Draft (26 May 2003)
33 pages

An Introduction to Domain Theory
Notes for a Short Course

Michael Mislove [1]

Department of Mathematics, Tulane University, New Orleans, LA 70118

Abstract

This set of notes arises from a short course given at the Universitá Degli Studi Udine, given in May, 2003. The aim of the short course is to introduce both undergraduate and PhD students to domain theory, providing some of its history, as well as some of its most recent advances.

1 Outline of the Course

The aim of this course is to present some of the basic results about domain theory. This will include the historically most significant contribution of the theory – the use of domains to provide a model of the untyped λ-calculus of Church and Curry. The course will proceed with the following sections:

(i) Some aspects of the λ-calculus, for motivation.

(ii) Some results from universal algebra and category theory for background.

(iii) ω-complete partial orders.

(iv) Solving domain equations, and, in particular, building a model of the untyped λ-calculus.

The last part of the course will include lectures that survey additional portions of domain theory, including:

- Continuous domains and topology, where we discover alternative views of domains.
- Measurements, in which we see how to generalize some of the fundamental results of domain theory, and
- Probabilistic models, where we discover the domain-theoretic approach to probabilistic computation.

[1] The author gratefully acknowledges the support of the National Science Foundation and the US Office of Naval Research.

This outline is being set down at the beginning of the course, and it isn't clear how much of this program we actually will accomplish. We'll be back to visit the list at the end and to see how we did.

2 The untyped λ-calculus

The untyped λ-calculus was discovered in the 1930s by Alonzo Church. Unfortunately, his original formulation was inconsistent, but it was put right later that decade by Haskell Curry. Church's motivation for devising the theory was an attempt to understand how functions evaluate their arguments. His approach was to make functions first-class objects in the theory, instead of using the more standard mathematical approach of starting with sets and building in functions at the next stage. This enterprise was inspired by Hilbert's program to mechanize mathematics. The Tenth Problem in Hilbert's famous problem set delivered to the International Congress of Mathematicians in 1899 sought a procedure to decide whether any Diophantine equation with integral coefficients either had a solution or not. This was the forerunner of his attempt to mechanize all of mathematics, a program that Gödel showed was impossible by proving his Incompleteness Theorem in 1930. Nonetheless, Hilbert's ideas inspired numerous researchers to look at the foundations of mathematics, and to seek ways to understand how to devise what we now call algorithms for solving mathematical problems in a mechanical fashion.

2.1 The Syntax

The syntax of the untyped λ-calculus is given as

$$t ::= (c \mid)\ x \mid tt \mid \lambda x.t,$$

where

- $c \in Const$ is the optional component consisting of a set $Const$ of constants,
- $x \in V$ is from a set of variables,
- tt represents the application of the left term on the right term, application being understood as functional application, and
- $\lambda x.t$ abstracts the term t on the variable x, thus making it a function on terms.

While the aim of the theory is to capture functions very abstractly, the grammar given above is not sufficient to do this. For example, the whole theory could be trivial (modulo the constants, if they're included). So we add *conversion rules* which capture more precisely how functions operate, and, most importantly, which show how application and abstraction relate to one another. Some of the standard rules are:

(α) $\lambda x.M \equiv \lambda y.M[y/x]$, provided y doesn't appear in M.

(β) $(\lambda x.M)N = M[N/x]$.

(η) $\lambda x.Mx \equiv M$ if x is not free in M.

The first rule shows that changing the variable of an abstraction to a fresh one (i.e., one not already mentioned) doesn't change the meaning of a term. The second gives the most important relationship between application and abstraction: abstracting on a variable x and applying the result to a term M is the same thing as substituting N for x in the original term M. The last expresses the idea that functions are *extensional* – they are determined completely by how they act on valid inputs. This axiom often is omitted; we'll indicated what effect that has on the domains we'll consider that are related to the untyped λ-calculus.

Of course there is more than just these rules. A detailed presentation requires a discussion of free and bound variables, and a precise presentation about substitution. Our goal is only to use the λ-calculus as a motivation for domain theory, so we will elide these more precise details, referring the reader to an expert presentation on the λ-calculus.

This version of the λ-calculus – i.e., without any types, or, more properly, with just one type, was devised by Church and Curry in the 1930s, but it did not attract wide interest, One reason was that there were no known *mathematical* models of the theory, other than the one-point model in which everything collapses. Nonetheless, when Dana Scott went to Oxford in the 1960s, he found Christopher Strachey and his school at the Programming Research Group using the calculus as a model for understanding programming languages. Scott protested that the calculus was purely syntax, and without any models in which to interpret the terms as functions and the operations as they are intended, it was a dry and uninteresting model. Still, in the end, Scott found the first model of the calculus, which is what led to its attaining a prominent role in programming, and what has caused domain theory as well to play a prominent role in the underlying mathematical theory.

2.2 What's a Model?

Logics are only as good as the models that represent them – hence the desire for soundness and completeness theorems, which show that what's internally true of a logic is exactly what can be demonstrated about it in some model. For programming languages as well, models play a crucial roles, since they allow one to make deductions about programs without getting bogged down in the details of a particular implementation. In either area, there are three basic types of models:

(i) *Mathematical:* An mathematical object M that supports the operations of the syntax and satisfies the laws assumed.

(ii) *Operational:* A set of rules that describe how an abstract machine would reduce each term, step-by-step.

(iii) *Logical:* A model that regards terms as logical formulae and makes deductions of when two formulae are equivalent.

In programming language semantics, the first two types often are closely related, and the *Full Abstraction Problem* amounts to showing that a given mathematical model corresponds exactly to a given operational model. The third approach has led to the Curry-Howard Isomorphism, which states that deductions from one term to another are the same thing as the proof that the first term entails the other as a logical formula. We'll focus mainly on the first type of model, since it is here that domains arise in considering the untyped λ-calculus.

3 Background

Now that we've discussed the untyped λ-calculus, and named the three modeling approach we might apply to it, we begin by recalling some background we'll need. This concerns two areas: universal algebra and category theory. We begin with the former.

3.1 Universal algebra

Universal algebra concerns what abstract algebraic objects have in common. It takes a very general view of such an object, and proves basic results about its structure. Our discussion will be restricted to *single-sorted* algebras – ones where all the operations are defined on the same underlying set. But similar results hold for multi-sorted algebras.

Definition 3.1 A *signature* is an indexed family $\Omega = \cup_{n\in\mathbb{N}}\Omega_n$. For each $n \in \mathbb{N}$, $\omega \in \Omega_n$ means ω takes as input n values, and returns one output. In this case, ω is said to have *arity* n.

For example, our definition of the untyped λ-calculus is given in terms of the signature Ω, where $\Omega_0 = C \cup V$, $\Omega_1 = \{\lambda x.- \mid x \in V\}$, $\Omega_2 = \{\cdot\}$ denotes application, and $\Omega_n = \emptyset$ for $n \geq 3$.

Definition 3.2 If Ω is a signature, then an Ω-algebra is a non-empty set S together with interpretations $\omega_S \colon S^{arr(\omega)} \to S$, where $arr(\omega)$ is the arity of ω.

Definition 3.3 If Ω is a signature, we can define the *term algebra* for Ω as $\mathcal{T}_\Omega = \cup_{n\in\mathbb{N}}\mathcal{T}_n$, where

- $\mathcal{T}_0 = \Omega_0$, the nullary operators of Ω, and
- $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{(\omega, t_1, \cdots, t_m) \mid \omega \in \Omega_m \ \&\ t_1 \ldots, t_m \in \mathcal{T}_n\}$.

Proposition 3.4 *For each signature Ω, $\mathcal{T}_\Omega$ is an Ω-algebra.*

Proof. We show that $\omega \colon \mathcal{T}_\Omega^{arr(\omega)} \to \mathcal{T}_\Omega$ is well-defined, for each $\omega \in \Omega$. Since $\Omega = \cup_m\Omega_m$, there is some m for which $\omega \in \Omega_m$.

If $\omega \in \Omega_0$, then ω is a nullary (i.e., constant) operator, and $\Omega_0 = \mathcal{T}_0 \subseteq \mathcal{T}_\Omega$ by definition.

Suppose that $\omega \in \Omega_m$, and let $t_1, \ldots, t_m \in \mathcal{T}_\Omega$. Then there is some m_i with $t_i \in \mathcal{T}_{m_i}$ for each $i = 1, \ldots, m$, and since $T_\Omega = \cup_n \mathcal{T}_n$ is an increasing union, there is some M with $t_i \in \mathcal{T}_M$ for each $i = 1, \ldots, m$. The definition of $\mathcal{T}_{M+1}$ implies $(\omega, t_1, \ldots, t_m) \in \mathcal{T}_{M+1}$, as required. □

Exercise 3.5 (i) In Proposition 3.4, we showed that each operator from Ω is defined at all appropriate inputs from Ω, but we didn't show that the operators on $\mathcal{T}_\Omega$ are well-defined. That is, for an operator $\omega \in \Omega$ and inputs $t_1, \ldots, t_m \in \mathcal{T}_\Omega$, where $arr(\omega) = m$, we didn't show there is only one possible value for $\omega(t_1, \ldots, t_m)$ in $\mathcal{T}_\Omega$. Show this.

(ii) If we let $\Omega_0 = \{0\}$, $\Omega_1 = \{s\}$ and all other $\Omega_n = \emptyset$, then show that the Ω-algebra $\mathcal{T}_\Omega$ is ω, the first infinite ordinal.

(iii) For an alphabet A (i.e., a non-empty set), a *string* over A is just a finite sequence $a_1 a_2 \cdots a_n$, where $a_1, \ldots, a_n \in A$ (note: the a_is need not be distinct). Two strings $s = a_1 \cdots a_m$ and $t = a'_1 \cdots a'_n$ can be concatenated, as in $s \cdot t = a_1 \cdots a_n a'_1 \cdots a'_m$. Give a signature Ω for the set of finite strings over the set A and concatenation as an operation.

Definition 3.6 Let Ω be a signature, and let S and T be Ω-algebras. The a map $f\colon S \to T$ is an Ω-*algebra homomorphism* if

$$f(\omega_S(s_1, \ldots, s_m)) = \omega_T(f(s_1), \ldots, f(s_m))$$

for every $\omega \in \Omega$, and for every $(s_1, \ldots, s_m) \in S$, where $m = arr(\omega)$.

Theorem 3.7 (Fundamental Theorem) *If Ω is a signature, then $\mathcal{T}_\Omega$ is the initial Ω-algebra: for any Ω-algebra S, there is a unique Ω-algebra map $\phi_S\colon \mathcal{T}_\Omega \to S$.*

Proof. We proceed to show that ϕ_S exists by induction on n, where $\mathcal{T}_\Omega = \cup_n \mathcal{T}_n$. Since S is an Ω-algebra, each $\omega \in \Omega_0$ has an interpretation, ω_S in S. So, we define $\phi(\omega) = \omega_S$ for each $\omega \in \Omega_0 = \mathcal{T}_0$.

Suppose that $\phi\colon \mathcal{T}_n \to S$ is defined; we can then extend $\phi\colon \mathcal{T}_{n+1} \to S$ by defining $\phi(\omega, t_1, \ldots, t_m) = \omega_S(\phi(t_1), \ldots, \phi(t_m))$, for each $\omega \in \Omega_m$ and elements $t_1, \ldots, t_n \in \mathcal{T}_n$. By induction, $\phi\colon \mathcal{T}_\Omega \to S$ is defined, and it is clearly an Ω-algebra homomorphism by definition. That ϕ is unique is again an inductive argument, based on the $\mathcal{T}_n$'s, which we leave as an exercise. □

Corollary 3.8 (Structural Induction) *Let Ω be a signature, and define an Ω-algebra structure on $\mathbb{N}$ by*

- $\omega_{\mathbb{N}} = 0$ $(\forall \omega \in \Omega_0$, *and*
- $\omega_{\mathbb{N}}\colon \mathbb{N}^m \to \mathbb{N}$ by $\omega_{\mathbb{N}}(n_1, \ldots, n_m) = \max\{n_1, \ldots, n_m\} + 1$ $(\forall \omega \in \Omega_m)$.

