nominal calculi, and to clarify their relationships. In particular, we focus on four alternative proposals, namely *(pre)sheaf categories*, the various approaches related to Fraenkel-Mostowski set theory, *permutation algebras*, and *named sets*.

Categories of functors over the category $\mathbb{I}$ of finite sets and injective functions, such as $Set^{\mathbb{I}}$, have been widely used for modeling "staged computations," indexed by the (finite) sets of names currently allocated. This approach has been introduced by O'Hearn and Tennent for modeling idealized Algol [19], and then followed by a number of authors, for example [7, 10, 16, 21, 23]. A variation of this approach considers only the subcategory of pullback-preserving functors, i.e. the so-called *Schanuel topos* [5, 10, 23]. These categories allow to extend the standard results about the existence of initial algebras/final coalgebras of polynomial functors also to functors dealing with names.

Actually, *permutation algebras* have been specifically introduced for the development of a theory of *structured coalgebras* in the line of algebraic specifications [3]. Permutation algebras are algebras over a signature whose function symbols form the underlying set of a group of permutations of an enumerable set. Most often, since the group of *all* permutations yields a non-countable signature, one can restrict the attention to countable subgroups, such as that of *finite* permutations. Moreover, in many cases it is necessary to restrict the attention to permutation algebras whose elements are *finitely supported*—for example, processes and terms with infinite free names are ruled out. Therefore, there are four possible theories of permutation algebras to consider.

Permutation algebras are strictly related also to *nominal sets*, an alternative approach stemming from Fraenkel-Mostowski permutation model of set theory with atoms. Several variants of this theory have been presented as *perm*($\mathbb{A}$)-*sets*, FM-sets, *nominal sets*, etc. [8, 20].

A different theory of sets with permutations is that of *named sets* [3]. A *named set* is a set in which each element is equipped with a finite set of names and name bijections. Named sets are supposed to be an implementation of permutation algebras; indeed, they are the basic building block of the operational model of History Dependent automata [17].

Thus, the named sets work is most accessible and directly applicable to computation, while the work on presheaf and sheaf categories has the most developed body of mathematics associated with it but is the least directly relevant to computation. Permutation algebras bear some of the characteristics of each of the other two, and can be seen as the bridge between them. So it is sensible to try to make precise the relationships between these models: such analysis can strengthen the body of mathematical theory underlying structures such as named sets and can give a clearer explanation of the computational significance of the more mathematically developed constructs such as sheaf categories. Techniques and constructions can be transferred among frameworks, thus cross-fertilizing each other, and apparently peculiar idiosyncrasies are either justified, or revealed to be inessential.

In this paper, we describe precisely the connections among these approaches. First, we prove that the four categories of permutation algebras subsume the several variants of FM-sets appeared in literature. Then, it is proved that finitely supported permutation algebras are equivalent to the category of pullback preserving functors $\mathbb{I} \to Set$, i.e., the Schanuel topos. This fact shall allow to transfer the constructions of polynomial functors from the Schanuel topos to permutation algebras.

Also the category of named sets turns out to be equivalent to the category of algebras with finite support, and hence to the Schanuel topos again. Therefore, named sets can be seen as an "implementation" of sheaves of the Schanuel topos, thus giving a sound base for realizing operational models of nominal calculi whose semantics can be given in these sheaf categories, like for example the π-calculus.

Admittedly, some of these results have been known in the community for a while, but often without proofs, or using different variants of the categories, with different names. In fact, the present paper aims to clean up and complete this picture, and to fit as much as possible these approaches in a uniform framework. Along the paper we will state clearly which results have already appeared in the literature.

Synopsis. In Section 2 we recall the basic definitions about permutation algebras, and develop some important properties about permutation algebras with finite support. These are proved to correspond to the sheaf of the Schanuel topos in Section 3. In Section 4 we show that named sets also form a category which is equivalent to the category of finite permutation algebras with finite support. In Section 5 we cast permutation algebras in the general theory of *continuous G-sets*. This will allow to fit all permutation algebras (with arbitrary support) in a uniform framework, and to have a different proof of the equivalence with the Schanuel topos. Conclusions and final remarks are in Section 6.

2. Permutation algebras

This section recalls the pivotal notion of *permutation algebras*. These definitions are mostly drawn from [17], with some additional references to the literature.

2.1. Permutation algebras

Definition 1 (*permutation group*). Let $\mathcal{N}$ be a set (of *names*). A *permutation* on $\mathcal{N}$ is a bijective endofunction on $\mathcal{N}$. The set of all such permutations on a given set $\mathcal{N}$ is denoted by $\mathrm{Aut}(\mathcal{N})$, and it forms the *permutation group* of $\mathcal{N}$, where the operation is function composition: For all $\pi_1, \pi_2 \in \mathrm{Aut}(\mathcal{N})$, $\pi_1\pi_2 \triangleq \pi_1 \circ \pi_2$.

Permutations on sets coincide with *automorphisms*, hence the notation denoting the permutation group. We stick however to the word ‘permutations’ since now this is almost the standard usage in theoretical computer science, and it is the term used in our main references: see [17, Section 2.1] and the initial paragraphs of [8, Section 3].

Following standard notation of algebraic specifications, we recall the definition of *permutation algebra* [17].

Definition 2 (*permutation signature and algebras*). For $\mathcal{N}$ a countable set, the *permutation signature* Σ_p on $\mathcal{N}$ is defined as follows

- the set of formal operators is $\{\widehat{\pi} : 1 \to 1 \mid \pi \in \mathrm{Aut}(\mathcal{N})\}$;
- the set of formal axiomatic equalities is

$$\{\widehat{id}(x) = x\} \cup \{\widehat{\pi}_1(\widehat{\pi}_2(x)) = \widehat{\pi_1\pi_2}(x) \mid \pi_1, \pi_2 \in \mathrm{Aut}(\mathcal{N})\}.$$

A *permutation algebra* $\mathcal{A} = (A, \{\widehat{\pi}_A : A \to A \mid \pi \in \mathrm{Aut}(\mathcal{N})\})$ is an algebra for Σ_p. A *permutation morphism* $\sigma : \mathcal{A} \to \mathcal{B}$ is an algebra morphism, i.e., a function $\sigma : A \to B$ such that $\sigma(\widehat{\pi}_A(x)) = \widehat{\pi}_B(\sigma(x))$. Finally, $Alg(\Sigma_p)$ (often shortened as Alg_p) denotes the category of permutation algebras and their morphisms.

Permutation algebras and their morphisms correspond trivially to Gabbay and Pitts' *perm($\mathbb{A}$)-sets* and *equivariant functions* [8]. An interesting (and recurring) example of permutation algebra is that for the π-calculus: the carrier contains all the processes, up-to structural congruence, and the interpretation of a permutation is the associated name substitution (see also [17, Definition 15 and Section 3]).

An unpleasant fact about Alg_p is that it has a non-countable set of operators and axioms. In order to have a more tractable signature, following [17, Section 2.1] we restrict our attention to *finite* permutations.

Definition 3 (*finite permutations*). Let $\mathcal{N}$ be a countable set, and let $\pi \in \mathrm{Aut}(\mathcal{N})$ be a permutation on $\mathcal{N}$. The *kernel* of π is defined as $\ker(\pi) \triangleq \{x \in \mathcal{N} \mid \pi(x) \neq x\}$.

A permutation π is *finite* if its kernel is finite. The set of all finite permutations is denoted by $\mathrm{Aut}^f(\mathcal{N})$ and it is a subgroup of $\mathrm{Aut}(\mathcal{N})$.

It is well-known from group theory that $\mathrm{Aut}^f(\mathcal{N})$ is characterized as the subgroup generated by all *transpositions*, which are permutations whose kernel has exactly 2 elements. Therefore, each finite permutation can be defined as the composition of a finite sequence of transpositions.

Definition 4 (*finite permutation signature and algebras*). The finite permutation signature Σ_p^f is obtained as the subsignature of Σ_p restricted to the unary operators induced by finite permutations.

The associated category of algebras is $Alg(\Sigma_p^f)$, shortened as Alg_p^f.

Each algebra in Alg_p^f has a *countable set* of operators and axioms, and thus it is more amenable to the standard results out of the algebraic specification mold. Each algebra in Alg_p can be cast trivially to an algebra in Alg_p^f (by forgetting the interpretation of non-finite permutations), and this inclusion is strict, as we shall see in Section 2.3.

2.2. Finitely supported permutation algebras

We provide now a final list of definitions, concerning the *finite support* property. They rephrase definitions in [17, Section 2.1], according to [8, Definition 3.3], and to our needs in the following sections.

Let us fix in this and the following sections a countable set $\mathcal{N}$, shortening $\mathrm{Aut}(\mathcal{N})$ and $\mathrm{Aut}^f(\mathcal{N})$ to Aut and Aut^f, respectively, usually putting a superscript $\_^f$ for definitions and notations concerning finite permutations. Moreover, subsets of $\mathcal{N}$ will be ranged over by X, Y.

Definition 5 (*support*). Let $\mathcal{A} \in Alg_p$ be a permutation algebra. For $a \in A$, the *isotropy group* of a is the set $\mathrm{fix}_A(a)$ of permutations fixing a in $\mathcal{A}$, i.e., $\mathrm{fix}_A(a) \triangleq \{\pi \in \mathrm{Aut} \mid \widehat{\pi}_A(a) = a\}$.

For $X \subseteq \mathcal{N}$, the *identity group* of X is the set $\mathrm{fix}(X)$ of permutations fixing X, i.e., $\mathrm{fix}(X) \triangleq \{\pi \in \mathrm{Aut} \mid \forall x \in X. \pi(x) = x\}$.

