Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science
Electronic Notes in Theoretical Computer Science 164 (2006) 67–84
www.elsevier.com/locate/entcs

Some Properties and Some Problems on Set Functors [4]

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Abstract

We study properties of functors on categories of sets (classes) together with set (class) functions. In particular, we investigate the notion of *inclusion preserving functor*, and we discuss various *monotonicity* and *continuity properties* of set functors. As a consequence of these properties, we show that some classes of set operators do not admit *functorial extensions*. Then, starting from Aczel's *Special Final Coalgebra Theorem*, we study the class of functors *uniform on maps*, we present and discuss various examples of functors which are *not* uniform on maps but still inclusion preserving, and we discuss simple characterization theorems of final coalgebras as *fixpoints*. We present a number of conjectures and problems.

Keywords: category of sets, set functor, inclusion preserving functor, κ-based functor, κ-reachable functor, functor uniform on maps, final coalgebra.

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[4] Research partially supported by the Project ART(PRIN 2005015824), and by the EU Project FP6-IST-510996 TYPES.

doi:10.1016/j.entcs.2006.06.005

1 Introduction

In recent years, *categories of sets*, i.e. categories where objects are sets (classes) of a possible non-wellfounded universe and morphisms are set (class) functions, have been used as convenient settings for studying the foundations of *coalgebraic semantics*, see e.g. [1,2,7,8,10,18,15,12,3,4].

In this paper, we discuss the structure of endofunctors on categories of sets, in the light of [3,9]. First, we investigate *inclusion preserving functors*, then we study continuity properties of set functors. The notion of *inclusion preserving functor* has been extensively investigated in [5]. The main result of [5] about inclusion preserving functors is the following:

Theorem Any *set endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor which is inclusion preserving.*

Building on the above theorem, we show the following strong continuity result:

Theorem (κ-continuity) Any *set endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G such that, for all $|X| = \kappa$ infinite,*

$$|GX| \leq \kappa \implies GX = \bigcup\{GY \mid |Y| < \kappa \ \wedge \ Y \subseteq X\} \ .$$

The κ-continuity result above strengthens Theorem 2.2 of [3].

As a consequence of the κ-continuity Theorem, we derive the following:

Theorem (Class Continuity) Any *class endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G which is continuous on classes, i.e., for all proper classes X, $GX = \bigcup\{GY \mid Y \text{ set} \wedge Y \subset X\}$.*

The above class continuity result corresponds to the fact that any class endofunctor is *set based*, as was recently proved in [3], and in [9], independently.

An interesting application of the general κ-continuity theorem above is the fact that, as we show in this paper, we can rule out the existence of functorial extensions of a class of set operators, satisfying certain cardinality conditions. This result can be viewed as a contribution to the theory of "non-existing functors" developed in [16,19].

By the Class Continuity Theorem above, using Aczel's Final Coalgebra Theorem [1,2], we obtain that *any* class endofunctor has final coalgebra. However, the construction of the final coalgebra given in [1,2] and in other existence theorems is quite abstract, and hence not particularly useful in applications, where we are interested in computing and studying properties of elements of a coinductive type, and in developing formal tools for reasoning about them. Thus, it is important to identify classes of functors whose final coalgebras have conceptually independent characterizations. The main result in this area is Aczel's Special Final Coalgebra Theorem, [1], which applies to a *non-wellfounded universe* and states that if a functor is *uniform on maps*, then the greatest fixpoint of the operator underlying the functor, together with the identity map, is a final coalgebra. Many of the functors which arise naturally are uniform on maps, or at least naturally isomorphic to a functor which is uniform on maps. However, we show that:

Theorem *There exist functors which are*

(i) not *naturally isomorphic to a functor uniform on maps;*

(ii) not *(naturally isomorphic to functors) uniform on maps, but admitting the greatest fixed point (together with the identity map) as final coalgebra;*

(iii) *inclusion preserving and whose final coalgebras are* not *the greatest fixpoints;*

(iv) *uniform on maps and isomorphic on objects, but not naturally isomorphic.*

All the counterexamples used for proving the above theorem rely on odd behaviours of functors on *non*-injective functions. Hence, we are led to the following open problems:

Problems

(i) *Are all functors naturally isomorphic on* injective functions *to a functor which is uniform on maps?*

(ii) *Let F, G coincide on objects and on injective functions. Do F, G admit the same final coalgebra?*

Moreover, as we will show, it is somewhat remarkable to notice that, if we work "up-to natural isomorphisms", the following results hold:

Theorem *Any endofunctor on a class category is naturally isomorphic to a functor whose* unique *fixpoint, together with the identity, is a final coalgebra.*

Theorem *Any endofunctor on a class category is naturally isomorphic to a functor which has* no *fixpoints.*

Summary

In Section 2, we provide set theory and categorical preliminaries. In Section 3, we investigate properties of inclusion preserving functors. In Section 4, we discuss continuity properties of set functors. In Section 5, using the continuity properties previously investigated, we rule out the existence of functorial extensions for a class of set operators. In Section 6, we investigate functors uniform on maps and characterizations of final coalgebras as (greatest) fixpoints. Final remarks and directions for future work appear in Section 7.

Acknowledgments

The authors thank Jiri Adamek for useful comments.

2 Preliminaries

2.1 Set Theory Preliminaries

We will work in a universe of sets and classes satisfying the theory NBG of von Neumann-Bernays-Gödel. The theory NBG is closely related to the more familiar set theory ZFC of Zermelo-Fraenkel with choice. The primary difference between ZFC and NBG is that NBG has proper classes among its objects. NBG and ZFC are actually equiconsistent, NBG being a conservative extension of ZFC. In NBG, the proper classes are differentiated from sets by the fact that they do not belong to other classes. Moreover, NBG classes satisfy the Axiom N of von Neumann, stating that all proper classes have the same cardinality of the set theoretic universe, which we denote by Ord.