Then there is a unique Ω-algebra map rank: $\mathcal{T}_\Omega \to \mathbb{N}$. □

Exercise 3.9 Use Corollary 3.7 to define a rank function for the set of finite strings over a set A, and show that this rank function gives exactly the length n of a string $s = a_1 \cdots a_n$.

Remark 3.10 We note that although everything we have done is for *anomic* algebras (i.e., ones without any laws or equations), the same theory holds for Ω-algebras with laws and (in)equations imposed on them.

Given a set of equations E, an (Ω, E)-algebra is an Ω-algebra for which all the equations in E are satisfied, and the initial (Ω, E)-algebra $\mathcal{T}_{(\Omega,E)}$ is the Ω-algebra quotient of $\mathcal{T}_\Omega$ by the congruence generated by the equations in E. Another way to arrive at the same thing is to take the quotient of $\mathcal{T}_\Omega$ by the congruence generated by Ω-algebra maps into (Ω, E)-algebras.

When inequations are included, then one has to add a partial order to the definition of an Ω-algebra, with the proviso that all operations $\omega \in \Omega$ are monotone with respect to this order. The construction then proceeds as in the case or (Ω, E)-algebras.

3.2 The λ-calculus revisited

We have seen that universal algebras can be used to generate term models for a given signature, and that the term model is initial. So, we know there is a term model for the untyped lambda calculus, even one that satisfies the laws given as (α) and (β). And the point of Scott's objection now can be appreciated more fully: without any models of this theory consisting of sets and functions, it wasn't clear whether the term model might be the only non-degenerate model of the theory. Let's consider more precisely what a model of the lambda calculus within sets and functions would have to be like.

We want a mathematical object – for now, let's just say a set – M with some interesting properties.[2] First, we expect that M has an operator $\cdot\colon M \times M \to M$ that represents application in the λ-calculus. This implies there is a mapping $p\colon M \to [M \to M]$, the space of selfmaps of M, such that $m \cdot m' = p(m)(m')$. Moreover, since selfmaps of M are supposed to be terms of M, then we also expect that there is a mapping $\iota\colon [M \to M] \to M$ which allows us to interpret selfmaps of M as inputs for (other) selfmaps of M. Moreover, since the term in M representing the function $f\colon M \to M$ is unique, we expect that

$$p \circ \iota\colon M \to M \text{ satisfies } p \circ \iota = 1_{[M \to M]}.$$

Dually, if we assume the extensionality rule (η), then two elements of M

[2] Throughout this discussion, we also assume that all elements of M are *denotable*, by which we mean they all lie in the image of the mapping from Λ to M. It turns out this is an unrealistic assumption, but it simplifies our discussion considerably at this point; when we have built up the necessary background, we'll weaken this assumption to a more realistic one.

cannot be represented by the same selfmap, and so we also expect

$$\iota \circ p \colon M \to M \text{ satisfies } \iota \circ p = 1_M.$$

In fact, there should be an Ω-algebra map $\phi\colon \Lambda \to M$ so that the following diagram commutes:

$$\begin{array}{ccc} \Lambda \times \Lambda & \xrightarrow{\Phi \times \phi} & [M \to M] \times M \\ \Big\downarrow \cdot & & \Big\downarrow \text{app} \\ \Lambda & \xrightarrow[\phi]{} & M \end{array}$$

where $\Phi = p \circ \phi$. The point is that we are now dealing not just with the set M, but also with its set of selfmaps. This is best understood if we bring some category theory to bear on the situation.

3.3 Category Theory

Our discussion of what we expect a model of the untyped λ-calculus to be has led us outside the realm of universal algebra to a setting in which we need to consider both objects and their space of selfmaps. This setting is best seen from the viewpoint of category theory, so we now introduce some of the basics of this area.

Definition 3.11 A *category* C consists of a collection $\operatorname{obj}\mathsf{C}$ of *objects* of C, and for each $C, C' \in \operatorname{obj}\mathsf{C}$ a set of *morphisms* $\mathsf{C}(C, C')$ which satisfy the following properties:

- For each $C \in \operatorname{obj}\mathsf{C}$, there is a distinguished morphism $1_C \in \mathsf{C}(C, C)$,
- For each $C, C', C'' \in \operatorname{obj}\mathsf{C}$, there is a mapping $\circ\colon \mathsf{C}(C, C') \times \mathsf{C}(C', C'') \to \mathsf{C}(C, C'')$ which is associative where it is defined, and
- If $C, C' \in \operatorname{obj}\mathsf{C}$ and $f \in \mathsf{C}(C, C')$, then

$$1_{C'} \circ f = f = f \circ 1_C.$$

Example 3.12 Some examples of categories are:

(i) Set, the category whose objects are sets, and whose morphisms are the functions between sets.

(ii) $p\mathsf{Set}$, whose objects are sets, and whose morphisms are *partial maps* between sets: a *partial map* $f\colon A \rightharpoonup B$ from the set A to the set B is a function whose domain of definition is a subset of A. For example, the function $f(x) = 1/x$ is a partial map from $\mathbb{R}$ to itself – it's domain of definition is the set of non-zero reals.

(iii) Let $(P, \leq)$ be a partially ordered set. Then $\leq$ is a reflexive, antisymmetric and transitive relation on P. We make P into a category C_P whose objects are the elements of P, and for each $x, y \in P$, we define

$$\mathsf{C}_P(x,y) = \begin{cases} \{x \to y\}, & \text{if } x \leq y, \text{ and} \\ \emptyset, & \text{otherwise.} \end{cases}$$

(iv) Pos, the category of partially ordered sets, and *monotone* maps – i.e., the morphisms are those mappings $f\colon P \to Q$ between partially ordered sets P and Q that preserve the order: if $x \leq_P y$, then $f(x) \leq_Q f(y)$.

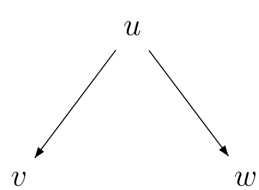
Exercise 3.13 Show that each of the examples above satisfies the conditions for being a category. In particular, show that each pair of objects in the category has a set of morphisms between them, verify that $\circ$ is associative where defined, and identify the identity map 1_C for each object C,

Definition 3.14 A *diagram* in a category C is a function $\Delta\colon D \to \mathsf{C}$ where $D = (V, E)$ is a directed graph, and Δ maps vertices $v \in V$ to objects $\Delta(v) \in \operatorname{obj} \mathsf{C}$, and edges $e = v \to w \in E$ to morphisms $\Delta(e) \in \mathsf{C}(\Delta(v, \Delta(w))$.

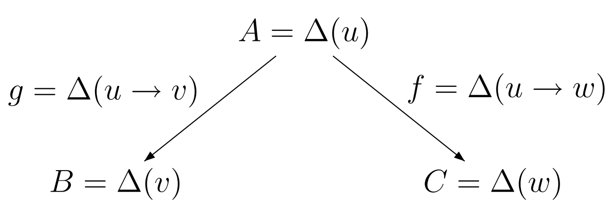
The diagram $\Delta\colon D \to \mathsf{C}$ is said to *commute* if, for all $u, v, w \in V$ if $e = u \to v \in E, e' = v \to w \in E$ and $e'' = u \to w \in E$, then $\Delta(e') \circ \Delta(e) = \Delta(e'')$.

Example 3.15 Here are some examples of diagrams in the category Set:

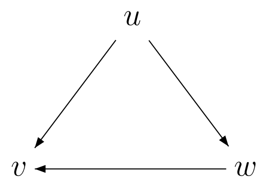
(i) Consider the diagram:



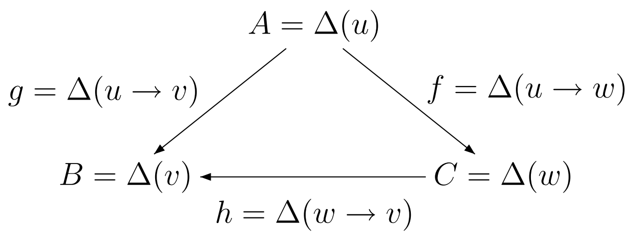
Then, its image in Set would be a diagram



(ii) Consider the diagram:



Then, its image in Set would be a diagram



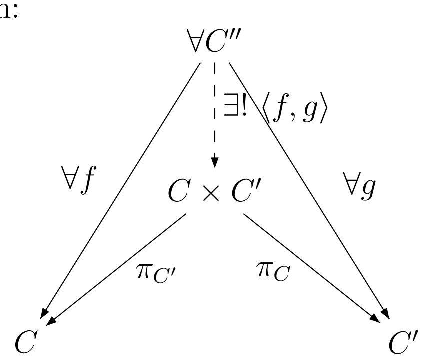
This diagram commutes iff $h \circ f = g$

Exercise 3.16 Let Ω be a signature. Give a diagram whose image in the category Ω-Alg of Ω-algebras and Ω-algebra homomorphisms expresses the fact that, for any Ω-algebra S, there is an Ω-algebra homomorphism from $\mathcal{T}_\Omega$ to S.

Note that we don't have a way to express uniqueness of this map, as the Fundamental Theorem tells us is true.

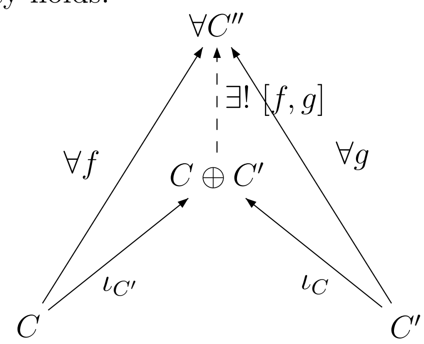
Diagrams allow us to express succinctly many desirable properties in category theory.

Definition 3.17 A category C *has (finite) products* if for every pair of objects $C, C' \in \text{obj}\,\mathsf{C}$, there is an object $C \times C' \in \text{obj}\,\mathsf{C}$ and mappings $\pi_C : C \times C' \to C$ and $\pi_{C'} : C \times C' \to C'$ with the universal property expressed by the following diagram:



In category theory, *dualizing* is a fundamental operation. Given a notion, the dual is obtained by reversing the arrows. For example:

Definition 3.18 A category C *has (finite) coproducts* if, for every pair of objects $C, C' \in \mathrm{obj}\,\mathsf{C}$, there is an object $C \oplus C' \in \mathrm{obj}\,\mathsf{C}$ and morphisms $\iota_C \colon C \to C \oplus C'$ and $\iota_{C'} \colon C' \to C \oplus C'$ such that the following universal property holds:



Exercise 3.19 (i) For each of the categories in Example 3.11, indicate whether finite products exist. In the case of the category C_P for a poset P, give necessary and sufficient conditions for products to exist.

(ii) Likewise, for each of the categories in Example 3.11, indicate whether finite coproducts exist. In the case of the category C_P for a poset P, give necessary and sufficient conditions for coproducts to exist.

(iii) Show that finite products are unique whenever they exist. Similarly, show that finite coproducts are unique if they exist.

Finally, we define two further dual notions.