We say that the subset X *supports* the element $a \in A$ if all permutations fixing X also fix a in $\mathcal{A}$ (i.e., if $\mathrm{fix}(X) \subseteq \mathrm{fix}_A(a)$).

The definition can be readily adapted to finite permutation algebras, by replacing Aut by Aut^f throughout.

The notion of support is a suitable generalization of that of "free variables" of terms, and of "free names" of processes: if X supports a, then a is affected only by the action of permutations over the set X.

Definition 6 (finitely supported algebras). A permutation algebra $\mathcal{A}$ is *finitely supported* if for each element of its carrier there exists a finite set supporting it.

The full subcategory of Alg_p of all finitely supported permutation algebras is denoted by $FSAlg(\Sigma_p)$, shortened as $FSAlg_p$.

The category $FSAlg_p^f$ of finitely supported, finite permutation algebras is defined similarly.

Finitely supported algebras over all permutations correspond trivially to the *perm($\mathbb{A}$)-sets with finite support* of [8]. On the other hand, it is easy to show that the category of finitely supported algebras over finite permutations corresponds to the category of *nominal sets* as defined in [20].

In general, an element of the carrier of an algebra may have different sets supporting it. The following proposition, mirroring [8, Proposition 3.4], ensures that a minimal support does exist.

Proposition 7. *Let $\mathcal{A}$ be a (finite) permutation algebra. If $a \in A$ is finitely supported, then there exists a least finite set supporting a, called the* support *of a and denoted by $supp_A(a)$.*

Remark 1. Not all algebras in Alg_p are finitely supported (hence, neither those in Alg_p^f). For example, let us consider the set $\mathcal{N} = \{0, 1, \ldots\}$, and the algebra $(\wp(\mathcal{N}), \{\widehat{\pi} \mid \pi \in \mathrm{Aut}^f\})$, where $\widehat{\pi}(X) = \{\pi(x) \mid x \in X\}$ for all $X \in \wp(\mathcal{N})$. The sets $\mathcal{N}_{even} = \{2i \mid i \geq 0\}$ and $\mathcal{N}_{odd} = \{2i + 1 \mid i \geq 0\}$, both elements of $\wp(\mathcal{N})$, are not finitely supported: for all X finite, we can always pick a (finite) permutation π fixing X but exchanging $\max(X) + 1$ and $\max(X) + 2$; then $\hat{\pi}(\mathcal{N}_{even}) \neq \mathcal{N}_{even}$.

2.3. Some properties of permutation algebras

In this subsection we present some important properties of the categories of permutation algebras defined in the previous subsections. In particular, we show that the two categories of algebras with finite support with either signatures are isomorphic—that is, we can restrict to the countable signature of finite permutations without changing the resulting category.

Recall the existence of the forgetful functor $U : Alg_p \to Alg_p^f$ which simply drops the interpretation of non-finite permutations, and that can be extended also to the finitely supported counterparts. Actually, all these categories are much more strictly related: in this section we prove this statement, presenting first some (likely folklore) results on permutation algebras.

Lemma 8 (preserving supports). *Let $\mathcal{A}$ be a permutation algebra, let $a \in A$, and let X be a subset supporting a in $\mathcal{A}$. Then*

(i) $\pi(X)$ supports $\widehat{\pi}_A(a)$, for all permutations $\pi \in \mathrm{Aut}$;
(ii) X supports $\sigma(a)$ in $\mathcal{B}$, for all homomorphisms $\sigma : \mathcal{A} \to \mathcal{B}$.

Both proofs are easy, and they are skipped. More interesting are their consequences on finitely supported elements.

Corollary 9. *Let $\mathcal{A}$ be a permutation algebra, and let $a \in A$ be finitely supported. Then*

(i) $supp_A(\widehat{\pi}_A(a)) = \pi(supp_A(a))$, *for all permutations* $\pi \in \mathrm{Aut}$;
(ii) $supp_B(\sigma(a)) \subseteq supp_A(a)$, *for all homomorphisms* $\sigma : \mathcal{A} \to \mathcal{B}$;
(iii) $\mathrm{fix}_A(a) \subseteq sp_A(a)$, *for* $sp_A(a) \triangleq \{\pi \mid \pi(supp_A(a)) = supp_A(a)\}$.

Proof: Points *(ii)* and *(iii)* are easy; so, let us consider *(i)*.

We know that $\pi(supp_A(a))$ supports $\widehat{\pi}_A(a)$ from *(i)* of Lemma 8. We have to prove that for any X, if X supports $\hat{\pi}_A(a)$ then $\pi^{-1}(X)$ supports a; equivalently, that $\mathrm{fix}(X) \subseteq \mathrm{fix}(\hat{\pi}_A(a))$ implies $\mathrm{fix}(\pi^{-1}(X)) \subseteq \mathrm{fix}(a)$. Let ρ be a permutation fixing $\pi^{-1}(X)$. Then $\pi\rho\pi^{-1}$ fixes X (because for all $x \in X : \pi\rho\pi^{-1}(x) = \pi\pi^{-1}(x) = x$), and hence $\pi\rho\pi^{-1}$ fixes $\hat{\pi}_A(a)$. This means that $\hat{\pi}_A(a) = \hat{\pi}_A\hat{\rho}_A\widehat{\pi^{-1}}_A(\hat{\pi}_A(a)) = \hat{\pi}_A\hat{\rho}_A(a)$, and hence, by applying π^{-1}, we get $a = \hat{\rho}_A(a)$. □

Note also that $sp_A(a)$ is obviously a group.

Proposition 10 (removing infinite supports). *The inclusion functor $FSAlg_p \to Alg_p$ admits a right adjoint.*

Proof: Given a permutation algebra $\mathcal{A}$, simply consider the sub-algebra obtained by dropping all the elements with infinite support: it is well-defined, thanks to (i) of Lemma 8, and it extends to a functor, thanks to (ii) of that same lemma. □

Let us now introduce some additional notation.

Definition 11 (completions). Let X be a subset, and let $\pi \in \mathrm{Aut}$: in the following, we denote by $\pi_{|X} : X \to \pi(X)$ the bijection obtained as a restriction of π on X.

Conversely, given subsets X, Y and a bijection $\rho : X \to Y$, we denote by $\rho^c \in \mathrm{Aut}^f$ any *completion* of ρ, i.e., any finite permutation such that $\rho^c_{|X} = \rho$.

Lemma 12 (equating supports). *Let $\mathcal{A}$ be a permutation algebra, let $a \in A$, and let $X \subseteq \mathcal{N}$ supporting a in $\mathcal{A}$. If two permutations $\pi, \kappa \in \mathrm{Aut}$ coincide on X (i.e., $\pi_{|X} = \kappa_{|X}$), then $\widehat{\pi}_A(a) = \widehat{\kappa}_A(a)$.*

The proof is easy. Simply note that $\kappa^{-1}\pi$ is the identity on X, hence $\kappa^{-1}\pi(a) = a$.

Now we prove that if we stick to algebras with finite support, the restriction to the countable signature does not change the models.

Proposition 13. *Categories $FSAlg_p$ and $FSAlg_p^f$ are isomorphic.*

Proof: Note that $U : Alg_p \to Alg_p^f$ restricts to $U : FSAlg_p \to FSAlg_p^f$; indeed, for any algebra $\mathcal{A} = (A, \{\hat{\pi} \mid \pi \in \mathrm{Aut}\}) \in Alg_p$, if $a \in A$ is supported by a finite subset X, then X supports a also in $U(\mathcal{A})$.

It is then enough to show that each finitely supported algebra over finite permutations can be uniquely extended to obtain an object of $FSAlg_p$. That is, given $\mathcal{A} \in FSAlg_p^f$ and $a \in A$, we must define the value $\widehat{\pi}_A(a)$ for all infinite permutations π.

To this end, let us choose a completion $\kappa = (\pi_{|supp_A(a)})^c \in \mathrm{Aut}^f$ for any $\pi \in \mathrm{Aut}$ and $a \in A$. Now, the interpretation of $\widehat{\pi}_A(a)$ must coincide with $\widehat{\kappa}_A(a)$: it is well-given, since thanks to Lemma 12 the choice of the actual completion is irrelevant; and thanks to (i) of Lemma 8 also the axioms of permutation signatures are satisfied. □

For nominal sets, this result has been mentioned (without proof) in [20, Section 3].

We now conclude with a remark on the categories of all algebras.

Proposition 14. *The forgetful functor $Alg_p \to Alg_p^f$ is not full on objects.*

Proof: We show a finite permutation algebra $\mathcal{A}$ which cannot be extended to all permutations, that is, such that it does not exist a $\mathcal{B} = (A, \{\hat{\pi} \mid \pi \in \mathrm{Aut}\}) \in Alg_p$ satisfying $U(\mathcal{B}) = \mathcal{A}$.