We will consider both set theoretic universes of *wellfounded sets* and *non-wellfounded sets*. In non-wellfounded universes, the Foundation Axiom is replaced by an Antifoundation Axiom, such as X_1 of Forti and Honsell, [11], or AFA by Aczel, [1]. Non-wellfounded sets, also called *hypersets*, have been used to model circular phenomena in Computer Science and in various other fields, see e.g. [8].

Throughout this paper, we will denote by V the universe of (non-)wellfounded sets.

Remark. In this paper we will refer not only to *large objects*, such as proper classes, but also to *very large* objects, such as functors over categories whose objects are classes. A foundational formal theory which can accommodate naturally all our arguments is not readily available. A substantial formalistic effort would be needed to "cross all our t's" properly. We shall therefore adopt a pragmatic attitude and freely assume that we have classes and functors over classes at hand. Worries concerning consistency can be eliminated by assuming that our ambient theory is a Set Theory with an inaccessible cardinal κ, and the model of our object theory consists of those sets whose hereditary cardinal is less than κ, V_κ say, the classes of our model are the subsets of V_κ, and functors live at the appropriate ranks of the ambient universe.

2.2 Categorical Preliminaries

In this paper, we will deal both with *categories of pure sets* and with *categories of classes*. By a *category of pure sets* we mean a category whose objects are the sets of a (possibly non-wellfounded) universe, and where morphisms are set theoretic functions. By a *category of classes*, we mean a category whose objects are sets and classes of a (possibly non-wellfounded) universe, and where morphisms are set (class) functions between them. We will use the term *category of sets* for indicating a category of pure sets or a category of classes, indifferently, and we will denote it by $\mathcal{C}$.

Throughout this paper, we use the following basic notation about functions:

Notation.

Let $f : X \to Y$ be any function on sets (or classes), and let $X' \subseteq X$, then:

- $gr(f)$ denotes the graph of f;
- $img\ f$ denotes the image of f;
- $f_{|img\ f} : X \to img\ f$ denotes the function obtained from f by restricting the

codomain Y to the image of f;

- $f_{X'} : X' \to Y$ denotes the function obtained from f by restricting the domain of f to X'.

3 Inclusion Preserving Functors

In this section, we focus on inclusion preserving functors, i.e. functors which preserve inclusion maps. Inclusion preserving functors satisfy many interesting properties, which we collect in this section. In particular, we provide an alternative proof of the main result of [5] about inclusion preserving functors, i.e. *any* functor on a set category is naturally isomorphic (up-to $\emptyset$) to a functor which is inclusion preserving. The results in Proposition 3.2 below appear also in [5], but for the fact that values on morphisms of inclusion preserving functors depend only on the graphs of the morphisms and not on the targets. Interestingly, this turns out to be a necessary and sufficient condition for a functor to be inclusion preserving.

In the next definition, we recall the notion of inclusion preserving functor:

Definition 3.1 $F : \mathcal{C} \to \mathcal{C}$ *is* inclusion preserving *if*

$$\forall X, Y.\ X \subseteq Y \implies F(\iota_{X,Y}) = \iota_{F(X),F(Y)}\ ,$$

where $\iota_{X,Y} : X \to Y$ is the inclusion map from X to Y.

Inclusion preserving functors satisfy the following properties:

Proposition 3.2 *Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving. Then*

(i) *The operator underlying F is monotone, i.e. $X \subseteq Y \implies FX \subseteq FY$.*

(ii) *F preserves images of functions, i.e., for any $f : X \to Y$, $F(img\ f) = img\ F(f)$.*

(iii) *The value of F on any morphism depends only on the graph of the morphism, i.e. for all $f : X \to Y$, $f' : X \to Y'$, $gr(f) = gr(f') \Rightarrow gr(F(f)) = gr(F(f'))$. Vice versa, if for all $f : X \to Y$, $f' : X \to Y'$, $gr(f) = gr(f') \Rightarrow gr(F(f)) = gr(F(f'))$, then F is inclusion preserving.*

(iv) *For all $X' \subseteq X$ and for all $f : X \to Y$, $gr(Ff_{X'}) = gr(Ff)_{FX'}$.*

(v) *F preserves non-empty* finite intersections*, i.e. for all X, Y such that $X \cap Y \neq \emptyset$, $F(X \cap Y) = FX \cap FY$.*

Proof.

(i) Straightforward.

(ii) Let $f : X \to Y$. Then $F(f) = F(\iota_{imgf,Y} \circ f_{|imgf}) = \iota_{F(imgf),FY} \circ F(f_{|imgf})$, since F is inclusion preserving. Therefore, $img\ F(f) = img\ F(f_{|imgf}) = F(img\ f)$, since $f_{|imgf}$ is surjective and any set functor preserves surjective functions.

(iii) $(\Rightarrow)$ Let $f : X \to Y$, $f' : X \to Y'$ be such that $gr(f) = gr(f')$. Then $img(f) =$

$img(f')$, hence $f_{|img\, f} = f'_{|img\, f'}$, $f = \iota_{img\, f,Y} \circ f_{|img\, f}$, and $f' = \iota_{img\, f,Y'} \circ f_{|img\, f}$. Hence, since F is inclusion preserving, $F(f) = \iota_{F(img\, f),FY} \circ F(f_{|img\, f})$ and $F(f') = \iota_{F(img\, f),FY'} \circ F(f_{|img\, f})$, i.e. $gr(F(f)) = gr(F(f'))$.
($\Leftarrow$) Let $X \subseteq Y$. Then $gr(\iota_{X,Y}) = gr(id_X)$, and hence $gr(F(\iota_{X,Y})) = gr(F(id_X)) = gr(id_{FX})$. Therefore, $F(X) \subseteq F(Y)$ and $gr(F(\iota_{X,Y})) = gr(\iota_{FX,FY})$, and hence $F(\iota_{X,Y}) = \iota_{FX,FY}$. □

(iv) The following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "id"] & Y \\ X' \arrow[u, "f_{X'}"] \arrow[r, "\iota"'] & X \arrow[u, "f"'] \end{tikzcd}$$

Hence, using the fact that F is inclusion preserving, also the following diagram commutes:

$$\begin{tikzcd} FY \arrow[r, "id"] & FY \\ FX' \arrow[u, "F(f_{X'})"] \arrow[r, "\iota"'] & FX \arrow[u, "Ff"'] \end{tikzcd}$$

Thus, $gr(Ff_{X'}) = gr(Ff)_{FX'}$.