Definition 3.20 An object $\mathsf{T} \in \mathsf{C}$ in a category C is *terminal* if for each object $C \in \mathrm{obj}\,\mathsf{C}$, there is a unique morphism $!_C \colon C \to \mathsf{T}$. Dually, the object $\mathrm{I} \in \mathsf{C}$ is *initial* if, for each object $C \in \mathrm{obj}\,\mathsf{C}$, there is a unique morphism $\iota_C \colon \mathrm{I} \to C$.

Example 3.21 For example, Theorem 3.7 shows that the category Ω-Alg has initial objects.

Exercise 3.22 (i) For each of the categories in Example 3.11, indicate whether the category has an initial object, and whether it has a terminal object. In each case where either exists, indicate what it is.

(ii) For P a partial order, give necessary and sufficient conditions for C_P to have an initial object, and, dually, necessary and sufficient conditions for C_P to have a terminal object.

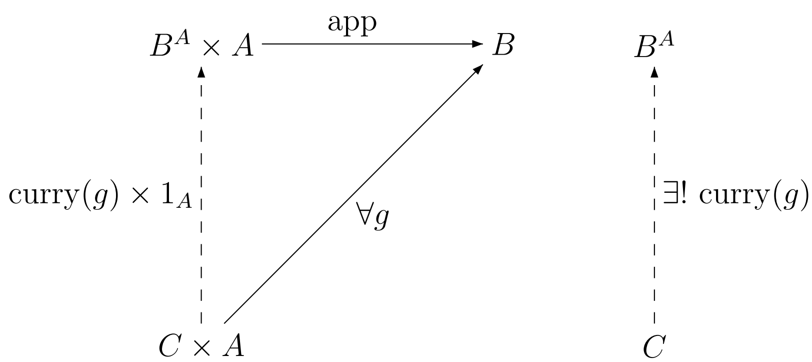
(iii) For any category C, show that initial objects and terminal objects are unique, if they exist.

3.4 Cartesian Closed Categories

Our next topic is rather advanced, but it contains a fundamental ingredient we need to model the λ-calculus.

Definition 3.23 The category C is *cartesian closed* if

- C has finite products,
- C has a terminal object, and
- $(\forall A, B \in \operatorname{obj} \mathsf{C})(\exists B^A \in \operatorname{obj} \mathsf{C})(\exists\, \mathrm{app} \in \mathsf{C}(B^A \times A, A))$ satisfying: $(\forall C \in \operatorname{obj} \mathsf{C})(\forall g \in \mathsf{C}(C \times B \to A))(\exists!\ \mathrm{curry}(g) \colon C \to B^A)$ so that the following commutes:



Example 3.24 • Set is cartesian closed.

- Pos is cartesian closed.
- If P is a poset, then C_P is a ccc iff P has finite meets, a largest element, and, for each $x, y \in P$, $x \Rightarrow y \equiv \vee\{z \in P \mid x \wedge z \leq y\}$ exists.
- Let V denote the category of finite dimensional vector spaces over $\mathbb{R}$ and linear maps between them. Show that V is cartesian closed:
  · V has finite products,
  · the 0-dimensional space $\{0\}$ is a terminal object for V, and
  · The internal hom is just the family of linear maps between the spaces – i.e., $W^V = \mathsf{V}(W, V) \in \mathsf{V}$..

  Also characterize the constant maps in $\mathsf{V}(W, V)$ for any vector spaces W and V.

Exercise 3.25 Show that each of the examples listed is a ccc.

We now list a number of results will be useful in what follows.

Exercise 3.26 Let C be a cartesian closed category, and let $A, B, C \in \operatorname{obj} \mathsf{C}$. Then,

(i) If $f, g \in C \to \mathsf{C}(A, B)$ and $\mathrm{app} \circ (f \times 1_A) = \mathrm{app} \circ (g \times 1_A)$, then $f = g$.

(ii) Taking $C = B$ in (i), and considering $\pi_B \colon B \times A \to B$, we get a unique mapping $\kappa^A_B \colon B \to B^A$ satisfying $\mathrm{app} \circ (\kappa^A_B \times 1_A) = \pi_B$. This is the so-called *constant picker* which associates to each $x \in B$ the constant map $a \mapsto x \colon A \to B$.

(iii) We next consider

$$\mathrm{app} \circ (1_{C^B} \times \mathrm{app}) \colon C^B \times (B^A \times A) \to C$$

and apply the universal property to define a unique map

$$\circ_{CBA}\colon C^B \times B^A \to C^A.$$

Show this is well-defined, and that, for any $D \in \operatorname{obj} \mathsf{C}$, we have

$$\circ_{D(CBA)}\colon D^C \times (C^B \times B^A) \to D^A$$

and

$$\circ_{(DCB)A}\colon (D^C \times C^B) \times B^A \to D^A$$

are the same.

Reflexive Objects

We now define those objects in a cartesian closed category that can serve as models for the untyped λ-calculus.

Definition 3.27 If C is a ccc, and object $C \in \operatorname{obj} \mathsf{C}$ is *reflexive* if

$$(\exists \iota\colon C^C \to C)(\exists r\colon C \to C^C)\ r \circ \iota = 1_{C^C}.$$

Note that this implies that ι is monic.

For example, any singleton set $\{x\}$ is reflexive in Set and in Pos (when endowed with the discrete order). However,

Proposition 3.28 (i) *If A is a set with more than one element, then A is not reflexive in* Set*.*

(ii) *Likewise, if P is a poset with more than one element, then P is not reflexive in* Pos*.*

Proof. For (i), we first note that a set A with $|A| \geq 2$ has the property that there is an injection $\phi\colon 2 = \{0, 1\} \to A$. Then, we define a mapping $\Phi\colon 2^A \to A^A$ by $\Phi(f) = \phi \circ f$. We claim that Φ is injective. Indeed, if $f \neq g\colon A \to 2$ are given, then for some $a \in A$, $f(a) \neq g(a)$. Since ϕ is injective, $\phi(f(a)) \neq \phi(g(a))$, so $\Phi(f) = \phi \circ f \neq \phi \circ g = \Phi(g)$.

Now, since there is an injection of 2^A into A^A, it follows that $|A^A| \geq |2^A| > |A|$. So, there cannot be an injection of A^A into A, and this means A cannot be reflexive in Set.

A similar argument can be made for Pos, relying on the fact, due to Gleason and Dilworth, that any poset with more than one element has the property that there is no injection of its set of lower sets into itself (this is the analog of Cantor's Lemma for sets in the category Pos). $\square$

4 Properties of a Model

We now compile some results about models of the untyped λ-calculus in a cartesian closed category. The main result will be to show that reflexive objects, under suitable assumptions, can serve as models.

4.1 Combinatory Results

We need some results about combinators in the λ-calculus. Recall the syntax of the untyped λ-calculus is given by

$$t ::= c \mid x \mid tt \mid \lambda x.t,$$

where $c \in C$ is a prescribed family of constants, and $x \in V$ ranges over a set of variables. Recall also the following reduction rules:

(α) $\lambda x.m \equiv \lambda y.m[y/x]$ if y is not free in m.

(β) $(\lambda x.m)n \equiv m[n/x]$

(η) $\lambda x.mx \equiv m$ if x is not free in m.

The following are standard combinators of the λ-calculus:

$K \equiv \lambda xy.x$

$S \equiv \lambda xyz.xz(yz)$

$Y \equiv \lambda f.(\lambda v.f(vv))(\lambda v.f(vv))$

$I \equiv \lambda x.x$

Proposition 4.1 *Assuming* (β)*:*

(i) $(SK)K \equiv I$*, so* S *and* K *generate.*

(ii) *For any term* f*,* $f(Yf) \equiv YF$*. This means every term has a fixed point in a model of the untyped* λ*-calculus that validates* (β)*.*

Proof. This is an exercise. □

The untyped λ-calculus also admits an internal definition of composition, as a derived operator.

Definition 4.2 Define $\circ \colon \Lambda \times \Lambda \to \Lambda$ by $\circ(m, n) = \lambda x.m(nx)$, where x is not free in m or n. We also write $m \circ n$ for $\circ(m, n)$.

Exercise 4.3 Show that, assuming (β), then $(m \circ n)p = m(np)$.

Exercise 4.4 An *ultra-metric* on a set X is a function $d \colon X \times X \to X$ satisfying, for all $x, y, z \in X$:

- $d(x, y) = 0$ iff $x = y$,
- $d(x, y) = d(y, x)$, and
- $d(x, z) = \max\{d(x, y), d(y, z)\}$

Recall that a *Cauchy sequence* in a metric space is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that, for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ with $d(x_m, x_n) < \epsilon$ $(\forall m, n \geq N)$. Further, a metric space is *complete* if each Cauchy sequence has a limit point.

Finally, a mapping $f\colon X \to Y$ between metric spaces is *non-expansive* if $d_Y(f(x), f(x')) \leq d(x, x')$ for all $x, x' \in X$.

Show that the category UMet of complete ultra-metric spaces and non-expansive maps is cartesian closed, where we define

$$d(f, g) = \sup\{|d(f(x), g(x))| \mid x \in X\}.$$

Furthermore, show that for any non-degenerate $X \in \operatorname{obj} \mathsf{UMet}$, there is a non-expansive selfmap $f\colon X \to X$ without a fixed point. This observation is due to Plotkin.
(Hint: show that, for any $\epsilon > 0$, $B_\epsilon(x) = \{y \mid d(x, y) \leq \epsilon\}$ is open and closed, for every $x \in X$. Conclude that the paradoxical combinator $Y \equiv \lambda f.(\lambda v, f(vv))(\lambda v.f(vv))$ has no interpretation in UMet.)

4.2 Semantic Environments

If we are given a term m with free variables, then we need to have values to assign to them if we want to evaluate the term. This role is played by environments in a model.

We begin with the assumption that we have a reflexive object $C \in \mathsf{C}$ in a cartesian closed category with (at least) finite products. We also assume that we are given an interpretation $c \mapsto [\![c]\!] \in C$ of each constant c in our syntax. We let C^V denote V-many copies of C; we note that if C has only finite products (it at least has these since it is a ccc), then this only makes sense if V is finite. We won't see any problem here – it lies in the fact that substitution is problematic in this case.

Definition 4.5 A *semantic environment* is an assignment $\sigma\colon V \to C$, assigning to each variable an element of C. Thus, C^V denotes the set of semantic environments.

Definition 4.6 Let $x \in C$, $v \in V$, and let $\sigma \in C^V$. Then we define a new environment $\sigma\{x/v\}$ by

$$\sigma\{x/v\}(w) = \begin{cases} \sigma(w), & \text{if } w \neq v, \\ x, & \text{if } w = v. \end{cases}$$

Thus, $\sigma\{x/v\}$ replaces σ's value at v with x.

Lemma 4.7 *Let $x, y \in C$, $u, v \in V$ and let $\sigma \in C^V$. Then*

$$\sigma\{x/u\}\{y/v\} = \begin{cases} \sigma\{y/v\}\{x/u\}, & \text{if } u \neq v, \\ \sigma\{y/v\}, & \text{if } u = v. \end{cases}$$

Proof. Exercise. □

This shows that changing the values of an environment in some of its variables acts as expected. The question is whether this can be done using morphisms in the category C. Certainly, for those variables $w \neq v$, $\sigma\{x/v\}(w) = \sigma(w) = \pi_w(\sigma)$ is a morphism of C, since C has finite products. The other component – v – requires more work.