Let us fix $\mathcal{N} = \{0, 1, 2, \dots\}$, and $\mathcal{N}_{even} = \{0, 2, 4 \dots\}$. We take $A \triangleq \{X \subseteq \mathcal{N} \mid X \cap \mathcal{N}_{even} \text{ infinite}\}$, and for $\pi \in \mathrm{Aut}^f$, let $\widehat{\pi}_A(X) = \{\pi(x) \mid x \in X\}$. Clearly, if X contains infinitely many even names, also $\widehat{\pi}_A(X)$ does, because π is a finite permutation. Let us consider the infinite permutation $\rho(x_{2i}) = x_{2i+1}$, $\rho(x_{2i+1}) = x_{2i}$ $(i \geq 0)$, swapping all odd and even names at once. By the axioms of permutation signatures, the interpretation of ρ must extend those of all finite permutations contained in it, therefore $\widehat{\rho}_A(X) = \{\rho(x) \mid x \in X\}$. But $\mathcal{N}_{even} \in A$, while $\widehat{\rho}_A(\mathcal{N}_{even}) = \{1, 3, 5, \dots\}$ which is not in A—absurd. □

3. Finitely supported algebras and sheaves

In this section, we begin to analyse how the pivotal notion of permutation algebra is connected with other models of nominal calculi. In Section 3.1 we show that the category of permutation algebras with finite support is equivalent to the Schanuel topos. This category, and the similar category of presheaves $Set^{\mathbb{I}}$, have been extensively used for defining semantic domains of calculi as final coalgebras of polynomial endofunctors; in Section 3.2 we recall these constructions in $\mathrm{Sh}(\mathbb{I}^{op})$, and relate with similar constructions on permutation algebras.

Then, we take advantage of this correspondence for transferring the constructions of polynomial "behavioural" functors from the Schanuel topos to the categories of permutation algebras.

3.1. Correspondence with sheaves

Recall that the category of *presheaves* over a small category C is the category of functors from $\mathcal{C}^{op}$ to *Set* and natural transformations. In this work, we are interested in the presheaf category $Set^{\mathbb{I}}$, where $\mathbb{I}$ is (without loss of generality) the category of finite subsets of $\mathcal{N}$ and *injective* maps. This category has been used by many authors for modeling the computational notion of dynamic allocation of names or locations; see for example [7, 10, 16, 19, 21, 23].

In order to show the correspondence between permutation algebras, we have to consider a particular subcategory of $Set^{\mathbb{I}}$, namely the category $\mathrm{Sh}(\mathbb{I}^{op})$ of *pullback preserving* functors only.[1] This condition has a precise meaning, which can be explained as follows. Let us consider a pullback $\begin{tikzcd} X \arrow[r, tail] \arrow[d, tail] & Y_1 \arrow[d, tail, "i_1"] \\ Y_2 \arrow[r, tail, "i_2"] & Z \end{tikzcd}$ in $\mathbb{I}$. Then, X is (isomorphic to) $i_1(Y_1) \cap i_2(Y_2)$. This pullback is mapped by a given functor $F : \mathbb{I} \to Set$ to a square $\begin{tikzcd} F_X \arrow[r, tail] \arrow[d, tail] & F_{Y_1} \arrow[d, tail, "F_{i_1}"] \\ F_{Y_2} \arrow[r, tail, "F_{i_2}"] & F_Z \end{tikzcd}$ in *Set*. If this square is a

[1] Clearly, a pullback-preserving functor is also mono preserving, but the converse is not true; see, for example, $P_\emptyset = \emptyset$ and $P_X = \mathcal{N}$ if $X \neq \emptyset$, and the pullback given by the inclusion of even and odd names in $\mathcal{N}$.

pullback, it means that for all $a \in F_Z$ is the image of some $a_1 \in F_{Y_1}, a_2 \in F_{Y_2}$, then there exists some $a_0 \in F_X$ which can be mapped to a_1, a_2 and ultimately to a. In other words, requiring F to be pullback preserving means that whenever two stages Y_1 and Y_2 are sufficient for "defining" an element a, also the intersection Y_1 and Y_2 must be. This seems to be a sensible condition.

The category of pullback-preserving functors from $\mathbb{I}$ to *Set* is also known as the category of *sheaves with respect to the atomic topology* [14, Sections 3.4, 3.9]; hence the notation $\mathrm{Sh}(\mathbb{I}^{op})$. We prefer the charaterization of sheaves as pullback-preserving functors, because it seems clearer than those based on topological notions; moreover it is easily generalizable to other index categories (see for example [15]). However, an interesting consequence is that $\mathrm{Sh}(\mathbb{I}^{op})$ is a (boolean) topos, and precisely the *Schanuel topos*.

The category $\mathrm{Sh}(\mathbb{I}^{op})$ can be used in place of $Set^{\mathbb{I}}$ for giving the semantics of languages with dynamic name allocations, as in [23, 24, 10, 1]. In fact, also the category of FM-sets with finite support (which correspond to $FSAlg_p$, as said before) is essentially equivalent to $\mathrm{Sh}(\mathbb{I}^{op})$, as mentioned briefly in [8, Section 7].

Here we give a direct proof that $FSAlg_p^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$ are equivalent. The first step is the definition of a categorical notion of "support".

Definition 15. Let $F : \mathbb{I} \to Set$, let $X \in \mathbb{I}$ and let $a \in F_X$. Then, $Y \subseteq X$ *supports* a if for all $h, k : X \rightarrowtail Z$ such that $h_{|Y} = k_{|Y}$ we have $F_h(a) = F_k(a)$.

In other words, the set Y supports $a \in F_X$ if we cannot tell apart the action on a of two given morphisms which agree on Y. This definition clearly generalizes the set-theoretical Definition 5 of support, as soon as we consider the case of $Z = X$ and $k = id_X$; in this case, h is a permutation of X and F_h corresponds to the operation $\widehat{h^c}$.

Proposition 16. *Let $F : \mathbb{I} \to Set$ be a sheaf, let $X \in \mathbb{I}$ and let $a \in F_X$. Let $i : Y \hookrightarrow X$ be a subset of X supporting a. Then, there exists a unique $b \in F_Y$ such that $a = F_i(b)$.*

Proof: It is easy to check that the following diagram in $\mathbb{I}$ is a pullback

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, hook, "i"] & X \arrow[d, "inl"] \\ X \arrow[r, "h"] & X \uplus X \end{tikzcd} \quad \text{where } h(x) = \begin{cases} inl(x) & \text{if } x \in Y \\ inr(x) & \text{otherwise.} \end{cases}$$

Since F is pullback preserving, the square in the following diagram in *Set* is a pullback:

$$\begin{tikzcd} 1 \arrow[dr, dotted, "b"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "a"'] & & \\ & F_Y \arrow[r, "F_i"] \arrow[d, "F_i"] & F_X \arrow[d, "F_{inl}"] \\ & F_X \arrow[r, "F_h"] & F_{X \uplus X} \end{tikzcd}$$

By hypothesis we know that $F_h(a) = F_{inl}(a)$, and hence, by the pullback property, there exists a unique $b \in F_Y$ such that $a = F_i(b)$. □

Lemma 17. *Let $F : \mathbb{I} \to Set$ be a sheaf, let $X \in \mathbb{I}$ and let $a \in F_X$. If both $Y_1, Y_2 \subseteq X$ support a, then also $Y_1 \cap Y_2$ supports a.*

Proof: The inclusion maps $i_1 : Y_1 \cap Y_2 \hookrightarrow Y_1$, $i_2 : Y_1 \cap Y_2 \hookrightarrow Y_2$ form a pullback of $j_1 : Y_1 \hookrightarrow X$, $j_2 : Y_2 \hookrightarrow X$. The sheaf F maps this pullback to the pullback square of the following diagram

$$\begin{tikzcd}
1 \arrow[dr, dotted, "b"] \arrow[drr, bend left, "b_1"] \arrow[ddr, bend right, "b_2"'] & & \\
& F_{Y_1\cap Y_2} \arrow[r, "F_{i_1}"] \arrow[d, "F_{i_2}"] & F_{Y_1} \arrow[d, "F_{j_1}"] \\
& F_{Y_2} \arrow[r, "F_{j_2}"] & F_X
\end{tikzcd}$$

where $b_1 \in F_{Y_1}$ and $b_2 \in F_{Y_2}$ are the elements given by Proposition 16 (that is, $F_{j_1}(b_1) = F_{j_2}(b_2) = a$. Due to the pullback there exists a unique $b \in F_{Y_1 \cap Y_2}$ such that $F_{j_1 \circ i_1}(b) = F_{j_2 \circ i_2}(b) = a$. Now, let $h, k : X \to Z$ be such that $h_{|Y_1 \cap Y_2} = k_{|Y_1 \cap Y_2}$; this means that $h \circ j_1 \circ i_1 = k \circ j_1 \circ i_1$. Then we have $F_h(a) = F_h(F_{j_1 \circ i_1}(b)) = F_k(F_{j_1 \circ i_1}(b)) = F_k(a)$. □

Proposition 18. *Let $F : \mathbb{I} \to Set$ be a sheaf, let $X \in \mathbb{I}$ and let $a \in F_X$. Then, there exists a least $Y \in \mathbb{I}$ supporting a.*

Proof: If both $i_1 : Y_1 \subset X$ and $i_2 : Y_2 \subset X$ support a, then also their pullback $Y_1 \cap Y_2$ does (Lemma 17). The cardinality of the pullback is $\leq \min\{|Y_1|, |Y_2|\}$, so by iteration we get the least support. □

Therefore, for all $X \in \mathbb{I}$ and $a \in F_X$, we can define $supp_X(a)$ as the least Y supporting a. Furthermore, we usually drop the subscript, since it is easy to check that such a least Y supporting a does not depend on the particular X the a comes from; that is, if $a \in F_X \cap F_Z$, then $supp_X(a) = supp_Z(a)$.

Proposition 19. *Categories $FSAlg_p^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$ are equivalent.*

Proof: Let us define first a functor $F : FSAlg_p^f \to \mathrm{Sh}(\mathbb{I}^{op})$. Let $\mathcal{A}$ be a finitely supported algebra over finite permutations. The corresponding functor $F\mathcal{A} : \mathbb{I} \to Set$ is defined

– on objects as $F\mathcal{A}_X \triangleq \{a \in A \mid supp_A(a) \subseteq X\}$;
– for $k : X \rightarrowtail Y$ in $\mathbb{I}$, $F\mathcal{A}_k : F\mathcal{A}_X \to F\mathcal{A}_Y$ maps $a \in A$ to $\widehat{\kappa}_A(a)$, where $\kappa \in \mathrm{Aut}^f$ is a(ny) finite permutation extending k to the whole $\mathcal{N}$. Since a has finite support, by Lemma 12 this is a good definition.