(v) By Proposition 2.1 of V. Trnkovà, [17], every set functor $F : \mathcal{C} \to \mathcal{C}$ preserves non-empty finite intersections in the following sense: let $X, Y \subseteq Z$ be such that $X \cap Y \neq \emptyset$, then $img(F(\iota_{X\cap Y,Z})) = img(F(\iota_{X,Z})) \cap img(F(\iota_{Y,Z}))$. By item (iii), $img(F(\iota_{X\cap Y,Z})) = F(X \cap Y)$, $img(F(\iota_{X,Z})) = FX$, $img(F(\iota_{Y,Z})) = FY$. Thus $F(X \cap Y) = FX \cap FY$.

□

Trivially, not every functor is inclusion preserving. Just consider any functor obtained by mapping isomorphically the value on a given set into a set which is disjoint from the value of the functor on a subset.

However, in the following, we will prove that *any* functor is naturally isomorphic to an inclusion preserving functor, up-to the value on $\emptyset$. To this end, we first introduce the notion of *strict functor*:

Definition 3.3 (Strict Functor) • *A functor $F : \mathcal{C} \to \mathcal{C}$ is* strict *if $F\emptyset = \emptyset$.*

- *Let $F : \mathcal{C} \to \mathcal{C}$ be a functor, and let $F_\emptyset : \mathcal{C} \to \mathcal{C}$ be the strict functor defined as F apart from the value on $\emptyset$, where $F_\emptyset\emptyset = \emptyset$; moreover, for any X and empty function $\epsilon_X : \emptyset \to X$, $F(\epsilon_X) = \epsilon_{FX}$.*

Theorem 3.4 *Any functor $F : \mathcal{C} \to \mathcal{C}$ is naturally isomorphic up-to-$\emptyset$ to a functor which is inclusion preserving.*

Proof. Let $G : \mathcal{C} \to \mathcal{C}$ be the functor defined by: for all X, $G(X) = img(F_\emptyset(\iota_{X,V}))$, and for all $f : X \to Y$, $G(f) = G(X) \to G(Y)$, $G(f) = F_\emptyset(\iota_{Y,V})_{|imgF_\emptyset(\iota_{Y,V})} \circ F_\emptyset(f) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1}$.

• We prove that G is well defined. By definition, G preserves identities. Now we show that G preserves composition. Let $f : X \to Y$ and $g : Y \to Z$,

$$\begin{aligned} G(g \circ f) &= F_\emptyset(\iota_{Z,V})_{|imgF_\emptyset(\iota_{Z,V})} \circ F_\emptyset(g \circ f) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \\ &= F_\emptyset(\iota_{Z,V})_{|imgF_\emptyset(\iota_{Z,V})} \circ F_\emptyset(g) \circ F_\emptyset(f) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \\ &= F_\emptyset(\iota_{Z,V})_{|imgF_\emptyset(\iota_{Z,V})} \circ F_\emptyset(g) \circ (F_\emptyset(\iota_{Y,V})_{|imgF_\emptyset(\iota_{Y,V})})^{-1} \circ \\ &\quad \circ F_\emptyset(\iota_{Y,V})_{|imgF_\emptyset(\iota_{Y,V})} \circ F_\emptyset(f) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \\ &= G(g) \circ G(f) \end{aligned}$$

• We prove that the functor G is naturally isomorphic to $F_\emptyset$. Let $\tau = \{\tau_X : GX \to F_\emptyset X\}_X$ be the family of bijective functions, which are defined by $\tau_X = (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1}$. We prove that τ is a natural isomorphism. Let $f : X \to Y$. We show that $\tau_Y \circ G(f) \circ \tau_X^{-1} = F(f)$. By substitution on Gf,

$$\begin{aligned} \tau_Y \circ G(f) \circ \tau_X^{-1} &= \tau_Y \circ F_\emptyset(\iota_{Y,V})_{|imgF_\emptyset(\iota_{Y,V})} \circ F_\emptyset(f) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \circ \tau_X^{-1} \\ &= F_\emptyset(f) \text{ by definition of } \tau_X, \tau_Y \ . \end{aligned}$$

• We are left to prove that G is inclusion preserving, that is, $G(\iota_{X,Y}) = \iota_{GX,GY}$. Let $X \subseteq Y$ and let $\iota_{X,Y} : X \to Y$. Then,

$$\begin{aligned} G(\iota_{X,Y}) &= F_\emptyset(\iota_{Y,V})_{|imgF_\emptyset(\iota_{Y,V})} \circ F_\emptyset(\iota_{X,Y}) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \\ &= F_\emptyset(\iota_{Y,V} \circ \iota_{X,Y})_{|imgF_\emptyset(\iota_{Y,V})} \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1} \\ &= (F_\emptyset(\iota_{X,V}) \circ (F_\emptyset(\iota_{X,V})_{|imgF_\emptyset(\iota_{X,V})})^{-1})_{|imgF_\emptyset(\iota_{Y,V})} \\ &= \iota_{GX,GY} \end{aligned}$$

□

4 Continuity Properties of Set Functors

This section is devoted to the investigation of continuity properties for general endofunctors on set categories. We introduce two notions of continuity at a cardinal κ: the notion of κ-*based* functor, which generalizes Aczel's notion of set based functor, and the apparently weaker notion of κ-*reachable* functor, which essentially corresponds to Koubek's notion of "attainable cardinal", [14]. The two notions of κ-based and κ-reachable functor turn out to be equivalent.