Proposition 4.8 *If $C \in \mathrm{obj}\,\mathsf{C}$ is a reflexive object in a cartesian closed category and $x \in C$, then $\kappa_x \colon C \to C$ by $\kappa_x(y) = x$ satisfies $\kappa_x \in C^C$.*

Proof. This is just Exercise 3.26. □

Corollary 4.9 *Let $C \in \mathrm{obj}\,\mathsf{C}$ be a reflexive object in the cartesian closed category C. Then the a mapping $\Phi \colon C \times V \times C^V \to C$ by $\Phi(x, v, \sigma) = \sigma\{x/v\}$ is in C.*

Proof.

$$\Phi(x, v, \sigma)(w) = \sigma\{x/v\}(w) = \begin{cases} \sigma(w) & \text{if } w \neq v, \\ x, & \text{if } w = v, \end{cases} = \begin{cases} \pi_w(\sigma), \text{if } w \neq v, \\ \kappa_x(\pi_v(\sigma)), \text{if } w = v. \end{cases}$$

□

Recall that the *pure untyped λ-calculus* has no constants – only variables, application and abstraction.

Theorem 4.10 *If $C \in \mathrm{obj}\,\mathsf{C}$ is a reflexive object in the cartesian closed category C, then $\mathsf{C}(C^V, C)$ is a model of the pure untyped λ-calculus.*

If there is an interpretation $c \mapsto [\![c]\!] \colon Const \to C$ of the constants in C, then $\mathsf{C}(C^V, C)$ is a model of the untyped λ-calculus with constants $Const$.

Proof. We regard the untyped λ-calculus as the initial algebra having the signature given by the syntax we have used to define the terms of the calculus. Theorem 3.7 then tells us it is sufficient to define interpretations of the operators of the calculus in $\mathsf{C}(C^V, C)$ in order to guarantee a unique algebra homomorphism from Λ to $\mathsf{C}(C^V, C)$. We define these interpretations as follows:

- For $c \in Const$ a constant, $[\![c]\!](\sigma) = [\![c]\!] = \kappa_{[\![c]\!]} \circ \sigma$.
- For $v \in V$, $[\![v]\!] \colon C^V \to C$ by $[\![v]\!](\sigma) = \sigma(v) = \pi_v(\sigma) \in C$.
- For terms $t, t' \in \Lambda$, $[\![t \cdot t']\!] \colon C^V \to C$ by $[\![t \cdot t']\!](\sigma) = [\![t]\!](\sigma) \cdot_C [\![t']\!](\sigma) \in C$, using the application operator $\cdot_C \colon C \times C \to C$.
- For $v \in V$ and $f \in \Lambda$, $\lambda v.f \colon C^V \to C$ is given by

$$(\lambda v.f)(\sigma) = j(x \mapsto [\![f]\!](\sigma\{x/v\})) = j([\![f]\!] \circ \Phi(-, v, \sigma)).$$

□

4.3 Functors and Natural Transformations

We now continue our discussion of category theory.

Definition 4.11 Let C and D be categories. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ is a family of mappings $F\colon \operatorname{obj}\mathsf{C} \to \operatorname{obj}\mathsf{D}$ and, for each $C, C' \in \operatorname{obj}\mathsf{C}$, a mapping $F\colon \mathsf{C}(C, C') \to \mathsf{D}(F(C), F(C''))$ satisfying

- $F(1_C) = 1_{F(C)}$ for each $C \in \operatorname{obj}\mathsf{C}$, and
- If $f \in \mathsf{C}(C, C')$ and $g \in \mathsf{C}(C', C'')$, then $F(g \circ f) = F(g) \circ F(f)$.

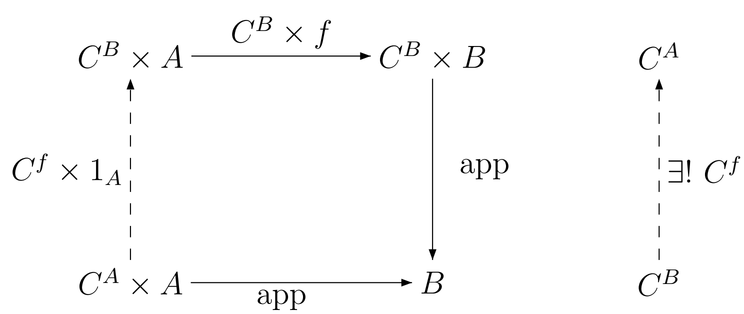
Example 4.12 (i) Let Grp denote the category of groups and group homomorphisms. Then there is a functor $|\ |\colon \mathsf{Grp} \to \mathsf{Set}$ which forgets the group structure on each group and simply remembers that a group is a set, and that a homomorphism between groups is a function between the underlying sets.

(ii) $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Pos}^{\mathrm{op}}$ by $\mathcal{P}(X)$ is the power set of X, and for $f\colon X \to Y$, $\mathcal{P}(f)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ is $\mathcal{P}(f)(A) = f^{-1}(A)$. This is an example of a *contravariant* functor: it reverses the direction of the morphisms. Hence the need to label Pos with the superscript op.

Exercise 4.13 Recall Exercise 3.26. The setting is a cartesian closed category C. We extend it with the following observations:

(i) If $C \in \operatorname{obj}\mathsf{C}$, then there is an endofunctor $\mathsf{C}_C\colon \mathsf{C} \to \mathsf{C}$ given by $\mathsf{C}_C(A) = A^C$ for $A \in \operatorname{obj}\mathsf{C}$, and $\mathsf{C}_C\colon \mathsf{C}(A, B) \to \mathsf{C}(A^C, B^C)$ by $\mathsf{C}_C(f) = f \circ -$.

(ii) Show that each morphism $f \in \mathsf{C}(A, B)$ and each object $C \in \mathsf{C}$ define a morphism $C^f\colon C^B \to C^A$ using the diagram:



(iii) Now use (ii) to show the following: if $C \in \operatorname{obj}\mathsf{C}$, then there is a contravariant functor $\mathsf{C}^C\colon \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ given by $\mathsf{C}^C(A) = C^A$ and $\mathsf{C}^C\colon \mathsf{C}(A, B) \to \mathsf{C}(C^B, C^A)$ by $\mathsf{C}^A(f) = - \circ f$, for $f \in \mathsf{C}(A, B)$.

Just as we use functors to relate categories, we want a way to relate functors.

Definition 4.14 If $F, G\colon \mathsf{C} \to D$ are functors, then a *natural transformation* from F to G, denoted $\eta\colon F \stackrel{\cdot}{\longrightarrow} G$ is a family of morphisms $\eta_C \in \mathsf{D}(F(C), G(C))$, one for each $C \in \operatorname{obj}\mathsf{C}$, such that the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ {\scriptstyle F(C)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\eta_{C'}]{} & F(C') \end{array}$$

Example 4.15 (i) Define $T\colon \mathsf{Grp} \to \mathsf{Set}$ by $T(G) = \{e_G\}$, the identity element of G, and $T(h)\colon TG \to T(H)$ by $T(h)(e_G) = e_H$. Then there is a natural transformation $\eta\colon | \stackrel{\cdot}{\longrightarrow} T$ (where $|\ |$ is the functor given in Example 4.12) given by $\eta_G\colon |G| \to \{e_G\}$ is the constant map.

(ii) Let C be a ccc, and let $C \in \mathrm{obj}\,\mathsf{C}$. Define $G_C\colon C \to C$ by $G_C(A) = A^C \times C$, and for $f \in \mathsf{C}(A, B)$, $G_C(f) \in \mathsf{C}(A^C \times A, B^C \times C)$ by $G_C(f) = f \circ -$. Then $\eta\colon G \stackrel{\cdot}{\longrightarrow} I$ by $\eta_A = \mathrm{app}\colon A^C \times C \to A$ is a natural transformation, where I denotes the identity functor.

Exercise 4.16 Verify that both of the examples listed in the Example are indeed natural transformations.

4.4 Adjunctions

Definition 4.17 An *adjunction* between a pair of categories C and D is a pair of functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ for which

- there is a natural transformation $\eta\colon I \stackrel{\cdot}{\longrightarrow} G \circ F$, and
- $(\forall f \in \mathsf{C}(C, GD)(\exists! \widehat{f} \in \mathsf{D}(FC, D))$ satisfying $G(\widehat{f}) \circ \eta_C = f$:

Exercise 4.18 Let $\Omega = \cup_n \Omega_n$ be a signature, and assume $\Omega_0 = \emptyset$. Define a family of signatures Ω^X by $\Omega^X = \cup_n \Omega_n^X$, where $\Omega_0^X = X$, and $\Omega_n^X = \Omega_n$ for $n \geq 1$. Let Ω-Alg denote the category of all Ω^X-algebras and all Ω^X-algebra homomorphisms. Define a functor $\mathcal{T}\colon \mathsf{Set} \to \Omega\text{-}\mathsf{Alg}$ by $\mathcal{T}(X) = T_{\Omega^X}$, the initial Ω^X-algebra. If $f\colon X \to Y$ is a function, then Theorem 3.7 guarantees f extends to an Ω-algebra map from T_{Ω^X} to T_{Ω^Y}.

Dually, there is a functor $G\colon \Omega\text{-}\mathsf{Alg} \to \mathsf{Set}$ which associates to an Ω-algebra its set Ω_0 of nullary operators. We can define $\eta\colon I \stackrel{\cdot}{\longrightarrow} GF$ by $\eta_X(x) = x \in G(T_{\Omega^X})$, which sends the element $x \in X$ to its interpretation as a constant in T_{Ω^X}.

Theorem 3.7 applies again to show that this defines an adjunction between Set and Ω-Alg.

The following greatly simplifies the process of showing that an adjunction exists.

Theorem 4.19 *Let $G\colon \mathsf{D} \to \mathsf{C}$ be a functor. Suppose there are:*

- *A mapping of objects* $F\colon \mathrm{obj}\,\mathsf{C} \to \mathrm{obj}\,\mathsf{D}$*,*
- *For each* $C \in \mathrm{obj}\,\mathsf{C}$*, a morphism* $\eta_C\colon C \to GF(C))$*, and*
- *For each* $D \in \mathrm{obj}\,\mathsf{D}$ *and each* $f \in \mathsf{C}(C, GD)$*, a unique morphism* $\widehat{f} \in \mathsf{D}(FC, D)$ *satisfying* $f = \widehat{f} \circ \eta_C$*.*

Then $F\colon \mathsf{C} \to \mathsf{D}$ *extends uniquely to a functor satisfying* (F, G) *is an adjunction with natural transformation* $\eta\colon I \stackrel{\cdot}{\longrightarrow} GF$*.*

Example 4.20 The Theorem eliminates some of the work from the previous exercise: it allows us to conclude that $\mathcal{T}_\Omega\colon \mathsf{Set} \to \Omega\text{-}\mathsf{Alg}$ is a functor just on the basis of Theorem 3.7.

4.5 Cones, Limit Cones and Colimit Cocones

Definition 4.21 Let $\Delta\colon D \to \mathsf{C}$ be a diagram in a category C. A *cone* for Δ consists of an object $C \in \mathrm{obj}\,\mathsf{C}$ and, for each $x \in D$, a morphism $f_x \in \mathsf{C}(C, \Delta(x))$ satisfying $\Delta(x \to y) \circ f_x = f_y$ for $x, y] \in D$ with $x \to y \in D$.

The family $(C, \{f_x\}_{x\in D}$ is a *limit cone* for D if, for any cone $(C', \{g_x\}_{x\in D})$, there is a unique mediating morphism $f \in \mathsf{C}(C', C)$ satisfying $g_x = f_x \circ f$ for each $x \in D$.

The notions of *cocone* and *colimit cocone* are defined dually.

Example 4.22 Products are examples of limit cones, and coproducts are examples of colimit cocones in any category where they exist.

Definition 4.23 The category is *complete* if all cones have limit cones, and it is *cocomplete* if all cocones have colimit cocones.