It is easy to check that this $F\mathcal{A}$ preserves pullbacks, thanks to Corollary 9(i); hence, it is a sheaf. Furthermore, let $\sigma : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism: the associated natural transformation $F\sigma : F\mathcal{A} \to F\mathcal{B}$ is defined as the obvious restriction $F\sigma_X \triangleq \sigma_{|F\mathcal{A}_X} : F\mathcal{A}_X \to F\mathcal{B}_X$ for all subsets X; it is well-defined thanks to Corollary 9(ii).

On the other hand, we define a functor $G : \mathrm{Sh}(\mathbb{I}^{op}) \to FSAlg_p^f$ as follows. Let $P : \mathbb{I} \to Set$ be any object of $\mathrm{Sh}(\mathbb{I}^{op})$; the carrier of the corresponding algebra $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is the set

$$A \triangleq \bigcup_{X \in \mathbb{I}} \{a \in P_X \mid supp(a) = X\}$$

For $\pi \in \mathrm{Aut}^f$, the map $\widehat{\pi}_A : A \to A$ is defined as

$$\widehat{\pi}_A(a) \triangleq P_{\pi_{|X}}(a) \ \text{ for } a \in P_X.$$

where $\pi_{|X} : X \rightarrowtail \pi(X)$ is the restriction of π to the finite X. It is trivial to check that if $a \in P_X$ then a is supported by X according to Definition 5. Finally, any natural transformation $\eta : P \to Q$ induces quite obviously a homomorphism between the corresponding algebras.

It is easy to check that there are two natural isomorphisms

$$\phi : GF \xrightarrow{\sim} Id_{FSAlg^f_p} \qquad \psi : FG \xrightarrow{\sim} Id_{\mathrm{Sh}(\mathbb{I}^{op})}.$$

Indeed, for any algebra $\mathcal{A}$ in $FSAlg^f_p$, the carrier of $GF\mathcal{A}$ is the set

$$\begin{aligned} |GF\mathcal{A}| &= \cup_{X\in\mathbb{I}}\{a \mid a \in (F\mathcal{A})_X, supp(a) = X\} \\ &= \cup_{X\in\mathbb{I}}\{a \mid supp_A(a) \subseteq X, supp(a) = X\} \\ &= \cup_{X\in\mathbb{I}}\{a \mid supp_A(a) = X\} \cong A \end{aligned}$$

where the third equality holds because $supp_A(a)$ supports a categorically (see Lemma 20 below) and hence $supp(a) \subseteq supp_A(a)$. The last equivalence holds because $\mathcal{A}$ is finitely supported.

On the other hand, for any sheaf $P : \mathbb{I} \to Set$, the carrier of the algebra GP is the set $\{a \mid Y \in \mathbb{I}, a \in P_Y, \mathrm{supp}(a) = Y\}$. Therefore, FGP is the presheaf mapping every X to the set

$$\begin{aligned} (FGP)_X &= \cup_{Y\in\mathbb{I}}\{a \mid a \in P_Y, supp(a) = Y, supp_{GP}(a) \subseteq X\} \\ &= \cup_{Y\in\mathbb{I}, Y\subseteq X}\{a \mid a \in P_Y, supp(a) = Y\} \\ &= \{a \mid a \in P_X, supp(a) \subseteq X\} = P_X \end{aligned}$$

where the last equivalence holds because by definition and thanks to Proposition 18 the support of $a \in F_X$ is a unique subset of X. $\square$

Lemma 20. *Let $\mathcal{A}$ be an algebra, and let $X \in \mathbb{I}$. Then, $supp_A(a)$ supports categorically a for all $a \in F\mathcal{A}_X$.*

Proof: Let $h, k : X \rightarrowtail Z$ in $\mathbb{I}$, such that $h_{|supp_A(a)} = k_{|supp_A(a)}$; we have to prove that $F\mathcal{A}_h(a) = F\mathcal{A}_k(a)$, that is, $\widehat{h^c}(a) = \widehat{k^c}(a)$, i.e., that $\widehat{h^c}^{-1}\widehat{k^c}$ fixes a, where $h^c, k^c \in \mathrm{Aut}(\mathcal{N})$ are two completions of h, k. By definition of support, it suffices to show that for all $x \in supp_A(a)$, $\widehat{h^c}^{-1}\widehat{k^c}(x) = x$, i.e., $\widehat{h^c}(x) = \widehat{k^c}(x)$; this holds because $h_{|supp_A(a)} = k_{|supp_A(a)}$ by hypothesis. $\square$

Proposition 19 has been mentioned (without proof) in the setting of FM-techniques, for example in [8, Section 7].

Remark 2. Let us consider now the presheaf category $Set^{\mathbb{B}}$, where $\mathbb{B}$ is the subcategory of $\mathbb{I}$ with only bijective maps. The inclusion functor $\mathbb{B} \hookrightarrow \mathbb{I}$ induces an obvious forgetful functor $|\_| : Set^{\mathbb{I}} \to Set^{\mathbb{B}}$, given by composition. As it is well known [14, Section 7], this functor has a left adjoint $(\_)_! : Set^{\mathbb{B}} \to Set^{\mathbb{I}}$, which in this case can be defined on objects as $(P_!)_X \triangleq \sum_{Y \subseteq X} P_Y$. In the unpublished work [4], Fiore proved that the Schanuel topos is equivalent to the Kleisli category of the monad $T : Set^{\mathbb{B}} \to Set^{\mathbb{B}}$ arising from this adjunction. More precisely, T is the composition $T_- = |(\_)_!|$, and the Kleisli category $\mathcal{K}(T)$ has as objects the objects of $Set^{\mathbb{B}}$, and for $P, Q : \mathbb{B} \to Set$, a morphism $\eta : P \to Q$ in $\mathcal{K}(T)$ is any natural transformation $\eta : P \to |Q_!|$ in $Set^{\mathbb{B}}$.

In fact, the correspondence in the proof of Proposition 19 can be easily strengthened to work also directly with $\mathcal{K}(T)$. Namely, each finitely supported permutation algebra $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is mapped to a functor $F\mathcal{A}$, object of $Set^{\mathbb{B}}$, defined as $F\mathcal{A}_X \triangleq \{a \in A \mid supp_A(a) = X\}$, and $F\mathcal{A}_\pi(a) \triangleq \widehat{\pi^c}_A(a)$ for $\pi : X \to X$ in $\mathbb{B}$. For $\sigma : \mathcal{A} \to \mathcal{B}$ in Alg_p^f, the corresponding morphism $F_\sigma : F\mathcal{A} \to F\mathcal{B}$ in $\mathcal{K}(T)$ is the natural transformation $\eta : F\mathcal{A} \to |(F\mathcal{B})_!|$ in $Set^{\mathbb{B}}$, defined as $\eta_X \triangleq \sigma_{|F\mathcal{A}_X} : \{a \in A \mid supp_A(a) = X\} \to \{b \in B \mid supp_B(b) \subseteq X\}$. This is a good definition in virtue of Lemma 9.

3.2. Behavioural functors over permutation algebras

As mentioned before, $Set^{\mathbb{I}}$ and $\mathrm{Sh}(\mathbb{I}^{op})$ have been widely used in the literature for definining the domain of meaning of name-passing calculi, such as the π-calculus. In these cases, the domain is obtained as the final coalgebra of a "behavioural" endofunctor $B : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is $Set^{\mathbb{I}}$ or $\mathrm{Sh}(\mathbb{I}^{op})$ (or a variant of them). The definition of B is usually polynomial, and this ensures the existence of the final coalgebra. Beside the usual constructors of polynomial functor (namely constants, finite sums and products and finite powersets), the categories $\mathrm{Sh}(\mathbb{I}^{op})$ and $Set^{\mathbb{I}}$ feature the peculiar constructors needed for giving semantics to name-passing calculi. We recall the definition of these constructors on $\mathrm{Sh}(\mathbb{I}^{op})$, which were used in for example [7, 10].

1. the *type of names* is the object $N \triangleq \mathbb{I}(1, \_)$ (for all $X \in \mathbb{I}$: $N_X \cong X$);
2. the *shift operator* is the functor $\delta : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ (defined as $\delta(P)_X \triangleq P_{X \uplus 1}$ on objects, and $\delta(P)_f \triangleq P_{f \uplus id}$ on arrows), a type constructor representing the dynamic generation of names;
3. the *finite powerset* $\wp_f : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ is defined pointwise;
4. the *name exponential* $(\_)^N : \mathrm{Sh}(\mathbb{I}^{op}) \longrightarrow \mathrm{Sh}(\mathbb{I}^{op})$ is defined as

$$(P^N)_X = Sh(\mathbb{I}^{op})(\mathbb{I}(X, \_) \times N, P) \cong (P_X)^X \times P_{X \uplus 1}$$

5. finally, the *partial name exponential* (useful for early semantics) $N \rightharpoonup \_ : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ is defined as

$$\begin{aligned}
(N \rightharpoonup P)_X &\triangleq (1 + P_X)^X \\
(N \rightharpoonup P)_f &: (1 + P_X)^X \to (1 + P_Y)^Y \qquad \text{for } f : X \rightarrowtail Y \\
& \quad u \mapsto \lambda y \in Y. \begin{cases} P_f(u(x)) & \text{if } f(x) = y \text{ and } u(x) \in P_X \\ * & \text{otherwise} \end{cases}
\end{aligned}$$

It is easy to check that any functor defined using these constructors (and finite sums and products) is accessible, and hence admits a final coalgebra [22]. For instance, following [7] the domain for late semantics of π-calculus can be defined as the final coalgebra of the functor $B : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$

$$BP \triangleq \wp_f(\overbrace{N \times P^N}^{\text{input}} + \overbrace{N \times N \times P}^{\text{output}} + \overbrace{N \times \delta_S P}^{\text{bound output}} + \overbrace{P}^{\tau})$$

$$(BP)_X = \wp_f(X \times (P_X)^X \times P_{X \uplus 1} + X \times X \times P_X + X \times P_{X \uplus 1} + P_X).$$

In virtue of the equivalence between $FSAlg_p^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$, it is possible to define these constructors also on $FSAlg_p^f$. Moreover, since the equivalence preserves both limits and colimits, we only need to check out the behaviour on the functor for names and on the shift operator.