The main result of this section is the κ-continuity Theorem stated in Section 1, i.e.:

Any set endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G such that, for all $|X| = \kappa$ infinite, if $|GX| \leq \kappa$, then $GX = \bigcup\{GY \mid |Y| < \kappa \ \wedge \ Y \subseteq X\}$.

The proof of this result is rather difficult and it makes use of a result on *almost disjoint systems.*

An interesting consequence of the above continuity result and of Axiom N is the following *class continuity* result, recently proved in [3,9]:

Any class endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G which is continuous on classes, i.e., for all proper classes X, $GX = \bigcup\{GY \mid Y \text{ set} \wedge Y \subset X\}$.

We start by introducing the definitions of κ-based and κ-reachable functor.

Definition 4.1 (κ-based) *Let κ be a cardinal. The functor $F : \mathcal{C} \to \mathcal{C}$ is* κ-based *if, for each object X and each $x \in FX$, there exists $Y \subseteq X$, $|Y| < \kappa$ such that $x \in img(F(\iota_{Y,X}))$, i.e. $FX = \bigcup\{img(F(\iota_{Y,X})) \mid Y \subseteq X \ \wedge \ |Y| < \kappa\}$.*

In virtue of Axiom N, Aczel-Mendler's notion of set based functor amounts to our notion of Ord-based functor.

Definition 4.2 (κ-reachable) *Let $\kappa > 1$.*

- *$x \in FX$ is* κ-reachable *from Y if $|Y| < \kappa$ and there exists $f : Y \to X$ such that $x \in img(Ff)$.*
- *$x \in FX$ is* κ-reachable *if there exists Y such that x is κ-reachable from Y.*
- *$x \in FX$ is* κ-unreachable *if it is not κ-reachable.*
- *FX is* κ-reachable *if for all $x \in FX$, x is κ-reachable.*
- *The functor F is* κ-reachable *if for all X, FX is κ-reachable.*

The following are immediate consequences of the above definition:

Lemma 4.3

(i) *If $|X| < \kappa$, then FX is κ-reachable.*

(ii) *If $|Y| \leq |X|$ and FX is κ-reachable, then also FY is κ-reachable.*

(iii) *If FX is κ-reachable, then $FX = \bigcup\{img(Ff) \mid f : Y \to X \ \wedge \ |Y| < \kappa\}$.*

Trivially, any κ-based functor is also κ-reachable. We show that also the converse implication holds, by exploiting properties of inclusion preserving functors and Theorem 3.4:

Lemma 4.4 *Let $F : \mathcal{C} \to \mathcal{C}$. Then F is κ-reachable if and only if F is κ-based.*

Proof.

($\Leftarrow$) By definition.

($\Rightarrow$) We prove the thesis for F inclusion preserving, then the thesis in its full generality follows from Theorem 3.4, since the property of being κ-based is preserved under natural isomorphism.
Let $x \in FX$. Since F is κ-reachable, for a set Y, $|Y| < \kappa$, there exists a function $f : Y \to X$ such that $x \in imgF(f)$. Let $Y_0 = img\ (f)$, then $|Y_0| < \kappa$ and, by Proposition 3.2 (ii), $F(f)_{|imgF(f)} : F(Y) \to F(Y_0)$. But if $x \in img(F(f))$, then $x \in F(Y_0) = img(F(\iota_{Y_0,X}))$. $\square$

Here we introduce the definition of *almost disjoint system*:

Definition 4.5 (Almost Disjoint System) *Let $|X| = \kappa$ infinite. An almost disjoint system $\mathcal{X}$ on X is a system $\mathcal{X} \subseteq \mathcal{P}X$ such that $|Y| = \kappa$, for all $Y \in \mathcal{X}$, and $|Y_1 \cap Y_2| < \kappa$, for all distinct $Y_1, Y_2 \in \mathcal{X}$.*

Generalizing Jech ([13], Lemma 24.7), we have:

Lemma 4.6 *Let $|X| = \kappa$, κ infinite. Then there exists an almost disjoint system $\mathcal{X}$ on X such that $|\mathcal{X}| > \kappa$.*

Proof. Following [13], Lemma 24.7, we prove a slightly more general statement, namely that there exists a family $\mathcal{F}$ of almost disjoint functions $f : \kappa \to \kappa$ such that $|\mathcal{F}| > \kappa$. Functions $f, g : \kappa \to \kappa$ are almost disjoint if $|\{\alpha \mid f(\alpha) = g(\alpha)\}| < \kappa$.
Assume that all maximal sets of almost disjoint functions over κ, which exist by Zorn's Lemma, have cardinality at most κ, then we derive a contradiction by diagonalization over κ. Let $|\{f_\alpha \mid \alpha < \kappa\}| \leq \kappa$ be a maximal set, we define a new element f by $f(\alpha) = \mu\beta.\forall\gamma < \alpha.\ \beta \neq f_\gamma(\alpha)$. Clearly, for all $\alpha < \kappa$ we have that $|\{\gamma \mid f(\gamma) = f_\alpha(\gamma)\}| = |\alpha| < \kappa$.
The original statement now is straightforward once we consider the graphs of the functions in $\mathcal{F}$, which are subsets of $\kappa \times \kappa = \kappa$. □

In order to prove the κ-continuity Theorem 4.9 below, we still need a couple of technical results:

Lemma 4.7 *Let $F : \mathcal{C} \to \mathcal{C}$ be inclusion preserving and let κ be infinite. Let $\{Y_\alpha\}_{\alpha \leq \kappa'}$ be a family of $\kappa' \geq 2$ sets such that $|Y_\alpha| = |X| = \kappa$. If FX is κ-unreachable, then for any family of injective functions $\{f_\alpha : Y_\alpha \to X\}_{\alpha \leq \kappa'}$ such that $|img(f_\alpha) \cap img(f_\beta)| < \kappa$ for all $\alpha \neq \beta$, there exists a family of κ' distinct κ-unreachable elements $\{x_\alpha \in F(img(f_\alpha))\}_{\alpha \leq \kappa'}$.*

Proof. We start by defining a family of functions $\{g_\alpha : X \to X\}_\alpha$ as follows: $g_\alpha(x) = \begin{cases} x & \text{if } x \in img(f_\alpha) \\ x_0 & \text{otherwise} \end{cases}$, where x_0 is any element in $img(f_\alpha)$. Now $|img(g_\alpha \circ f_\alpha)| = |img(f_\alpha)| = \kappa$, since f_α is injective, while $|img(g_\alpha \circ f_\beta)| < \kappa$, for $\alpha \neq \beta$, since $|img(f_\alpha) \cap img(f_\beta)| < \kappa$ and κ is infinite. Thus $F(img(g_\alpha \circ f_\beta))$ is κ-reachable for $\alpha \neq \beta$, while $F(img(g_\alpha \circ f_\alpha))$ contains a κ-unreachable element $\overline{x}_\alpha$. By Proposition 3.2 (ii), $F(img(g_\alpha \circ f_\alpha)) = img(F(g_\alpha \circ f_\alpha))$, hence there exists $x_\alpha \in img(F(f_\alpha))$ κ-unreachable such that $g_\alpha(x_\alpha) = \overline{x}_\alpha$. Now we show that $x_\alpha \notin img(F(f_\beta))$, for all $\beta \neq \alpha$. Namely, assume by contradiction that $x_\alpha \in img(F(f_\beta))$, then $\overline{x}_\alpha \in img(F(g_\alpha \circ f_\beta))$. That is, using Proposition 3.2 (ii), $\overline{x}_\alpha \in F(img(g_\alpha \circ f_\beta))$. Hence, since $\overline{x}_\alpha$ is κ-reachable as an element of $F(img(g_\alpha \circ f_\beta))$, then it is also κ-reachable as an element of $F(img(g_\alpha \circ f_\alpha))$, through the inclusion map $i : img(g_\alpha \circ f_\beta) \to img(g_\alpha \circ f_\alpha)$. Contradiction [5]. □

The following proposition [6] is a consequence of Lemma 4.6 and of the above

[5] This proof uses the hypothesis κ infinite. However, Lemma 4.7 holds also if we drop this hypothesis. In this case, the proof can be carried out by induction on κ.

[6] An essentially equivalent statement appears in [19], but without proof, and in [14], but under GCH.

Lemma:

Proposition 4.8 *Let $|X| = \kappa$ infinite. If FX contains κ-unreachable elements, then there exists $\kappa^{'} > \kappa$ such that FX contains $\kappa^{'}$ κ-unreachable elements.*

Proof. We prove the thesis in the case of F inclusion preserving. The result for a generic functor follows then from Theorem 3.4, by observing that κ-unreachable elements are preserved under natural isomorphism. Let F be inclusion preserving. By Lemma 4.6, there exists an almost disjoint system $\mathcal{X}$ on X such that $|\mathcal{X}| = \kappa' > \kappa$. For each $Y \in \mathcal{X}$, let us consider the inclusion map $\iota_{Y,X} : Y \to X$. By Lemma 4.7, we get that FX contains $\kappa^{'}$ κ-unreachable elements. □

Finally, by Proposition 4.8 and Theorem 3.4, we have:

Theorem 4.9 (κ-continuity) *Any set endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G such that, for all $|X| = \kappa$ infinite,*

$$|GX| \leq \kappa \implies GX = \bigcup\{GY \mid |Y| < \kappa \ \wedge \ Y \subseteq X\} \ .$$

A consequence of the κ-continuity Theorem 4.9 above is:

Theorem 4.10 (Class Continuity) *Any class endofunctor is naturally isomorphic up-to-$\emptyset$ to a functor G which is continuous on classes, i.e., for all proper classes X, $GX = \bigcup\{GY \mid Y \text{ set} \wedge Y \subset X\}$.*

Proof. We proceed by contradiction, i.e. we assume that F is not Ord-based. That is, there exists a class X such that FX contains Ord-unreachable elements. Then, by Theorem 4.9, $|FX| > Ord$, contradicting the Axiom N of NBG, i.e. the fact that all classes have cardinality Ord. □

Remark. Notice that, if GCH does not hold, then Theorem 4.9 above can be strengthened to the case where $|FX| \leq \kappa' \leq 2^{|X|}$, for $\kappa' > |X|$, under the hypothesis that there exists an almost disjoint system of cardinality κ'. Such cases are known to exist.