5 ω-Complete Partial Orders

We now introduce a category in which we actually can construct a model of the untyped λ-calculus. We choose perhaps the simplest such category – there are several others whose definitions require more intricate concepts.

Definition 5.1 A partially ordered set P is ω-complete if every increasing chain $\{x_n\}_{n\in\mathbb{N}} \subseteq P$ has a least upper bound in P.

Example 5.2

- For any set X, the power set $(\mathcal{P}(X), \subseteq)$ is ω-complete. More generally, any complete lattice is ω-complete.
- $([0, 1], \leq)$ – the unit interval in the usual order is ω-complete.
- Likewise, $([0, 1) \cup \{2\}, \leq)$ is ω-complete, where now $\sqcup_n(1 - 1/n) = 2$.
- On the other hand, $(\mathbb{R}, \leq)$ is *not* ω-complete; nor is $(\mathbb{Q} \cap [0, 1], \leq)$, the rationals in the unit interval.

Definition 5.3 A function $f\colon P \to Q$ is *continuous* if f is monotone and f preserves suprema of increasing chains.

We let $\Omega\!-\!\mathsf{Pos}$ denote the category of ω-complete partial orders and continuous maps.

Here are some examples of continuous maps between ω-continuous partial orders.

Example 5.4

- $f\colon ([0,1],\leq) \to ([0,1)\cup\{2\},\leq)$ by

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ 2, & \text{if } x = 1. \end{cases}$$

- For any set X, $f\colon (\mathcal{P}(X),\subseteq) \to (\mathcal{P}(X),\supseteq)$ by $f(A) = X \setminus A$.
- If P is an ωcomplete partial order, then so is $(P,=)$, P endowed with the *discrete order*. Then the identity map $1_P\colon (P,=) \to (P,\leq)$ is continuous.

Theorem 5.5 $\Omega\!-\!\mathsf{Pos}$ *has all products and coproducts.*

Proof. We show the result for products, and for finite coproducts.

For products, let $\{P_i\}_{i\in I}$ be family of ω-complete partial orders. We take $\prod_i P_i$ to be the usual product (i.e., the one from Set), and we endow this with the partial order: $\{x_i\}_{i\in I} \leq \{y_i\}_{i\in I}$ iff $x_i \leq y_i$ for every $i \in I$. Clearly, this is the same as saying $x \leq y \in \prod_i P_i$ iff $\pi_i(x) \leq \pi_i(y$ for every $i \in I$, so that the projection maps are not only monotone, they in fact define the order.

If $\{x_n\}_{n\in\mathbb{N}}$ is an increasing chain in $\prod_i P_i$, then $m \leq n \in \mathbb{N}$ implies $\pi_i(x_m) \leq \pi_i(xn)$, so $\{\pi_i(x_m)\}_{m\in\mathbb{N}}$ is an increasing chain in P_i. As such, it has a least upper bound, $x_i \in P_i$, and then $x \in \prod_i P_i$ by $\pi_i(x) = x_i$ is clearly an upper bound for $\{x_n\}_{n\in\mathbb{N}}$. And, if $x_n \leq y \in \prod_i P_i$ for every $n \in \mathbb{N}$, then $\pi_i(x_n) \leq]pi_i(y)$ for every n, for each $i \in I$. Then $x_i = \sqcup_n \pi_i(x_n) \leq \pi_i(y)$, so $x \leq y$. Thus $\{x_n\}_{n\in\mathbb{N}}$ has a least upper bound in $\prod_i P_I$, and we also can conclude that each projection $\pi_i\colon \prod_i P_I \to P_i$ is continuous.

Now, if $Q \in \Omega\!-\!\mathsf{Pos}$ and if $f_i\colon Q \to P_i$ is continuous for each $i \in I$, then we define $\langle f_i\rangle_{i\in I}\colon Q \to \prod_i P_i$ by $\langle f_i\rangle_{i\in I}(q) = \{f_i(q)\}_{i\in I}$. By definition, $\pi_i \circ \langle f_i\rangle_{i\in I} = f_i$, and since f_i is continuous and the π_is define the order on $\prod_i P_i$, we conclude that $\langle f_i\rangle_{i\in I}$ is continuous. The mapping clearly is the unique $f\colon Q \to \prod_i P_i$ satisfying $\pi_i \circ f$ by definition. This shows $\prod_i P_i$ deserves its name.

For coproducts, let $P, Q \in \operatorname{obj}\Omega\!-\!\mathsf{Pos}$. Define $P+Q = \{0\}\times P \cup \{1\}\times Q$, where we assume $0,1 \notin P \cup Q$. Then this is a disjoint union. We define $(a,b) \leq (c,d)$ in $P+Q$ iff $a = c$ and $b \leq d$ in whichever component they both reside in – for $a = c = 0$, they are in P, while if $a = c = 1$, then they both are in Q. This is clearly a partial order, and any ω-chain in $P+Q$ must either be in P or in Q; in either case, the chain has its least upper bound defined as in P or Q, respectively.

Define mappings $\iota_P\colon P \to P+Q$ by $\iota_P(x) = (0,x,$ and likewise, $\iota_Q\colon Q \to$

$P + Q$ by $\iota_Q(y) = (1, y)$. It is routine to verify these mappings are ω-continuous.

Finally, given continuous maps $f\colon P \to Q$ and $g\colon Q \to R$, we define $[f, g]\colon P + Q \to R$ by $[f, g](0, x) = f(x)$, and $[f, g](1, y) = g(y)$. This defines a unique continuous map from $P+Q$ to R whose composition with the coproduct maps ι_P and $]iota_Q$ yields f and g, respectively. This shows $P + Q$ is the coproduct of P and Q in $\Omega{-}\mathsf{Pos}$. □

Exercise 5.6 Generalize the definition of the coproduct given in the previous proof to show that $\Omega{-}\mathsf{Pos}$ has all coproducts.

5.1 Function Spaces

We now show that the function space of continuous maps between ω-complete partial orders is another such.

Definition 5.7 If $f, g \in \Omega{-}\mathsf{Pos}(P, Q)$, then we define $f \leq g$ iff $f(x) \leq g(x)$ for every $x \in P$.

Theorem 5.8 $\Omega{-}\mathsf{Pos}(P, A) \in \Omega{-}\mathsf{Pos}$ *for every* $P, Q \in \operatorname{obj} \Omega{-}\mathsf{Pos}$.

Proof. Let $f_n \in \Omega{-}\mathsf{Pos}(P, Q)$ for each $n \in \mathbb{N}$ with $f_m \leq f_n$ for $m \leq n$. Define $f\colon P \to Q$ by $f(x) = \sqcup_f f_n(x)$, for each $x \in P$. This is well-defined, because $\{f_n(x)\}_{n\in\mathbb{N}}$ is increasing in Q, which is ω-complete. It also is clear that f is monotone. Let $\{x_m]\_m \in \mathbb{N} \subseteq P$ be an increasing chain, and let $x = \sqcup_m x_m$. Then

$$
\begin{aligned}
f(x) &= (\sqcup_n f_n)(x) = \sqcup_n f_n(x) && \text{by definition} \\
&= \sqcup_n f_n(\sqcup_m x_m) = \sqcup_n(\sqcup_m f_n(x_m)) && \text{since } f_n \text{ is continuous} \\
&= \sqcup_{m,n} f_n(x_m) = \sqcup_m(\sqcup_n f_n(x_m)) && \text{since suprema commute} \\
&= \sqcup_m f(x_m).
\end{aligned}
$$

□

It's not only true that $\Omega{-}\mathsf{Pos}$ has function spaces, but in fact it's easy for functions defined on a product of ω-complete posets to be continuous.

Proposition 5.9 *Let* $f\colon P \times Q \to R$ *be a function from the product of* ω-*complete posets to another such, The following are equivalent:*

(i) f *is continuous.*

(ii) f *is continuous in each variable separately.*

Proof. Suppose f is continuous in each variable separately. It is routine to show this implies f is monotone. Now, let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq P \times Q$ be an increasing chain. Then

$$\begin{aligned} f(\sqcup_n(x_n, y_n)) &= f((\sqcup_m x_m, \sqcup_n y_n)) \\ &= \sqcup_m f((x_m, \sqcup_n y_n)) \\ &= \sqcup_m \sqcup_n f((x_m, y_n)) \\ &= \sqcup_n f((x_n, y_n)), \end{aligned}$$

where the first and last equalities follow from the facts that x_m, y_n are increasing and f is monotone, and the middle two rely on the separate continuity of f □

5.2 $\Omega-$Pos is Cartesian Closed

We now turn our attention to showing that $\Omega-$Pos is cartesian closed. We already have most of the ingredients – $\Omega-$Pos has an internal hom, the function space, and it is closed under products, so it also has a terminal object, the one-point space. It only remains to verify the required properties hold.

Theorem 5.10 $\Omega-$Pos *is cartesian closed.*

Proof. We've already remarked that all the ingredients we need are available – products, a terminal object and function spaces. So, support that $P, Q, R \in \text{obj}\,\Omega-\mathsf{Pos}$. Then $Q^P \equiv \Omega-\mathsf{Pos}(P, Q) \in \text{obj}\,\Omega-\mathsf{Pos}$, and we define $\text{app}\colon Q^P \times P \to Q$ by $\text{app}(f, x) = f(x)$. This is clearly well-defined, and it's easy to see it is monotone. If $f_n \in Q^P$ is increasing and $f = \sqcup_n f_n$, then for any $x \in P$,

$$\text{app}(f, x) = f(x) = (\sqcup_n f_n)(x) = \sqcup_n f_n(x),$$

by definition of $\sqcup_n f_n$. Likewise, if $f \in Q^P$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq P$ is increasing with $x = \sqcup_n x_n$, then

$$\text{app}(f, x) f(\sqcup_n x_n) = \sqcup_n f(x_n)$$

since f is continuous. This shows that app is continuous in each variable separately, and so Proposition 5.9 implies app is jointly continuous.

Now suppose $g\colon R \times P \to Q$, and define $\text{curry}(g)\colon R \top Q^P$ by $\text{curry}(g)(x)\colon P \to Q$ by $\text{curry}(g)(x)(y) = g(x, y)$. Proposition 5.9 implies $\text{curry}(g)(x) \in \Omega-\mathsf{Pos}(P, Q)$, and, equally, that this assignment is continuous in $x \in R$. Clearly $\text{app} \circ \text{curry}(g) = g$, and certainly this defines $\text{curry}(g)$ uniquely. □

5.3 Tarski's Theorem

Of course, we also are interested in finding non-degenerate reflexive objects in $\Omega-$Pos, but to do that, we need more category theory. Before pursuing that, however, we first prove a classical result about ω-complete partial orders, which also provides a hint of how we'll seek to prove there are non-degenerate models on the untyped λ-calculus in $\Omega-$Pos.

Theorem 5.11 (Tarksi, Knaster, Scott) *Let* $f \in \Omega-\mathsf{Pos}$ *where* P *has a least element. Then* f *has a least fixed point* $\text{fix}(f) = \sqcup_n f^n(\bot)$.

Proof. Since $\perp$ is the least element of P, we have $\perp \leq f(\perp)$. Since f is monotone, applying f to this inequality shows that $f^n(\perp) \leq f^{n+1}(\perp)$, by induction on n. Hence $\{f^n(\perp)\}_{n\in\mathbb{N}}$ is an increasing sequence in P, and so it has a least upper bound, which we denote by $\mathrm{fix}(f)$. Indeed, since f is continuous,

$$f(\mathrm{fix}(f)) = f(\sqcup_n f^n(\perp) = \sqcup_n f^{n+1}(\perp) = \mathrm{fix}(f).$$

The fact that $\mathrm{fix}(f)$ is f's least fixed point is an easy argument. □

Exercise 5.12 Show that $\mathrm{fix}(f)$ is indeed the least fixed point of f.