1. The *algebra of names* is given by $\mathcal{N} = (\mathcal{N}, \mathrm{Aut}^f)$.
2. The *shift* operator $\delta_A : FSAlg_p^f \to FSAlg_p^f$ is defined as follows. If $\mathcal{A} = (A, \{\widehat{\pi}_A \mid \pi \in \mathrm{Aut}^f\})$ is a permutation algebra, we define

   $$\delta(\mathcal{A}) \triangleq (A, \{\widehat{\pi^{+1}}_A \mid \pi \in \mathrm{Aut}^f\})$$

   where for $\pi \in \mathrm{Aut}^f$, $\pi^{+1} \in \mathrm{Aut}^f$ is defined as

   $$(\pi^{+1})(s_0) = s_0 \qquad (\pi^{+1})(s_{n+1}) = succ(\pi(s_n)).$$

   for any fixed enumeration $\mathcal{N} = \{s_0, s_1, s_2, \ldots\}$.[2]
   For any morphism $\sigma : \mathcal{A} \to \mathcal{B}$, we put $\delta_A \sigma = \sigma$; indeed, for $\pi \in \mathrm{Aut}^f$, we have $\sigma \circ \widehat{\pi^{+1}}_A = \widehat{\pi^{+1}}_B \circ \sigma$ by definition of σ. It is easy to check that δ_A is an endofunctor on $FSAlg_p^f$.
3. Finite powersets, products and coproducts are defined pointwise.
4. By exploiting the equivalence between $\mathrm{Sh}(\mathbb{I}^{op})$ and $FSAlg_p^f$, we can derive the definition of $\mathcal{A}^{\mathcal{N}}$ whose carrier is the set

   $$\{f : X \to A_X \mid X \subset \mathcal{N} \text{ finite}\} \times A$$

   where $A_X \triangleq \{a \in A \mid \mathrm{supp}_A(a) \subseteq X\}$. For $\pi \in \mathrm{Aut}^f$, the corresponding operator $\widehat{\pi}_{A^{\mathcal{N}}}$ maps each pair $(f : X \to A_X, a)$ to $(\widehat{\pi}_A \circ f \circ \pi^{-1} : L \to A_L, \widehat{\pi^{+1}}_A)$, where $L = \pi(X)$.
5. Finally, the *partial name exponential* on algebras is defined again by taking advantage of the equivalence with $\mathrm{Sh}(\mathbb{I}^{op})$. For an algebra $\mathcal{A}$, the carrier of the algebra $\mathcal{N} \rightharpoonup \mathcal{A}$ is the set of partial functions

   $$B = \{f : X \rightharpoonup A_X \mid X \subset \mathcal{N},\ \text{finite}\}$$

   and for $\pi \in \mathrm{Aut}^f$, the operator $\hat{\pi}_B : B \to B$ maps a partial function $u : X \rightharpoonup A_X$ to the partial function $v : Y \rightharpoonup A_Y$ where $Y \triangleq \pi(X)$ and for all $y \in Y$

   $$v(y) \triangleq \begin{cases} \hat{\pi}_A(u(\pi^{-1}(y))) & \text{if } u(\pi^{-1}(y)) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

[2] This is one of the literally infinite possible definitions of δ_A; it corresponds to de Bruijn indexes, where the newly created (i.e., locally bound) name is always s_0.

The coalgebras of the functors over $FSAlg_p^f$ are a particular class of the *structured coalgebras* studied for instance in [2, 18]. Moreover, these functors correspond exactly to the polynomial functors over $\mathrm{Sh}(\mathbb{I}^{op})$ defined using the constructors listed above.

Proposition 21. *Let* $B : \mathrm{Sh}(\mathbb{I}^{op}) \to \mathrm{Sh}(\mathbb{I}^{op})$ *be a polynomial endofunctor. Then, there exists a functor* $\bar{B} : FSAlg_p^f \to FSAlg_p^f$ *such that the category* $\mathbf{Coalg}(B)$ *is isomorphic to* $\mathbf{Coalg}(\bar{B})$*, and* vice versa.

Proof: It is sufficient to check that the functors F and G between $FSAlg_p^f$ and $\mathrm{Sh}(\mathbb{I}^{op})$ commute with the constructors of the polynomial functors. This can be proved easily by inspection. □

4. Finitely supported algebras and named sets

In this section we compare finitely supported algebras and *named sets*, which were introduced as the building blocks of *HD-automata*.

4.1. Named sets

The definitions below are drawn from [3, Section 3.1], and simplified according to our needs.

Definition 22 (*named sets*). A *named set* N is a triple

$$N = \left\langle Q_N, \|\cdot\|_N : Q_N \to \wp_f(\mathcal{N}), G_N : \prod_{q \in Q_N} \wp(\mathrm{Aut}(\|q\|_N)) \right\rangle$$

where Q_N is a set of *states*; $\|\cdot\|_N$ is the *enumerating* function; and for all $q \in Q_N$, the set $G_N(q)$ is a subgroup of $\mathrm{Aut}(\|q\|_N)$, and it is called the *permutation group* of q.

Intuitively, a state in Q_N represents a process, and thus the function $\|\cdot\|_N$ assigns to each state the finite set of variables possibly occurring free in it. Finally, G_N denotes for each state the group of renamings under which it is preserved, i.e., those permutations on names that do not interfere with its possible behavior.

Definition 23 (*category of named sets*). Let N, M be named sets. A *named function* $L : N \to M$ is a pair

$$L = \left\langle l : Q_N \to Q_M, \Lambda : \prod_{q \in Q_N} \wp(\mathbb{I}(\|l(q)\|_M, \|q\|_N)) \right\rangle$$

for l a function and $\Lambda(q)$ a (non-empty) set of injections from $\|l(q)\|_M$ to $\|q\|_N$, satisfying the additional condition

$$G_N(q) \circ \lambda \subseteq \Lambda(q) = \lambda \circ G_M(l(q)) \qquad \forall \lambda \in \Lambda(q) .$$

Finally, *NSet* denotes the category of named sets and their morphisms.

So, a named function is a state function, equipped with a set of injective renamings for each $q \in Q_N$, which is somewhat compatible with the permutations in $G_N(q)$ and $G_M(l(q))$. More precisely, "the whole set of $\Lambda(q)$ must be generated by saturating any of its elements by the permutation group of $l(q)$, and the result must be invariant with respect to the permutation group of q" [3, Section 3.1]. The category is clearly well-defined: in particular, the identity on N is $\langle id_{Q_N}, G_N \rangle$, and composition is defined as expected.

Remark 3. We simplified the definition in [3], since we did not restrict the enumerating function to taking value in prefixes of $\{0, 1, \ldots\}$: this would correspond to fix a canonical choice of free variables for each state, and albeit important for verification purposes, it is not relevant here. We do not further discuss the matter, referring the reader to [9] for a correspondence between permutation algebras and that alternative presentation of named sets.

Note also that an alternative, yet equivalent definition for a named function $\langle l, \Lambda \rangle : N \to M$ would be to require for each $q \in Q_N$ the existence of an injection $\lambda \in \mathbb{I}(\|l(q)\|_M, \|q\|_N)$ such that $\Lambda(q) = \lambda \circ G_M(l(q))$ holds and moreover $G_N(q) \circ \Lambda(q) = \Lambda(q) \circ G_M(l(q))$.

Example 24. Let us consider the singleton set $\{q\}$. Then, both $N_1 = \langle \{q\}, \|q\| = \{x\}, \mathrm{Aut}(q) = \{id_x\} \rangle$ and $N_2^p = \langle \{q\}, \|q\| = \{x_1, x_2\}, \mathrm{Aut}(q) = \mathrm{Aut}(\{x_1, x_2\}) \rangle$ are named sets: same set of states, different enumerating functions. Instead, $N_2^i = \langle \{q\}, \|q\| = \{x_1, x_2\}, \mathrm{Aut}(q) = \{id_{x_1,x_2}\} \rangle$ is a named set with the same set of states and the same enumerating function of N_2^p, but with a different permutation group.

Notice that there is no named function from N_2^p to N_1, since any injection λ, when post-composed with $\mathrm{Aut}(\{x_1, x_2\})$, generates the whole $\mathbb{I}(\{x\}, \{x_1, x_2\})$. Instead, denoting I_j the set containing only the injection mapping x to x_j, then $\langle id, I_j \rangle$ is a named function from N_2^i to N_1, while $\langle id, I_0 \cup I_1 = \mathbb{I}(\{x\}, \{x_1, x_2\}) \rangle$ is not.