5 Non-existing Functors

Since set functors preserve injective maps with non-empty domain, then the operator underlying any functor is *monotone* w.r.t. *object cardinalities*, up-to-$\emptyset$. Thus non-monotone operators, such as $X \mapsto \{x \mid x \cap X = \emptyset\}$, are not extensible to functors. In Section 4, we have proved that operators underlying set functors also satisfy various continuity properties. Here we capitalize on these, and we show that some (monotone) set operators are not extensible to functors. Or, equivalently, that functors satisfying certain (cardinality) conditions on the object part do not exist. This problem has been addressed in previous papers for different classes of functors satisfying a system of *cardinality equations* on the object part, [16,19]. Here we focus on the case of cardinality conditions expressed by *inequations*. For example, we are able to prove that there is no functor $F : \mathcal{C} \to \mathcal{C}$ such that

$$|FX| = \begin{cases} 1 & \text{if } |X| < \omega \\ 2 & \text{if } |X| \geq \omega \end{cases}$$

Namely, assume by contradiction that such an F exists. By Theorem 3.4, w.l.o.g. we may assume F to be inclusion preserving. Now, let $X \geq \omega$ and let $FX = \{x_1, x_2\}$. Then, by Theorem 4.9, FX is ω-based. That is, there must be two finite sets $Y_1, Y_2 \subseteq X$ such that $FY_1 = \{x_1\}$ and $FY_2 = \{x_2\}$. But then, since F is monotone, $F(Y_1 \cup Y_2) = \{x_1, x_2\}$, contradicting the fact that $|Y_1 \cup Y_2| < \omega$.

The above example of a non-existing functor is an instance of a more general situation captured in Theorem 5.1 below. We recall that a cardinal κ is weakly inaccessible if it is a limit cardinal which is regular, i.e. κ is greater than the sup of a chain of length less than κ of cardinals less than κ. The proof of Theorem 5.1 follows the same line of the proof of the example above.

Theorem 5.1 *There is no functor $F : \mathcal{C} \to \mathcal{C}$ such that*

$$|FX| = \begin{cases} \kappa_1 & \textit{if } |X| < \kappa_1' \\ \kappa_2 & \textit{otherwise} \end{cases},$$

where $\kappa_1 < \kappa_2 < \kappa_1'$ and κ_1, κ_1' are successor and weakly inaccessible cardinals, respectively.

6 Characterizations of Final Coalgebras

In this section, we discuss characterizations of final coalgebras as (greatest) fixpoints. In particular, we start by analyzing Aczel's Special Final Coalgebra Theorem [1], and the notion of functor uniform on maps. We present and discuss various examples of functors which are not (naturally isomorphic to a functor) uniform on maps, and we provide alternative characterization theorems for various classes of functors. Aczel's Special Final Coalgebra Theorem focusses on inclusion preserving functors on $\mathcal{C}lass^*$ which are *uniform on maps*. We recall that the category $\mathcal{C}lass^*$ is the category of classes on a *non-wellfounded universe* of sets. Aczel's Theorem provides a characterization of the final coalgebra as the *greatest fixpoint* of the operator underlying the functor.

An instance of a functor uniform on maps is the powerset $\mathcal{P} : \mathcal{C}lass^* \to \mathcal{C}lass^*$. It is well-known that the non-wellfounded universe of sets V can be viewed both as the *greatest fixpoint* of the powerset, and as a *final $\mathcal{P}$-coalgebra*, (V, id_V), see e.g. [1,18] for more details. Before introducing the definition of functor uniform on maps, we recall that the *universe expanded with atoms in A* is a universe of sets whose elements can be atoms of a given class A. The non-wellfounded universe expanded with atoms in A, denoted by V_A, turns out to be the greatest fixpoint of the functor $\mathcal{P}(A + \_)$, i.e. $V_A = \nu X.\mathcal{P}(A + X)$. Moreover, one can prove that (V_A, id) is a final $\mathcal{P}(A + \_)$-coalgebra. Actually, also the functor $\mathcal{P}(A + \_)$ is an instance of a functor uniform on maps.

Definition 6.1 (Uniform on Maps) *Let $F : \mathcal{C}lass^* \to \mathcal{C}lass^*$. Then F is* uniform maps *if $\forall A \,.\, \exists \phi_A : F(A) \to V_A$ such that* $\forall f : A \to B, \forall a \in F(A),\ F(f)(a) \;=\; \widehat{f} \circ \phi_A(a)$, *i.e.:*

$$\begin{tikzcd} F(A) \arrow[r, "\phi_A"] \arrow[d, "F(f)"'] & V_A \arrow[d, "\widehat{f}"] \\ F(B) \arrow[r, "\iota_{F(B),V}"'] & V \end{tikzcd}$$

where V_A is the expanded universe with atoms in A, and, for any $f : A \to V$, $\widehat{f} : V_A \to V$ is defined by

$$\widehat{f}(x) = \{f(y) \mid y \in x \cap A\} \cup \{\widehat{f}(y) \mid y \in x \cap V_A\}\ .$$ [7]

Notice that the above definition of functor uniform on maps is not exactly the definition of [1], the difference being that in the original definition the commutativity of the diagram is required only for functions $f : A \to V$. Our definition has the advantage of implying that any functor uniform on maps is also inclusion preserving. Otherwise, this has to be added as hypothesis in the Special Final Coalgebra Theorem.

A functor uniform on maps is a functor for which the behaviour on morphisms is determined by the behaviour on objects. Namely, by Definition 6.1, the action of F on morphisms is given in terms of the family $\{\phi_A\}_A$. But one can show that this family is in turn determined by the action of F on objects, see [9] for more details.

Aczel's Special Final Coalgebra Theorem generalizes the fact that the greatest fixpoint of the powerset functor together with the identity map is a final coalgebra:

Theorem 6.2 (Special Final Coalgebra Theorem, [1]) *Let $F : \mathcal{C}lass^* \to \mathcal{C}lass^*$ be uniform on maps. Then (J_F, id) is a final F-coalgebra, where J_F is the greatest fixpoint of the set operator underlying F, i.e. $J_F = \nu X.FX$.*

The definition of functor uniform on maps and the proof of the above theorem make essential use of the fact that the universe of sets underlying $\mathcal{C}lass^*$ is non-wellfounded (see [1] for more details). Thus the Special Final Coalgebra Theorem is not directly generalizable to a generic class category.