Exercise 5.13 Consider $[\mathbb{N} \rightharpoonup \mathbb{N}]$, the space of partially defined selfmaps of $\mathbb{N}$, where we give $\mathbb{N}$ the flat order.

(i) Order $[\mathbb{N} \rightharpoonup \mathbb{N}]$ by

$$f \leq g \;\Leftrightarrow\; \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \;\&\; g|_{\mathrm{dom}(f)} = f.$$

Show that $[\mathbb{N} \rightharpoonup \mathbb{N}] \in \Omega{-}\mathsf{Pos}$.

(ii) Define $F\colon [\mathbb{N} \rightharpoonup \mathbb{N}] \to [\mathbb{N} \rightharpoonup \mathbb{N}]$ by

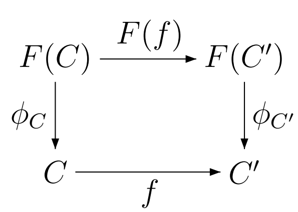
$$F(f)(n) = \begin{cases} 1, & \text{if } n = 0 \\ (n+1) * f(n), & \text{if } n > 0. \end{cases}$$

Show that F is ω-continuous, and $\mathrm{fix}(F) = \mathrm{fac}\colon \mathbb{N} \to \mathbb{N}$.

6 F-algebras and Continuous Functors

Definition 6.1 Let $F\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor on a category C. An object $C \in \mathsf{C}$ is an *F-algebra* if there is a map $\phi_C\colon F(A) \to A$.

For two F-algebras C, C', an F-algebra homomorphism is a mapping $f\colon C \to C'$ such that the following commutes:



Example 6.2 Consider the category Ω-Alg of Ω-algebras and Ω-algebra homomorphisms, and the functor $T_\Omega\colon \mathsf{Set} \to \Omega$-$\mathsf{Alg}$. If we compose T_Ω with the forgetful functor back to Set, then any set X which is an Ω algebra admits a unique Ω-algebra map from $T_\Omega(X)$ which extends the identity map. Thus Ω-algebras are $|\ |\circ T_\Omega$-algebras.

Theorem 6.3 *For an endofunctor $F\colon \mathsf{C} \to \mathsf{C}$, if $C \in \operatorname{obj}\mathsf{C}$ is an initial F-algebra, then the mapping $\phi_C\colon F(C) \to C$ is an isomorphism.*

Proof. Exercise. □

6.1 Adjunctions Between Posets

Definition 6.4 A pair of mappings $f\colon P \to Q$ and $g\colon Q \to P$ form an *adjunction* if

$$(\forall x \in P)(\forall y \in Q) \quad x \leq_P g(y) \;\Leftrightarrow\; f(x) \leq_Q y.$$

Example 6.5

(i) Suppose P has a least element, $\bot$, and define $\kappa\colon P \to \mathsf{Pos}(P,P)$ by $\kappa(x)(y) = x$, and $!\colon \mathsf{Pos}(P,P) \to P$ by $!(f) = f(\bot)$. Then

$$x \leq_P !f \;\Leftrightarrow\; x \leq_P f(\bot) \;\Leftrightarrow\; x \leq f(y)\ (\forall y) \;\Leftrightarrow\; \kappa(x) \leq f.$$

(ii) Let A be a poset with a least element, and for a poset P, let $Q = A \times P$ in the product order. Define $f\colon Q \to P$ to be the projection map, and let $g\colon P \to Q$ by $g(x) = (\bot_A, x)$. Then (f,g) is an adjunction:

$$(a, x') \leq g(x) \;\Leftrightarrow\; x' = f(a, x') \leq x.$$

(iii) Let P have a least element, $\bot$, and define $\kappa\colon P \to \mathsf{Pos}(P,P)$ by $\kappa(x)(y) = x$, and $!\colon \mathsf{Pos}(P,P) \to P$ by $!(f) = f(\bot)$. Then

$$x \leq_P !f \;\Leftrightarrow\; x \leq_P f(\bot) \;\Leftrightarrow\; x \leq f(y)\ (\forall y) \;\Leftrightarrow\; \kappa(x) \leq f.$$

Theorem 6.6 *Given a pair of monotone maps $f\colon P \to Q$ and $g\colon Q \to P$, the following are equivalent:*

(i) *The pair (f,g) is an adjunction.*

(ii) *$f \circ g \leq 1_Q$ and $g \circ f \geq 1_P$.*

(iii) *$(\forall x \in P)\, f(x) = \inf g^{-1}(\uparrow x)$, and $\forall y \in Q)\, g(y) = \sup f^{-1}(\downarrow y)$.*

In this case, f preserves all existing suprema, and g preserves all existing infima. Finally, each of these conditions implies

- *$f = f \circ g \circ f$ and $g = g \circ f \circ g$.*

Proof. Exercise. □

Definition 6.7 The upper adjoint f is called an *embedding* if $g \circ f = 1_P$. In this case, g is called a *projection.*

Corollary 6.8 *The functors listed above cut down to a dual equivalence between the categories $\Omega{-}\mathsf{Pos}_e$ and $\Omega{-}\mathsf{Pos}_p$ whose morphisms are embedding, respectively, projections.*

Exercise 6.9 Show that embeddings are one-to-one, and that projections are onto.

Remark 6.10 Let $f\colon P \to Q$ and $g\colon Q \to P$ be an adjunction. Then,

(i) f is called the *lower adjoint* and g is called the *upper adjoint*.

(ii) This result does *not* require that f or g be continuous – in fact, it shows that f must be continuous.

Exercise 6.11 Show that the conditions above are equivalent to $f\colon P \to Q$ and $g\colon Q \to P$ being adjoint functors, if P and Q are viewed as categories.

We define:
$\Omega{-}\mathsf{Pos}_U$ – ω-complete posets and upper adjoints.
$\Omega{-}\mathsf{Pos}_L$ – ω-complete posets and lower adjoints.

Theorem 6.12 *There are adjoint functors*

$$\widehat{\ }^U\colon \Omega{-}\mathsf{Pos}_L \to \Omega{-}\mathsf{Pos}_U^{\mathrm{op}} \text{ and } \widehat{\ }^L\colon \Omega{-}\mathsf{Pos}_U^{\mathrm{op}} \to \Omega{-}\mathsf{Pos}_L$$

which form a dual equivalence, where:

- $\widehat{P}^L = P$ *and* $\widehat{f}^L = g$*, the lower adjoint of* f*, and*
- $\widehat{P}^U = P$ *and* $\widehat{g}^U = f$*, the upper adjoint of* g*.*

Proof. Straightforward. □

6.2 Projective Limits

Definition 6.13 A an ordered set D is *directed* if every finite subset $F \subseteq D$ has a least upper bound in D. A *projective diagram* in a category C is a diagram $\Delta\colon D \to \mathsf{C}$ in which D is a directed set.

Proposition 6.14 *The category* $\Omega{-}\mathsf{Pos}_p$ *has projective limits.*

Proof. Recall $\prod_{i\in D} \Delta(i)$ is in $\Omega{-}\mathsf{Pos}$. Given the diagram $\Delta\colon D \to \Omega{-}\mathsf{Pos}_p$, consider the set

$$P = \{(x_i)_{i\in D} \in \prod_i \Delta(i) \mid i \le j \in D \;\Rightarrow\; f_{ij}(x_j) = x_i\}.$$

This is closed in $\prod_i \Delta(i)$ because the bonding maps - f_{ij} - are continuous. Define $\pi_i|_P\colon P \to \Delta(i)$ – this is an upper adjoint to

$$e_i\colon \Delta(i) \to P \text{ by } (e_i(x))j = f_{jk} \circ g_{ik}(x) \text{ where } i, j \le k \in D.$$

Suppose $Q \in \operatorname{obj} \Omega{-}\mathsf{Pos}$ and $h_i \in \Omega{-}\mathsf{Pos}(Q, P_i)$ with $f_{ij} \circ h_j = h_i$ for each $i \le j \in D$. Defining

$$h \in \Omega{-}\mathsf{Pos}(Q, P) \text{ by } (h(x))_i = h_i(x),$$

yields a unique continuous map with $g_{ij}(h(x)_j) = h(x)_i$. □

Exercise 6.15

(i) Show that $\pi_i\colon P \to P_i$ is surjective for each $i \in D$. Conclude that $P \neq \emptyset$.

(ii) Show that the mapping $e_i\colon \Delta(i) \to P$ is well-defined – i.e., show that, given any $j \in D$, for any choices $k, k' \in D$ with $i, j \leq k, k'$, we have $f_{jk} \circ g_{ik} = f_{jk'} \circ g_{ik'}$.

(iii) Show that the mediating map $h\colon Q \to P$ does satisfy the conditions needed.

6.3 Inductive Limits

Theorem 6.16 *If $(P_i, g_{ij})_{i\leq j\in D}$ is a projective system in $\Omega{-}\mathsf{Pos}_p$, and $(P_i, f_{ij})_{i\leq j\in D}$ is the associated system of lower adjoints in $\Omega{-}\mathsf{Pos}_e$, then*

$$\lim(P_i, g_{ij})_{i\leq j\in D} \simeq \operatorname{colim}(P_i, f_{ij})_{i\leq j\in D}.$$

Proof. We know $\lim(P_i, g_{ij})_{i\leq j\in D}$ exists, and since $\widehat{\ }^U$ and $\widehat{\ }^L$ form a dual equivalence, the latter takes existing limits to colimits in the dual category. But $\widehat{\ }^U$ and $\widehat{\ }^L$ are both the identity on objects, so if $P = \lim(P_i, g_{ij})_{i\leq j\in D}$, then

$$P = \widehat{P}^L = (\operatorname{colim} P_i, f_{ij})_{i\leq j\in D}.$$

□

Remark 6.17 It is important to notice here that the limit and colimit have the same underlying set. This is an example of the so-called "limit-colimit coincidence:" The fact that limits with projections not only correspond to colimits with the associated embeddings – they are in fact the same sets.

6.4 Enriched Categories

Definition 6.18 A category C is *enriched over* Pos if $\mathsf{C}(P,Q) \in \mathsf{Pos}$ for each $P, Q \in \operatorname{obj}\mathsf{C}$, and $\circ\colon \mathsf{C}(C,C') \times \mathsf{C}(C',C'') \to \mathsf{C}(C,C'')$ preserves the order.

Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor between categories enriched over Pos. Then F is *locally monotone* if $F\colon \mathsf{C}(P,Q) \to \mathsf{D}(F(P), F(Q))$ preserves the order.

Example 6.19 Let $L\colon \Omega{-}\mathsf{Pos} \to \Omega{-}\mathsf{Pos}_!$ be defined by $L(P) = P \cup \{\bot_P\}$, and $L(f)\colon L(P) \to L(Q)$ is

$$L(f)(x) = \begin{cases} f(x), & \text{if } x \in P, \\ \bot_Q, & \text{if } x = \bot_P\ . \end{cases}$$

Then L is locally monotone: $f \leq g \ \Rightarrow\ L(f) \leq L(g)$.