Similarly, there is no named function from N_2^p to N_2^i. In general, it is easy to see that, given named sets $\langle Q, \| \cdot \|, G_1 \rangle$ and $\langle Q, \| \cdot \|, G_2 \rangle$ (i.e., same state set and enumerating function, different permutation groups), with $G_1(q)$ a subgroup of $G_2(q)$ for all $q \in Q$, then $\langle id, G_2 \rangle$ is a well-defined named function from the former named set to the latter.

In the remaining of this section we relate $FSAlg_p^f$ and *NSet*, the category of named sets. We plan to sharpen and make more concise some of the results presented in [17, Section 6].

Summarizing, Propositions 25 and 28 prove the existence of suitable functors between those categories, generalizing the functions given in Definitions 49 and 50 in [17], respectively. Moreover, Proposition 29 extends to a categorical equivalence the correspondence proved in Theorem 51 of the same paper.

4.2. From named sets to permutation algebras

In order to define the functor from named sets to (finite) permutation algebras, we first introduce some notation. Generalising Definition 11, for any pair of subsets X, Y and $\lambda \in \mathbb{I}(X, Y)$, we denote by $\lambda^c \in \mathrm{Aut}^f$ a completion of λ, i.e., a finite permutation such that $\lambda^c(x) = \lambda(x)$ for all $x \in X$; and for $I \subseteq \mathbb{I}(X, Y)$, we denote by I^c the set of all the finite permutations obtained by closing the injections of I.

Proposition 25 (from sets to algebras). *Let us consider the following definition of a functor $F : NSet \to FSAlg_p^f$.*

F maps each named set N to the finite permutation algebra with carrier the set of pairs $\{\langle q, \pi \circ G_N(q)^c \rangle \mid q \in Q_N, \pi \in \mathrm{Aut}^f\}$ *and family of functions given by* $\widehat{\pi}_{F(N)}(\langle q, I \rangle) = \langle q, \pi \circ I \rangle$.

F maps each named function $\langle l, \Lambda \rangle : N \to M$ *to the homomorphism associating to each* $\langle q, I \rangle$ *the element* $\langle l(q), I \circ \Lambda(q)^c \rangle$.

The functor F is well defined.

Proof: The resulting algebra has finite support, since each element $\langle q, I \rangle$ is supported by the set $\pi(\|q\|_N)$, for any $\pi \in \mathrm{Aut}^f$ such that $I = \pi \circ (G_N(q))^c$. In order to prove this, we must show that for any permutation κ fixing $\pi(\|q\|_N)$, then $\pi \circ G_N(q)^c = \kappa \circ \pi \circ G_N(q)^c$ holds. Note that $\forall x \in \pi(\|q\|_N) : \kappa(x) = x$ implies $\forall k \in \|q\|_N : \kappa(\pi(k)) = \pi(k)$, and then we are done, since by definition all the permutations in $G_N(q)^c$ preserve $\|q\|_N$.

Let us now consider a named set function $L : N \to M$, and let us consider the function associating to each $\langle q, I \rangle$ the pair $\langle l(q), I \circ \Lambda(q)^c \rangle$. It is well-defined, since by the definition of named function we have $G_N(q) \circ \Lambda(q) = \Lambda(q) = \Lambda(q) \circ G_M(l(q))$, and the equality can be lifted to their respective closures. It is immediate to check that the function is also an homomorphism. □

4.3. From permutation algebras to named sets

Recall now that, according to (ii) of Lemma 8, given an algebra homomorphism $\sigma : \mathcal{A} \to \mathcal{B}$, and a finitely supported element $a \in A$, then $supp_B(\sigma(a)) \subseteq supp_A(a)$. So, let $in_\sigma(a)$ be the uniquely associated injection: this remark is sufficient for defining a functor I from finitely supported permutation algebras to named sets.

Proposition 26 (from algebras to sets). *Let us consider the following definition of a functor* $I : FSAlg_p^f \to NSet$.

I *maps each* $\mathcal{A} \in FSAlg_p^f$ *to the named set* $\langle A, supp_A(\cdot), G_{I(\mathcal{A})} \rangle$, *where* $G_{I(\mathcal{A})}(a) \triangleq \{\pi_{|supp_A(a)} \mid \pi \in \mathrm{fix}_A(a)\}$.

Let $\sigma : \mathcal{A} \to \mathcal{B}$, *and let* $in_\sigma(a) : supp_B(\sigma(a)) \to supp_A(a)$ *be the uniquely induced arrow. Hence, let* $I(\sigma)$ *be the named function* $\langle l_\sigma, \Lambda_\sigma \rangle$ *given by the obvious function from A to B and by the sets of injections* $\Lambda_\sigma(a) = in_\sigma(a) \circ G_{I(\mathcal{B})}(\sigma(a))$ *for all* $a \in A$.

The functor I is well defined.

Proof: It is easy to see that $G_{I(\mathcal{A})}(a)$ is well-defined, since $\mathrm{fix}_A(a) \subseteq sp_A(a)$ holds by (iii) of Corollary 9, hence $\pi_{|supp_A(a)} \in \mathrm{Aut}(supp_A(a))$; moreover, it is a group, since $\mathrm{fix}_A(a)$ is so.

It is now enough to prove that for all $a \in A$

$$G_{I(\mathcal{A})}(a) \circ in_\sigma(a) \subseteq in_\sigma(a) \circ G_{I(\mathcal{B})}(\sigma(a)).$$

This is equivalent to ask that for all $\pi \in G_{I(\mathcal{A})}(a)$ there exists a $\kappa \in G_{I(\mathcal{B})}(\sigma(a))$ such that $\pi \circ in_\sigma(a) = in_\sigma(a) \circ \kappa$. A possible choice is $\pi_{|supp_B(\sigma(a))}$: in fact, since $\mathrm{fix}_A(a) \subseteq \mathrm{fix}_B(\sigma(a))$, any π^c also fixes $\sigma(a)$ in $\mathcal{B}$; and since $\mathrm{fix}_B(\sigma(a)) \subseteq sp_B(\sigma(a))$, then $\pi_{|supp_B(\sigma(a))}$ is well-defined and satisfies the requirements.

Identities and composition are preserved, hence the result holds. □

Unfortunately, the functor I just defined does not allow establishing an equivalence between the categories. Intuitively, the reason is that the functor F relates to each state of a named set a whole set of elements of the associated algebra, obtained *via* permutations of its free names. Thus, we introduce a final concept, the *orbit* of an element, consisting of the family of all the elements of the carrier of an algebra which can be reached from the given element.

Definition 27 (*orbits*). Let $\mathcal{A} \in Alg_p$ and let $a \in A$. The *orbit* of a is the set $Orb_A(a) \triangleq \{\widehat{\pi}_A(a) \mid \pi \in \text{Aut}\}$.

Orbits obviously partition a permutation algebra. So, let us assume the existence for each orbit $Orb_A(a)$ of a canonical representative a_O (we come back on this later on, in Remark 4), and let $A_O \triangleq \{a_O \mid a \in A\}$.

Proposition 28 (from algebras to sets, II). *Let us consider the following definition of a functor $\widehat{I} : FSAlg_p^f \to NSet$.*

$\widehat{I}_O$ maps each $\mathcal{A} \in FSAlg_p^f$ to to the named set $\langle A_O, supp_A(\cdot), G_{I(\mathcal{A})}\rangle$.

Let $\sigma : \mathcal{A} \to \mathcal{B}$, let $in_\sigma(a_O) : supp_B(\sigma(a_O)) \to supp_A(a_O)$ be the uniquely induced arrow, and let I_{a_0} be the set of bijections defined as $\{\pi_{|supp_B(\sigma(a_O)_O)} \mid \widehat{\pi}_B(\sigma(a_O)_O) = \sigma(a_O)\}$. Hence, let $\widehat{I}_A$ be the function associating to σ the named function $\langle l_\sigma, \Lambda_\sigma\rangle$ such that $l_\sigma(a_O) = \sigma(a_O)_O$ and $\Lambda_\sigma(a_O) = in_\sigma(a_O) \circ I_{a_O}$ for all $a_O \in A$.

The pair $\widehat{I} = \langle \widehat{I}_O, \widehat{I}_A\rangle$ defines a functor from $FSAlg_p^f$ to NSet.

Proof: First, note that clearly $I_{a_O} \subseteq \mathbb{I}(supp_B(\sigma(a_O)_O), supp_B(\sigma(a_O)))$ by (i) of Lemma 8, so that $\Lambda(a_O)$ is a well-defined set of injections.

Then, the key remark for the correctness of Λ_σ is the obvious coincidence between $\lambda \circ \text{fix}_B(\sigma(a_O)_O)$ and $\text{fix}_B(\sigma(a_O)) \circ \lambda$, for any $\lambda \in Hom_B[\sigma(a_O)_O, \sigma(a_O)]$; so that the equality $\Lambda_\sigma(a_O) \circ G_{\widehat{I}(\mathcal{B})}(\sigma(a_O)_O) = in_\sigma(a_O) \circ G_{I(\mathcal{B})}(\sigma(a_O)) \circ I_{a_O}$ holds. Then, it is enough to mimic the proof for Proposition 26. □

Using the previously defined functor, it is easy to realize that named sets are just a different presentation for finite permutation algebras.

Proposition 29. *Categories NSet and $FSAlg_p^f$ are equivalent.*

Proof: Let N be a named set. First, notice that

$$\|q\|_N = supp_{F(N)}(\langle q, G_N(q)^c\rangle) .$$

Since the choice of the canonical representative is irrelevant, we may choose precisely $\langle q, G_N(q)^c\rangle$. Hence, it is easy to prove that $\widehat{I}(F(N))$ is naturally isomorphic to N.