Various questions naturally arise on the collection of functors uniform on maps. It is well-known that there are functors which are not uniform on maps, i.e. the Identity functor *Id*. However, *Id* is naturally isomorphic to a functor uniform on maps, i.e. the functor $\mathcal{S}ing$ defined by $\mathcal{S}ing(X) = \{\{x\} \mid x \in X\}$. So, the question naturally arises whether *all* functors on $\mathcal{C}lass^*$ are naturally isomorphic to a functor which is uniform on maps. The answer is negative; in what follows we give various counterexamples. The first counterexample is given by the subfunctor of the powerset, which yields all subsets of a given cardinality. Other counterexamples are obtained by considering the functor R_3, which amounts to the 3-bounded powerset,

[7] By a slight abuse of notation, in the diagram above, we use the same symbol f to denote both the function $f : A \to B$ and the function $f : A \to V$.

i.e. the powerset of all sets of at most three elements. The functor R_3 is studied in [6] for different reasons. This functor is special, because, contrary to all the other functors R_n for $n \neq 3$, it admits two variants, the functors F_1, F_2, with the same behaviour on objects, but with a different behaviour on morphisms. While R_3 is uniform on maps, neither F_1 nor F_2 are naturally isomorphic to a functor uniform on maps. Finally, our last example of a functor which is not naturally isomorphic to a functor uniform on maps, is a variant of the iterated powerset functor. Here we provide precise definitions:

Definition 6.3

(i) **κ-Powerset**. *Let $B_\kappa : \mathcal{C} \to \mathcal{C}$ for any cardinal κ, be defined by: $B_\kappa X = \{u \subseteq X \mid |u| = \kappa\} \cup \{\emptyset\}$. For all $f : X \to Y$, $B_\kappa f : B_\kappa X \to B_\kappa Y$ is defined by*

$$B_\kappa f(u) = \begin{cases} f^+(u) = \{fx \mid x \in u\} & \textit{if } |fu| = \kappa \\ \emptyset & \textit{otherwise} . \end{cases}$$

(ii) **3-Bounded Powerset**. *Let $R_3 : \mathcal{C} \to \mathcal{C}$ be such that $R_3 X = \{u \mid 0 < |u| \leq 3\}$, and, for any $f : X \to Y$, $R_3 f = f^+$.*
Let $F_1 X = R_3 X = \{u \subseteq X \mid 0 < |u| \leq 3\}$. For all $f : X \to Y$,

$$F_1 f(u) = \begin{cases} \{fa\} & \textit{if } |f^+(u)| = 2 \ \wedge \ u = \{a, b, c\} \ \wedge \ fa = fb \\ f^+(u) & \textit{otherwise} . \end{cases}$$

Let $F_2 X = R_3 X = \{u \subseteq X \mid 0 < |u| \leq 3\}$. For all $f : X \to Y$,

$$F_2 f(u) = \begin{cases} \{fc\} & \textit{if } |f^+(u)| = 2 \ \wedge \ u = \{a, b, c\} \ \wedge \ fa = fb \\ f^+(u) & \textit{otherwise} . \end{cases}$$

(iii) **Iterated Powerset**. *Let G be the iterated contravariant powerset functor defined by $G(X) = P(P(X))$ for all classes X, and for all $f : X \to Y$, $Gf : P(P(X)) \to P(P(Y))$, is such that $G(f) = f^\circ$, where f° is defined by*

$$f^\circ(\alpha) = \{y \subseteq f^+(\cup\alpha) \ \mid \ f^{-1}(y) \in \alpha\},$$

where by $f^{-1}(y)$, for $y \in \mathcal{P}Y$, we denote the inverse image of y under f.

One can check that the functors above are all inclusion preserving. However,

Proposition 6.4 *The functors $B_\kappa, F_1, F_2, G : \mathcal{C}lass^* \to \mathcal{C}lass^*$, for $2 \leq \kappa < \omega$, are not naturally isomorphic to a functor uniform on maps.*

Proof.

B_κ We proceed by contradiction. Let $F : \mathcal{C} \to \mathcal{C}$ be a uniform on maps functor such that $B_\kappa \cong F$. Then there exists $\{\tau_X : B_\kappa X \xrightarrow{\sim} FX\}_X$. In particular, for any $f : X \to X$, we have:

$$\begin{tikzcd}
B_\kappa X \arrow[r, "\tau_X"] \arrow[d, "B_\kappa f"'] & FX \arrow[r, "\phi_X"] \arrow[d, "Ff"'] & V_X \arrow[d, "\widehat{f}"] \\
B_\kappa X \arrow[r, "\tau_X"'] & FX \arrow[r, "\iota"'] & V
\end{tikzcd}$$

Let $f : X \to X$ be the identity on X, then, by the commutativity of the above diagrams we have: $\phi_X \circ \tau_X(u) = \tau_X(u)$, for all $u \in B_\kappa X$. Thus, in particular $\phi_X \circ \tau_X$ is injective. Moreover, one can easily check that $\phi_X \circ \tau_X(\emptyset) = \tau_X(\emptyset) = \emptyset$ (by considering a suitable non-injective function $f' : X \to X$). Now, let $u \in B_\kappa X$, $|u| = \kappa$, and let f' be such that $|f'^{+}(u)| = \kappa - 1$. Then $B_\kappa f'(u) = \emptyset$ and, by the commutativity of the above diagrams, $\widehat{f'} \circ \phi_X \circ \tau_X(u) = \tau_X(\emptyset)$. Now, since $u \neq \emptyset$ and $\phi_X \circ \tau_X$ is injective, then $\phi_X \circ \tau_X(u) \neq \emptyset$ and hence $\widehat{f'} \circ \phi_X \circ \tau_X(u) \neq \emptyset$, contradicting the fact that $\tau_X(\emptyset) = \emptyset$.