Lemma 6.20 *Let $F\colon \mathsf{C} \to \mathsf{D}$ be locally monotone, where C and D are subcategories of $\Omega{-}\mathsf{Pos}$. If $f \in \mathsf{C}(P,Q)$ is the upper adjoint of $g\colon Q \to P$ in C, then $F(f)\colon F(P) \to F(Q)$ is upper adjoint to $F(g)\colon F(Q) \to F(P)$ in D.*

Proof. Suppose that $f\colon P \to Q$ is a lower adjoint to $g\colon Q \to P$. Then $f \circ g \leq 1_Q$ and $g \circ f \geq 1_P$. Then, applying the locally monotone functor F, we have

$$F(f) \circ F(g) = F(f \circ g) \leq F(1_P) = 1_{F(P)},$$

and

$$F(g) \circ F(f) = F(g \circ f) \geq F(1_Q) = 1_{F(Q)}.$$

□

6.5 Calculating Limits

Definition 6.21 Let A be a subcategory of $\Omega{-}\mathsf{Pos}$. We say *A is pro-complete* if A is closed in $\Omega{-}\mathsf{Pos}$ under the formation of projective limits. Likewise, A is *ind-complete* if A is closed in $\Omega{-}\mathsf{Pos}$ under the formation of inductive limits.

Exercise 6.22

(i) Let A be a pro-complete subcategory of $\Omega{-}\mathsf{Pos}$. Then A_p, the subcategory of objects of A with projections in A as morphisms, also is pro-complete.

(ii) Likewise, if A is ind-complete, the so is the subcategory A_e, whose objects are the same as those of A, and whose morphisms are embeddings.

Theorem 6.23 *Let A and B be pro-complete subcategories of $\Omega{-}\mathsf{Pos}$, and let $F\colon \mathsf{A} \to \mathsf{B}$ be a locally monotone functor which is a left adjoint. Then the restriction $F_p\colon \mathsf{A}_p \to \mathsf{B}_p$ is pro-continuous.*

Proof. The previous lemma shows F has a restriction F_p. If $(P_i, g_{ij})_{i\leq j\in D}$ is a projective system in A_p, then the associated inductive system $(P_i, f_{ij})_{i\leq j\in D}$ lies in A_e. Moreover, if

$$(P, g_i)_{i\in D} = \lim(P_i, g_{ij})_{i\leq j\in D},$$

then a previous theorem implies

$$(P, f_i)_{i\in D} = \operatorname{colim}(P_i, f_{ij})_{i\leq j\in D}.$$

Since $F\colon \mathsf{A} \to \mathsf{B}$ is a left adjoint, we have

$$\begin{aligned}
F_p(\lim(P_i, g_{ij})) &= F(\lim(P_i, g_{ij})) && \mathsf{A}\ \text{pro}-\text{complete}\\
&\simeq F(\operatorname{colim}(P_i, f_{ij})) && \text{previous Theorem}\\
&\simeq \operatorname{colim}(F(P_i), F(f_{ij})) && \text{left adjoints preserve colimits}\\
&\simeq \lim(F(P), F(g_{ij})) && F\ \text{locally monotone and a previous Theorem}\\
&\simeq \lim(F_p(P_i), F_p(g_{ij})) && \mathsf{B}\ \text{pro}-\text{complete.}
\end{aligned}$$

□

Exercise 6.24 Show that if $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ form an adjunction between categories, then the left adjoint F preserves all existing colimits, and the right adjoint G preserves all existing limits.

6.6 Main Result

Corollary 6.25 *Let A be a pro-complete subcategory of $\Omega{-}\mathsf{Pos}$ containing an initial object $\bot$. If $F\colon \mathsf{A} \to \mathsf{A}$ is a locally monotone, left adjoint and if $!\colon\ \bot \to F(\bot)$ is an embedding, then there is an initial F-algebra in A, which is necessarily a fixed point for F.*

Proof. Since $!\colon\ \bot \to F(\bot)$ is an embedding, there is an upper adjoint $\widehat{!}\colon F(\bot) \to \bot$, and so we have a projective sequence $(F^n(\bot), F^m(!) \circ \cdots \circ F^{n-1}(!))_{m\leq n\in\mathbb{N}}$. The previous results imply this sequence has a limit, and that F preserves it:

$$\begin{aligned} &F(\lim(F^{n-1}(\bot), F^m(!) \circ \cdots \circ F^{n-1}(!))_{m\leq n\in\mathbb{N}} \\ \simeq\ &\lim(F^n(\bot), F^{m+1}(!) \circ \cdots \circ F^n(!))_{m\leq n\in\mathbb{N}} \\ \simeq\ &(F^n(\bot), F^m(!) \circ \cdots \circ F^{n-1}(!))_{m\leq n\in\mathbb{N}} \end{aligned}$$

which proves our result. □

Exercise 6.26 Recall $\Omega{-}\mathsf{Pos}_!$, the category of $\Omega{-}\mathsf{Pos}$ objects having least elements, and continuous maps preserving the least element. We defined $L\colon \Omega{-}\mathsf{Pos} \to \Omega{-}\mathsf{Pos}_!$ by $L(P) = P \cup \{\bot_P\}$, where $L(f)\colon L(P) \to L(Q)$ is

$$L(f)(x) = \begin{cases} f(x), & \text{if } x \in P, \\ \bot_Q, & \text{if } x = \bot_P\ . \end{cases}$$

Then

- L is locally monotone: $f \leq g \ \Rightarrow\ L(f) \leq L(g)$ and $\circ\colon \Omega{-}\mathsf{Pos}(P,Q) \times \Omega{-}\mathsf{Pos}(Q.R) \to \Omega{-}\mathsf{Pos}(P,Q)$ is as well.
- $\Omega{-}\mathsf{Pos}$ has an initial object $\{\bot\}$ for which $!\colon \{\bot\} \to P$ is an embedding with upper adjoint $x \mapsto \bot\colon P \to \{\bot\}$.

Hence, an initial L-algebra is

$$\lim(L^n(\{\bot\}), L^m(\widehat{!}) \circ \cdots \circ L^{n-1}(\widehat{!}))_{m\leq n\in\mathbb{N}} \simeq \mathbb{N}^{\top}.$$

6.7 The Problem for $\Omega{-}\mathsf{Pos}(P,P)$

We would like to apply our main result to a functor which, on objects is given by $P \mapsto \Omega{-}\mathsf{Pos}(P,P)$. However, there's no clear way to extend this a functor (check this!). Still the following shows us how to proceed:

We define the category

$\Omega{-}\mathsf{Pos}_{ep}$ – ω-complete posets and embedding-projection pairs.

The following diagram contains the crucial information:

$$\begin{array}{ccc}
P & \Omega{-}\mathsf{Pos}(P,P) & \Omega{-}\mathsf{Pos}(P,P) \\
e\big\downarrow\ \big\uparrow p & e\circ - \circ p \big\downarrow & \big\uparrow p \circ - \circ e \\
Q & \Omega{-}\mathsf{Pos}(Q,Q) & \Omega{-}\mathsf{Pos}(Q)
\end{array}$$

Exercise 6.27

(i) Show that the mappings $f \mapsto e \circ f \circ p \colon \Omega{-}\mathsf{Pos}(P,P) \to \Omega{-}\mathsf{Pos}(Q,Q)$ and $g \mapsto p \circ g \circ e \colon \Omega{-}\mathsf{Pos}(Q,Q) \to \Omega{-}\mathsf{Pos}(P,P)$ form an adjunction.

(ii) Show this defines an endofunctor of $\Omega{-}\mathsf{Pos}_{ep}$.

But even so, we can't apply our technology, since we don't know that the internal hom functor defined on $\Omega{-}\mathsf{Pos}_{ep}$ is the restriction of a left adjoint to this subcategory of $\Omega{-}\mathsf{Pos}$. However, Plotkin and Smyth developed a more general approach for just this situation.

Definition 6.28 We say a category C is *enriched over* $\Omega{-}\mathsf{Pos}$ if each hom set $\mathsf{C}(P,Q) \in \operatorname{obj}\Omega{-}\mathsf{Pos}$ and $\circ\colon \mathsf{C}(C,C') \times \mathsf{C}(C',C'') \to \mathsf{C}(C,C'')$ is continuous.

Further, a functor $F\colon \mathsf{C} \to \mathsf{D}$ between $\Omega{-}\mathsf{Pos}$-enriched categories is *locally continuous* if for each $C, C' \in \operatorname{obj}\mathsf{C}$, we have $F\colon \mathsf{C}(C,C') \to \mathsf{D}(F(C), D(C''))$ is continuous.

Theorem 6.29 (Plotkin & Smyth) *If* C, D *are enriched over* $\Omega{-}\mathsf{Pos}$ *and* $F\colon \mathsf{C} \to \mathsf{D}$ *is* locally continuous*, then* F *preserves limits of embedding-projection pairs.*

Exercise 6.30

(i) Show that we have defined an embedding-projection pair.

(ii) Also, show the same result applies to adjoint pairs in general.

Problem: But there's still a problem:

$$\Omega{-}\mathsf{Pos}(\{\perp\},\{\perp\}) \simeq \{\perp\}.$$

Cure: Use $\mathsf{2} \equiv \{0,1\}$:

$$e\colon \Omega{-}\mathsf{Pos}(\mathsf{2},\mathsf{2}) \leftrightarrow \Omega{-}\mathsf{Pos}(\mathsf{2},\mathsf{2})\colon p$$

by

$$e(x) = \kappa_x \text{ and } p(f) = f(0).$$

Exercise 6.31

(i) Use the e-p pair

$$e\colon \Omega{-}\mathsf{Pos}(\mathsf{2},\mathsf{2}) \longleftrightarrow \Omega{-}\mathsf{Pos}(\mathsf{2},\mathsf{2})\colon p$$

to generate sequence

$$(F^n(2), F^m(p) \circ \cdots \circ F^{n-1}(p))_{m \le n \in \mathbb{N}}$$

and show this sequence has a limit with

$$F(\lim(F^n(2), F^m(p) \circ \cdots \circ F^{n-1}(p))_{m \le n \in \mathbb{N}}) \simeq (F^n(2), F^m(p) \circ \cdots \circ F^{n-1}(p))_{m \le n \in \mathbb{N}}.$$

This is the so-called D_∞ model that Scott first devised for the untyped λ-calculus.

(ii) More generally, if A is a set of constants over which we define the untyped λ-calculus, then define

$$F\colon \Omega{-}\mathsf{Pos}_{ep} \to \Omega{-}\mathsf{Pos}_{ep} \text{ by } F(P) = A + \Omega{-}\mathsf{Pos}(P, P).$$

Use above technology to find initial solution:

$$P \simeq F(P) = A + \Omega{-}\mathsf{Pos}(P, P).$$

7 Collected Exercises

Here are a number of related exercises from the notes collected together. The original number of the exercise is included for references (this sometimes can be the number of a result such as a Theorem, Proposition, etc., in case the exercise was to prove the result):

Strings For an alphabet A (i.e., a non-empty set), a *string* over A is just a finite sequence $a_1 a_2 \cdots a_n$, where $a_1, \ldots, a_n \in A$ (note: the a_is need not be distinct). Two strings $s = a_1 \cdots a_m$ and $t = a'_1 \cdots a'_n$ can be concatenated, as in $s \cdot t = a_1 \cdots a_n a'_1 \cdots a'_m$.

(i) [3.5(iii)] Give a signature Ω for the set of finite strings over the set A and concatenation as an operation.

(ii) [3.9] Use Corollary 3.7 to define a rank function for the set of finite strings over a set A, and show that this rank function gives exactly the length n of a string $s = a_1 \cdots a_n$.

(iii) [4.18] Using the technique of outlined in Exercise 4.18, show that the signature Ω you defined in the first part can be used to define a family of signatures Ω^A by $\Omega^A = \cup_n \Omega^A_n$, where $\Omega^A_0 = A$, and $\Omega^A_n = \Omega_n$ for $n \ge 1$.