Analogous considerations hold for the isomorphism $F(\widehat{I}(\mathcal{A})) \to \mathcal{A}$ on algebras, which is obtained as the obvious extension of the function mapping each term $\langle a_O, G_{\widehat{I}(\mathcal{A})}(a_O)\rangle$ into a_O. □

Remark 4. The canonical representative a_O of each orbit can be constructively defined, if the underlying set $\mathcal{N}$ is totally ordered, as in the original definition [3]. In fact, this property naturally allows both $\wp_{fin}$ and Aut^f to be equipped with a total order, and the latter is then lifted to sets of permutations. Hence, for each orbit an element a_c can be chosen, such that $supp_A(a_c)$ is minimal, and which has the minimal permutation group associated to it. The element a_c is well-defined, since it is easy to prove that $\text{fix}_A(a) = \text{fix}_A(b)$ implies $a = b$ for all finitely supported $a, b \in A$ such that $b \in Orb_A(a)$.

Remark 5. An equivalence result relating the category of named sets and the Schanuel topos, thus corresponding to the concatenation of our Propositions 19 and 29, has been proved independently in [6]. Their paper however has a different focus than ours, since its aim is the characterization of a class of transition systems, considered as coalgebras over pre-shaves, corresponding to history-dependent automata.

Moreover, they consider a slightly different notion of named set, namely "a pair (A, f) where A is a set and for all $a \in A$, $f(a)$ is a subgroup of Aut" [6, Definition 4.2]. This definition is simpler than our Definition 22, because it basically lacks the enumerating function for each element; nevertheless, the notion of "supporting set" can be recovered by stating that $X \subseteq \mathcal{N}$ supports $a \in A$ if and only if $\text{fix}(X) \subseteq f(a)$.

According to this alternative definition, a named set is not finitely supported *a priori*, but the property must be required explicitly; instead, all the named sets of Definition 22 are always finitely supported. In fact, in [6] the subcategory of finitely supported named sets is proved equivalent to the Schanuel topos, and hence, by the results above, to the category *NSet* of Definition 23. Hence, we can see the more explicit Definition 23 as an "implementation-oriented" notion of named sets, while the more compact definition used in [6] appears to be more "theoretical-oriented".

5. Permutation algebras and continuous G-sets

In the previous sections we proved the equivalences

$$FSAlg_p \cong FSAlg_p^f \cong \text{Sh}(\mathbb{I}^{op})(\cong NSet)$$

by providing directly suitable equivalence functors. In this section we re-analyze these correspondences in the light of a well-known theory from algebraic topology, namely that of *(continuous) G-sets*. This allows the categories Alg_p and Alg_p^f, which were omitted in the previous analysis, to be accommodated in a single framework as well.

5.1. Continuous G-sets

In this subsection we recall some standard definitions and results about continuous G-sets; see for example [12] for a general introduction, and [13, Section 5.9] and [14, Chapter II] for a discussion in category and topos theory.

Definition 30 (G-sets). Let G be a group. A *G-set* is a pair $(X, \cdot_X)$ where X is a set and $\cdot_X : X \times G \to X$ is a *right G-action*, that is

$$x \cdot_X id = x \qquad (x \cdot_X g_1) \cdot_X g_2 = x \cdot_X (g_1 g_2)$$

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ between G-sets is a function $f : X \to Y$ such that $f(x \cdot_X g) = f(x) \cdot_Y g$ for all $x \in X$.

The G-sets and their morphisms form a category denoted by $\mathbf{B}G^\delta$.

For instance, the $perm(\mathbb{A})$-sets and equivariant functions used in [8] form the category $\mathbf{B}perm(\mathbb{A})^\delta$.

More generally, we are interested in G-sets where G is a *topological group*, i.e., its carrier is equipped with a topology and multiplication and inverse are continuous. For sake of completeness, we recall next some basic definitions of topology.

Definition 31. A *topological space* is a pair $(X, \mathcal{O}(X))$ for X a set and $\mathcal{O}(X) \subseteq \wp(X)$ (the *topology* over X) is closed with respect to arbitrary union and finite intersection, and $\emptyset, X \in \mathcal{O}(X)$.

A function $f : X \to Y$ is a *continuous map* $f : (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$ if $f^{-1}(U) \in \mathcal{O}(X)$ for all $U \in \mathcal{O}(Y)$.

The elements of $\mathcal{O}(X)$ are referred to as the *open sets* of the topology.

The finest topology is the *discrete topology*, where $\mathcal{O}(X) = \wp(X)$. A topology is discrete if and only if $\{x\} \in \mathcal{O}(X)$ for all $x \in X$, i.e., if every point is separated from the others (hence the name). Clearly, every function is continuous with respect to the discrete topology.

The category of topological spaces is complete and cocomplete [13, Section 5.9]. In particular, given a family of topological spaces $(X_i, \mathcal{O}(X_i))$, indexed by $i \in I$, the product $\prod_{i\in I}(X_i, \mathcal{O}(X_i))$ is the topological space whose space is $X = \prod_{i\in I} X_i$, and the topology is the smallest topology such that the projections $\pi_i : X \to X_i$ are continuous. If I is finite, then $\mathcal{O}(X) = \prod_{i\in I} \mathcal{O}(X_i)$. This does not hold for I infinite, in general.

Finally, we recall the last standard definition we need for our development, which generalizes Definition 30.

Definition 32 (topological groups and continuous G-sets). A group G is a *topological group* if its carrier is equipped with a topology, and its multiplication and inverse are continuous with respect to this topology.

A G-set $(X, \cdot_X)$ is *continuous* if G is topological and the action $\cdot_X : X \times G \to G$ is continuous with respect to X equipped with the discrete topology.

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ between continuous G-sets is a function $f : X \to Y$ which respects the actions.

For a given topological group G, continuous G-sets and their morphisms form a category, denoted by $\mathbf{B}G$.

Notice that for any group G, the category of *all* G-sets is the category of continuous G-sets where G is taken with the discrete topology—hence the notation $\mathbf{B}G^\delta$ from [14] that we used in Definition 30.

A useful characterization of continuous G-sets is given by the following lemma [14, I, Exercise 6].

Lemma 33.1. *Let G be a topological group, let $(X, \cdot_X)$ be a G-set, and for each $x \in X$ let* $\mathrm{fix}_X(x) \triangleq \{g \in G \mid x \cdot_X g = x\}$ *be the* isotropy group *of x. Then, $(X, \cdot_X)$ is continuous iff all its isotropy groups are open sets in G.*

5.2. Permutation algebras as G-sets

For any countably infinite set of names $\mathcal{N}$, a permutation $\pi \in \mathrm{Aut}(\mathcal{N})$ is equivalent to a permutation over the set of natural numbers $\mathbf{N}$. Therefore, in the rest of this section we assume, without loss of generality, that $\mathcal{N} = \mathbf{N}$.

Let us consider the G-sets when G is either Aut or Aut^f. Clearly, every Aut-set is also a Aut^f-set (just by restricting the action to the finite permutations), mimicking the correspondence between Alg_p and Alg_p^f. In fact, a stronger equivalence holds, as shown by the next result.

Proposition 34. *Categories Alg_p and $\mathbf{B}\mathrm{Aut}^\delta$ are isomorphic.*

Proof: Let us define an isomorphism functor $G : Alg_p \to \mathbf{B}\mathrm{Aut}^\delta$.

Let $\mathcal{A}$ be a permutation algebra. The corresponding Aut-set is $G(\mathcal{A}) = (A, \cdot_{G(A)})$, where $a \cdot_{G(A)} \pi \triangleq \widehat{\pi}_A(a)$ for all $a \in A$. On the other hand, if $(X, \cdot_X)$ is a Aut-set, the corresponding algebra $\mathcal{X} = (X, \{\hat{\pi}_X\})$ is defined by taking $\widehat{\pi}_X(x) \triangleq x \cdot_X \pi$ for $\pi \in \mathrm{Aut}$.

Let $\mathcal{A}, \mathcal{B}$ be two permutation algebras. A function $f : A \to B$ is a morphism $f : \mathcal{A} \to \mathcal{B}$ in Alg_p iff $f(\widehat{\pi}_A(a)) = \widehat{\pi}_B(f(a))$ for all permutations π and $a \in A$, which in turn holds iff $f(a \cdot_{G(A)} \pi) = f(a) \cdot_{G(B)} \pi$ for all π and a, which equivalently states that $f : (A, \cdot_{G(A)}) \to (B, \cdot_{G(B)})$ is a morphism in $\mathbf{B}\mathrm{Aut}^\delta$. □

Notice that the functor G can be restricted to Alg_p^f and $\mathbf{B}\mathrm{Aut}^{f^\delta}$, and hence we have that Alg_p^f and $\mathbf{B}\mathrm{Aut}^{f^\delta}$ are isomorphic as well.

Also the categories of algebras with finite support, possibly over only finite permutations, can be recast in the general setting of G-sets, but to this end we need to equip the groups Aut and Aut^f with a topology.

Let us consider the space $\mathbf{N}$, given as the set of natural numbers equipped with the discrete topology. The *Baire space* is the topological space $\prod_{i=0}^{\infty} \mathbf{N} = \mathbf{N}^\omega$, equipped with the infinite product topology. A base of this topology is given by the sets of the form $\prod_{i=0}^{\infty} X_i$ where $X_i \neq \mathbf{N}$ only for *finitely many* indexes i.

Let us now consider the groups Aut and Aut^f. The carriers of these groups are subspaces of the Baire space, where each π corresponds to the infinite list $(\pi(0), \pi(1), \pi(2), \ldots)$, as described in [14, Section 3.9] for Aut. Therefore, both Aut and Aut^f inherit a topology from $\mathbf{N}^\omega$: their open sets are of the form $U \cap \mathrm{Aut}$ and $U \cap \mathrm{Aut}^f$, for U open set of $\mathbf{N}^\omega$.