(a) Let Ω-Alg denote the category of all Ω^A-algebras and all Ω^A-algebra homomorphisms. Define a functor $\mathcal{T}\colon \mathsf{Set} \to \Omega\text{-}\mathsf{Alg}$ by $\mathcal{T}(A) = T_{\Omega^A}$, the initial Ω^A-algebra. If $f\colon A \to B$ is a function, then use Theorem 3.7 to show f extends to an Ω-algebra map from T_{Ω^A} to T_{Ω^B}.

(b) Dually, show there is a functor $G\colon \Omega\text{-}\mathsf{Alg} \to \mathsf{Set}$ which associates to an Ω-algebra its set of Ω_0 of nullary operators. We can define $\eta\colon I \stackrel{\cdot}{\longrightarrow} GF$ by $\eta_A(x) = x \in G(T_{\Omega^A})$, which sends the element $x \in A$ to its interpretation as a constant in T_{Ω^A}.

(c) Use Theorem 3.7 to show that this defines an adjunction between Set and Ω-Alg.

3.26 Let C be a cartesian closed category, and let $A, B, C \in \operatorname{obj} \mathsf{C}$. Then,

(i) If $f, g \in C \to \mathsf{C}(A, B)$ and $\mathrm{app} \circ (f \times 1_A) = \mathrm{app} \circ (g \times 1_A)$, then $f = g$.

(ii) Taking $C = B$ in (i), and considering $\pi_B \colon B \times A \to B$, we get a unique mapping $\kappa_B^A \colon B \to B^A$ satisfying $\mathrm{app} \circ (\kappa_B^A \times 1_A) = \pi_B$. This is the so-called *constant picker* which associates to each $x \in B$ the constant map $a \mapsto x \colon A \to B$.

(iii) We next consider

$$\mathrm{app} \circ (1_{C^B} \times \mathrm{app}) \colon C^B \times (B^A \times A) \to C$$

and apply the universal property to define a unique map

$$\circ_{CBA} \colon C^B \times B^A \to C^A.$$

Show this is well-defined, and that, for any $D \in \operatorname{obj} \mathsf{C}$, we have

$$\circ_{D(CBA)} \colon D^C \times (C^B \times B^A) \to D^A$$

and

$$\circ_{(DCB)A} \colon (D^C \times C^B) \times B^A \to D^A$$

are the same.

4.1ff This is an exercise on the combinators of the λ-calculus. Assuming (β), show:

(i) $(SK)K \equiv I$, so S and K generate.

(ii) For any term f, $f(Yf) \equiv YF$. This means every term has a fixed point in a model of the untyped λ-calculus that validates (β).

(iii) If $\circ \colon \Lambda \times \Lambda \to \Lambda$ is the composition operator defined in Definition 4.2, then for all $m, n, p \in \Lambda$, $(m \circ n)p = m(np)$.

4.7ff Let C be a cartesian closed category and let $C \in \operatorname{obj} \mathsf{C}$ be a reflexive object in C.

(i) Let $x, y \in C$, $u, v \in V$ and let $\sigma \in C^V$. Then

$$\sigma\{x/u\}\{y/v\} = \begin{cases} \sigma\{y/v\}\{x/u\}, & \text{if } u \neq v, \\ \sigma\{y/v\}, & \text{if } u = v. \end{cases}$$

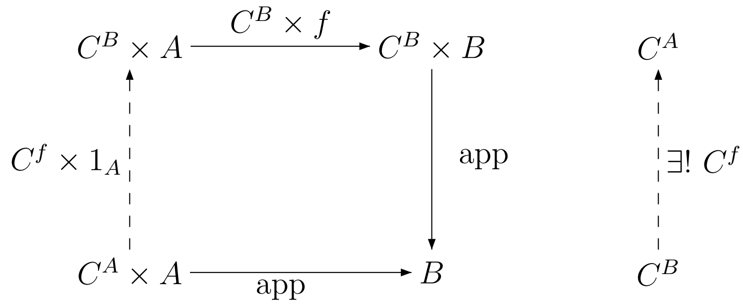
(ii) Use (i) to show that (α)(see 4.1) and (β)(see 4.1) both hold for the interpretation of the untyped λ-calculus in C (see 4.10).

(iii) * What about (η)(see 4.1)?

4.13 The setting is a cartesian closed category C. We extend it with the following observations:

(i) If $C \in \operatorname{obj} \mathsf{C}$, then there is an endofunctor $\mathsf{C}_C \colon \mathsf{C} \to \mathsf{C}$ given by $\mathsf{C}_C(A) = A^C$ for $A \in \operatorname{obj} \mathsf{C}$, and $\mathsf{C}_C \colon \mathsf{C}(A, B) \to \mathsf{C}(A^C, B^C)$ by $\mathsf{C}_C(f) = f \circ -$.

(ii) Show that each morphism $f \in \mathsf{C}(A, B)$ and each object $C \in \mathsf{C}$ define a morphism $C^f \colon C^B \to C^A$ using the diagram:



(iii) Now use (ii) to show the following: if $C \in \operatorname{obj} \mathsf{C}$, then there is a contravariant functor $\mathsf{C}^C \colon \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ given by $\mathsf{C}^C(A) = C^A$ and $\mathsf{C}^C \colon \mathsf{C}(A, B) \to \mathsf{C}(C^B, C^A)$ by $\mathsf{C}^A(f) = - \circ f$, for $f \in \mathsf{C}(A, B)$.

4.15(ii) Let C be a ccc, and let $C \in \operatorname{obj} \mathsf{C}$. Define $G_C \colon C \to C$ by $G_C(A) = A^C \times C$, and for $f \in \mathsf{C}(A, B)$, $G_C(f) \in \mathsf{C}(A^C \times A, B^C \times C)$ by $G_C(f) = f \circ -$. Then $\eta \colon G \stackrel{\cdot}{\longrightarrow} I$ by $\eta_A = \mathrm{app} \colon A^C \times C \to A$ is a natural transformation, where I denotes the identity functor.

5.12ff

(i) Show that $\Omega{-}\mathsf{Pos}$ has all coproducts.

(ii) Show that $\mathrm{fix}(f)$ is the least fixed point of a selfmap $f \colon P \to P$, if $P \in \operatorname{obj} \Omega{-}\mathsf{Pos}$ has a least element.

(iii) [5.13] Consider $[\mathbb{N} \rightharpoonup \mathbb{N}]$, the space of partially defined selfmaps of $\mathbb{N}$, where we give $\mathbb{N}$ the flat order.

(a) Order $[\mathbb{N} \rightharpoonup \mathbb{N}]$ by

$$f \leq g \ \Leftrightarrow\ \mathrm{dom}(f) \subseteq \mathrm{dom}(g)\ \&\ g|_{\mathrm{dom}(f)} = f.$$

Show that $[\mathbb{N} \rightharpoonup \mathbb{N}] \in \Omega{-}\mathsf{Pos}$.

(b) Define $F \colon [\mathbb{N} \rightharpoonup \mathbb{N}] \to [\mathbb{N} \rightharpoonup \mathbb{N}]$ by

$$F(f)(n) = \begin{cases} 1, & \text{if } n = 0 \\ (n+1) * f(n), & \text{if } n > 0. \end{cases}$$

Show that F is ω-continuous, and $\mathrm{fix}(F) = \mathrm{fac} \colon \mathbb{N} \to \mathbb{N}$.

6.3 For an endofunctor $F \colon \mathsf{C} \to \mathsf{C}$, if $C \in \operatorname{obj} \mathsf{C}$ is an initial F-algebra, then the mapping $\phi_C \colon F(C) \to C$ is an isomorphism.

6.6 Given a pair of monotone maps $f \colon P \to Q$ and $g \colon Q \to P$, the following are equivalent:

(i) The pair (f, g) is an adjunction.

(ii) $f \circ g \leq 1_Q$ and $g \circ f \geq 1_P$.

(iii) $(\forall x \in P)\, f(x) = \inf g^{-1}(\uparrow x)$, and $(\forall y \in Q)\, g(y) = \sup f^{-1}(\downarrow y)$.

In this case, f preserves all existing suprema, and g preserves all existing infima. Finally, each of these conditions implies

- $f = f \circ g \circ f$ and $g = g \circ f \circ g$.

6.15 * This is in the context of a projective family $(P_i, f_{ij})_{i\leq j\in D}$.

(i) Show that $\pi_i \colon P \to P_i$ is surjective for each $i \in D$. Conclude that $P \neq \emptyset$.

(ii) Show that the mapping $e_i \colon \Delta(i) \to P$ is well-defined – i.e., show that, given any $j \in D$, for any choices $k, k' \in D$ with $i, j \leq k, k'$, we have $f_{jk} \circ g_{ik} = f_{jk'} \circ g_{ik'}$.

(iii) Show that the mediating map $h \colon Q \to P$ does satisfy the conditions needed.

6.26 Recall $\Omega{-}\mathsf{Pos}_!$, the category of $\Omega{-}\mathsf{Pos}$ objects having least elements, and continuous maps preserving the least element. We defined $L \colon \Omega{-}\mathsf{Pos} \to \Omega{-}\mathsf{Pos}_!$ by $L(P) = P \cup \{\bot_P\}$, where $L(f) \colon L(P) \to L(Q)$ is

$$L(f)(x) = \begin{cases} f(x), & \text{if } x \in P, \\ \bot_Q, & \text{if } x = \bot_P \ . \end{cases}$$

Then

- L is locally monotone: $f \leq g \;\Rightarrow\; L(f) \leq L(g)$ and $\circ \colon \Omega{-}\mathsf{Pos}(P,Q) \times \Omega{-}\mathsf{Pos}(Q.R) \to \Omega{-}\mathsf{Pos}(P,Q)$ is as well.
- $\Omega{-}\mathsf{Pos}$ has an initial object $\{\bot\}$ for which $! \colon \{\bot\} \to P$ is an embedding with upper adjoint $x \mapsto \bot \colon P \to \{\bot\}$.

Hence, an initial L-algebra is

$$\lim(L^n(\{\bot\}), L^m(\widehat{!}) \circ \cdots \circ L^{n-1}(\widehat{!}))_{m\leq n\in\mathbb{N}} \simeq \mathbb{N}^\top.$$

6.27 This is in the context of defining a functor on $\Omega{-}\mathsf{Pos}_{ep}$ using the internal hom.

(i) Show that the mappings $f \mapsto e \circ f \circ p \colon \Omega{-}\mathsf{Pos}(P,P) \to \Omega{-}\mathsf{Pos}(Q,Q)$ and $g \mapsto p \circ g \circ e \colon \Omega{-}\mathsf{Pos}(Q,Q) \to \Omega{-}\mathsf{Pos}(P,P)$ form an adjunction.

(ii) Show this defines an endofunctor of $\Omega{-}\mathsf{Pos}_{ep}$.

6.28ff

(i) Show that $\Omega{-}\mathsf{Pos}$ is enriched over $\Omega{-}\mathsf{Pos}$.

(ii) Show that for a given set A of constants over which we define the untyped λ-calculus, then define

$$F \colon \Omega{-}\mathsf{Pos}_{ep} \to \Omega{-}\mathsf{Pos}_{ep} \text{ by } F(P) = A + \Omega{-}\mathsf{Pos}(P,P).$$

Use above technology to find initial solution:

$$P \simeq F(P) = A + \Omega{-}\mathsf{Pos}(P,P).$$

7.1 Exam Questions

For the examination in the course, work through the following exercises listed above: ***Strings, 3.26, 4.7ff, 4.13, 5.12ff, 6.6 and 6.15.*** Note that * exercises are *optional*; this means 4.7(iii) and 6.15 are not required, but are recommended.

8 Bibliography

Here are some general references for domain theory and related topics. This is a minimal list, but it should provide sufficient resources to find the wealth of material available on the subject. For example, there is an especially well-annotated bibliography in [7], and there is a very extensive bibliography in [4].

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