We can thus consider the categories $\mathbf{B}\mathrm{Aut}$ and $\mathbf{B}\mathrm{Aut}^f$ of continuous Aut-sets and continuous Aut^f-sets, respectively. For the former category there is a famous characterization result [14, Section 3.9, Corollary 3].

Proposition 35. *Categories $\mathbf{B}\mathrm{Aut}$ and $\mathrm{Sh}(\mathbb{I}^{op})$ are equivalent.*

By Proposition 13, we have that $FSAlg_p \cong \mathbf{B}\mathrm{Aut} \cong FSAlg_p^f$. But actually this equivalence can be extended to $\mathbf{B}\mathrm{Aut}^f$ as well, as a consequence of the following result.

Theorem 36. *Categories $FSAlg_p^f$ and $\mathbf{B}\mathrm{Aut}^f$ are equivalent.*

Proof: We show that the functor G of Proposition 34 maps finite permutation algebras with finite support to continuous Aut^f-sets, and *vice versa*.

Let $\mathcal{A} = (A, \{\hat{\pi}_A\})$ be an algebra in $FSAlg_p^f$; the corresponding Aut^f-set is $(A, \cdot_{G(A)})$, where $a \cdot_{G(A)} \pi \triangleq \hat{\pi}_A(a)$ for all $a \in A$. For Lemma 33, $G(\mathcal{A})$ is continuous if and only if $\mathrm{fix}_A(a)$ is open for all $a \in A$: this is proved by a suitable characterization of $\mathrm{fix}(a)$, given by

$$
\begin{aligned}
\mathrm{fix}_A(a) &= \bigcup_{\pi \in \mathrm{fix}_A(a)} \prod_{i=0}^{\infty} \{\pi(i)\} \\
&= \bigcup_{\pi \in \mathrm{fix}_A(a)} \left(\prod_{i=0}^{\infty} A_i^{\pi} \right) \cap \mathrm{Aut}^f \quad \text{for } A_i^{\pi} \triangleq \begin{cases} \{\pi(i)\} & \text{if } i \in supp_A(a) \\ \mathbf{N} & \text{otherwise} \end{cases} \\
&= \left(\bigcup_{\pi \in \mathrm{fix}_A(a)} \prod_{i=0}^{\infty} A_i^{\pi} \right) \cap \mathrm{Aut}^f
\end{aligned}
$$

where the second equality holds because $fix_A(a) \subseteq sp_A(a)$ and by Lemma 12, while the latter expression clearly denotes an open set in Aut^f because each $\prod_{i=0}^{\infty} A_i^{\pi}$ is open in $\mathbf{N}^{\omega}$ since $supp_A(a)$ is finite and thus only finitely many A_i^{π}'s are not equal to $\mathbf{N}$.

On the other hand, let $(X, \cdot_X)$ be a continuous Aut^f-set; we prove that $\mathcal{X} = (X, \{\widehat{\pi}_X\})$ is in $FSAlg_p^f$. Clearly $\mathcal{X}$ is a finite permutation algebra. By Lemma 33, for any $x \in X$, $\mathrm{fix}_X(x)$ is an open set of Aut^f, hence $\mathrm{fix}_X(x) = U \cap \mathrm{Aut}^f$ for some U open set of $\mathbf{N}^{\omega}$. More explicitly, $\mathrm{fix}_X(x)$ can be written as

$$
\mathrm{fix}_X(x) = \left(\bigcup_{i \in I} \prod_{j=0}^{\infty} X_{ij} \right) \cap \mathrm{Aut}^f
$$

for some family of indexes I, and where for each $i \in I$ there exists a finite $J_i \subset \omega$ such that $X_{ij} \neq \mathbf{N}$ only for $j \in J_i$. Since $id \in \mathrm{fix}_X(x)$ (it is a group), there exists $i_0 \in I$ such that $id \in \prod_{j=0}^{\infty} X_{i_0 j}$. We prove that the finite set $J \triangleq J_{i_0}$ supports x. Let $\pi \in \mathrm{fix}_X(J) \cap \mathrm{Aut}^f$. For all $j \in \omega$, if $j \in J$ then $\pi(j) = j \in X_{i_0 j}$, otherwise $X_{i_0 j} = \mathbf{N}$. In both cases, $\pi(j) \in X_{i_0 j}$. So $\pi \in \prod_{j=0}^{\infty} X_{i_0 j}$, and therefore $\pi \in \mathrm{fix}_X(x)$, i.e. $\widehat{\pi}_X(x) = x \cdot_X \pi = x$, hence the thesis. □

Corollary 37. *Categories* $\mathbf{B}\mathrm{Aut}^f$ *and* $\mathrm{Sh}(\mathbb{I}^{op})$ *are equivalent.*

Actually, the proof of Proposition 13 suggests a direct proof of this last result. Corollary 37 can be proved along the same pattern of the argument following [14, 3.9, Theorem 2], just restricting to finite permutations. The argument works in the restricted case because any monomorphism $\beta : L \rightarrowtail K$ in $\mathbb{I}$ can be extended to a *finite kernel* isomorphism on $\mathbf{N}$, that is, to an object $\bar{\beta} \in \mathrm{Aut}^f$, for example as

$$
\bar{\beta}(i) \triangleq \begin{cases} \beta(i) & \text{if } i \in L \\ (i + 1 - j)\text{-th element of } \mathbf{N} \setminus \beta(L) & \text{otherwise,} \end{cases}
$$

where $j = |\{l \in L \mid l < i\}|$. Clearly $\bar{\beta}$ is a permutation, and it is easy to see that $|\ker(\bar{\beta})| \leq \max(L \cup K) + 1$, and hence it is finite. See [9] for a detailed description of this proof.

$$
\begin{tikzcd}
Alg_p \arrow[rr, hook] \arrow[dr, "\cong"] & & Alg_p^f \arrow[dr, "\cong"] & \\
& \mathbf{B}\mathrm{Aut}^{\delta} \arrow[rr, hook] \arrow[dd, shift left, "\dashv"'] & & \mathbf{B}\mathrm{Aut}^{f\delta} \arrow[dd, shift left, "\dashv"'] \\
FSAlg_p \arrow[uu, hook] \arrow[rr, "\cong"] \arrow[dr, "\cong"] & & FSAlg_p^f \cong NSet \arrow[uu, hook] \arrow[dr, "\cong"] & \\
& \mathbf{B}\mathrm{Aut} \cong \mathrm{Sh}(\mathbb{I}^{op}) \arrow[uu, hook, shift left] \arrow[rr, "\cong"] & & \mathbf{B}\mathrm{Aut}^{f} \arrow[uu, hook, shift left]
\end{tikzcd}
$$

Fig. 1 The permutation algebra cube

It is interesting to notice that both the inclusion functor $\mathbf{B}\mathrm{Aut} \hookrightarrow \mathbf{B}\mathrm{Aut}^{\delta}$ and its counterpart for finite permutations have a right adjoint; the latter is defined on the objects as follows

$$
\begin{aligned}
r : \mathbf{B}\mathrm{Aut}^{\delta} &\to \mathbf{B}\mathrm{Aut} \\
(X, \cdot_X) &\mapsto (\{x \in X \mid \mathrm{fix}(x) \text{ open for Aut}\}, \cdot_X)
\end{aligned}
$$

and it is the restriction on morphisms. Therefore, r maps every $\mathbf{B}\mathrm{Aut}^{\delta}$-set to the largest continuous $\mathbf{B}\mathrm{Aut}$-set contained in it. Translating r to permutation algebras along the equivalences, this is equivalent to the existence of the functor

$$
r' : Alg_p \to FSAlg_p \qquad (A, \{\hat{\pi}_A\}) \mapsto (B, \{\hat{\pi}_A|_B\})
$$

where $B \triangleq \{a \in A \mid \mathrm{fix}_A(a) \text{ open for Aut}\}$. Now, $\mathrm{fix}_A(a)$ is open iff there exists a finite $J \subset \omega$ such that for any π, if $\pi(i) = i$ for all $i \in J$ then $\pi \in \mathrm{fix}_A(a)$ (see the proof of Theorem 36). This corresponds exactly to say that a has finite support, hence we can define directly $r'(A) = \{a \in A \mid supp_A(a) \text{ finite}\}$.

6. Conclusions

In this paper we surveyed four main approaches to the treatment of nominal calculi. We compared models based on (pre)sheaf categories, on named sets, and on permutation algebras, which in turn subsume those approaches based on Fraenkel-Mostowski set theory (such as nominal sets, FM-sets and alike). We proved that the category of named sets is equivalent to the category of permutation algebras with finite support (either on the signature with all permutations or with only finite ones) which in turn is equivalent to the category of sheaves over $\mathbb{I}$, that is the Schanuel topos. Figure 1 summarizes these relationships. These results confirm that permutation algebras and named sets can be used as algebraic specification and "implementation version" of sheaves on the Schanuel topos. Moreover, these equivalences allow for "importing" into the category of finitely supported permutation algebras, the known algebra/coalgebra machinery and constructions of the sheaf category.

As future work, it would be interesting to investigate a suitable internal language for the models analyzed here. The connection with Fraenkel-Mostowski set theory, lead us to consider some variant (possibly higher-order) of Pitts' Nominal Logic [20], or the Theory of Contexts [11]. Interesting future work are also to investigate how, and under which conditions, we can extend the basic (finite) permutation signature with other operators and axioms; for instance, these operators may represent object language constructors, or other operations over names such as (non-injective) substitutions.

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