F_1 We proceed by contradiction. Let F be a uniform on maps functor such that $\tau : F_1 \cong F$. Then, in particular, for any $f : X \to X$, we have:

$$\begin{tikzcd}
F_1 X \arrow[r, "\tau_X"] \arrow[d, "F_1 f"'] & FX \arrow[r, "\phi_X"] \arrow[d, "Ff"'] & V_X \arrow[d, "\widehat{f}"'] \\
F_1 X \arrow[r, "\tau_X"'] & FX \arrow[r, "\iota"'] & V
\end{tikzcd}$$

Let $f : X \to X$ be the identity on X, then we have: $\phi_X \circ \tau_X(u) = \tau_X(u)$, for all $u \in F_1 X$. Thus $\phi_X \circ \tau_X$ is injective. Now, let $u = \{a, b, c\}, u' = \{a, b, c'\}$, $u, u' \subseteq X$, and let $f' : X \to X$ be the identity on X, but for $f' b = a'$. Since $\phi_X \circ \tau_X$ is injective, $\phi_X \circ \tau_X(u) \neq \phi_X \circ \tau_X(u')$. But, by commutativity of the above diagrams, we have: $\widehat{f'} \circ \phi_X \circ \tau_X(u) = \widehat{f'} \circ \phi_X \circ \tau_X(u')$. Thus, since $\phi_X \circ \tau_X$ is injective, then $\phi_X \circ \tau_X(u) = \{a\} \cup v$ and $\phi_X \circ \tau_X(u') = \{b\} \cup v$, for some v for which $a \notin v \ \vee \ b \notin v$ (or vice versa).
Now one can check that, by considering $f'' : X \to X$ to be the identity on X, but for $f'' c = f'' c' = b$, we get a contradiction.

F_2 Analogous to F_1 above.

G The functor G behaves as $\mathcal{P}^2$, where $\mathcal{P}^2$ is the composition of the standard powerset functor with itself, on injective functions, while on non-injective functions f, $G(f)$ behaves as $\mathcal{P}^2(f)$ only on *f-saturated* elements, otherwise it yields $\emptyset$. By an *f-saturated* element $u \in \mathcal{P}^2(X)$ we mean an element such that $\mathcal{P}(f)(\alpha) = \mathcal{P}(f)(\beta) \ \wedge \ \alpha \in u \ \Rightarrow \ \beta \in u$. Hence G cannot be naturally isomorphic to a functor which is uniform on maps. □

Now the question naturally arises whether the functors above still admit the greatest fixpoint as final coalgebra.

The answer is positive, but for the case of the functor B_2 for which the least fixpoint is a final coalgebra:

Proposition 6.5

(i) *The set $\{\emptyset\}$ is the least fixpoint of B_κ, for all $\kappa \geq 2$. Moreover $(\{\emptyset\}, id)$ is a* *final coalgebra for B_κ, for all $\kappa \geq 2$.*

(ii) *For all $\kappa > 2$, $\{\emptyset\}$ is the unique fixpoint of B_κ; while B_2 admits other fixpoints. The greatest fixpoint of B_2 is the least set A closed under the following rules:* $\emptyset \in A \quad \dfrac{x, y \in A}{\{x, y\} \in A} \quad \dfrac{x \in A}{z = \{x, z\} \in A}$.[8]

(iii) *The set $\{x\}$, where x is a self-singleton set, i.e. $x = \{x\}$, is the greatest fixpoint of both F_1 and F_2. Moreover, $(\{x\}, id)$ is a final coalgebra for F_1, F_2.*

(iv) *The universe V is the greatest fixpoint of G and (V, id) is a final G-coalgebra.*

Notice that, however, in all the above cases, final coalgebras are fixpoints. When this is the case, and the functor is inclusion preserving, one can prove that the greatest fixpoint is a *weakly* final coalgebra. This holds on a generic class category:

Theorem 6.6 *Let F be an inclusion preserving endofunctor on a class category $\mathcal{C}$[9]. If F admits a fixpoint as final coalgebra, then the greatest fixpoint J_F together with the identity is a weakly final coalgebra.*

Proof. Let $\bar{X}$ be a fixpoint of F such that $(\bar{X}, id)$ is final F-coalgebra, let (X, α) be an F-coalgebra, and let $f : (X, \alpha) \to (\bar{X}, id)$ be the final morphism. Then diagram (1) below commutes since $(\bar{X}, id)$ is final, while diagram (2) commutes since F is inclusion preserving. Thus, $\iota_{\bar{X}, J_F} \circ f$ is an F-coalgebra morphism from (X, α) into (J_F, id).

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "(1)"] & \bar{X} \arrow[r, "\iota_{\bar{X},J_F}"] \arrow[d, "id"] \arrow[dr, phantom, "(2)"] & J_F \arrow[d, "id"] \\
FX \arrow[r, "Ff"'] & F\bar{X} \arrow[r, "F(\iota_{\bar{X},J_F})"'] & FJ_F
\end{tikzcd}$$

□

Both in the cases of the functors B_κ and F_1, F_2 in Proposition 6.5 above, the fixpoints which are final coalgebras amount to singleton sets. Actually, these are instances of a general situation: when a functor on a class category has a singleton fixpoint, then this is a final coalgebra.

Theorem 6.7 *Let F be an endofunctor on a class category $\mathcal{C}$. If F admits a fixpoint $\bar{X}$ which is a singleton set, then $(\bar{X}, id)$ is a final F-coalgebra.*

Proof. Let (X, α) be an F-coalgebra. Then the unique coalgebra morphism from (X, α) to $(\bar{X}, id)$ is the function $f : X \to \bar{X}$ which maps all elements of X into the unique element $\bar{X}$. □

If we work "up-to natural isomorphism", then the following strong result holds:

[8] Another fixpoint of B_2 is $x = \{x, \emptyset\}$.

[9] The assumption $\mathcal{C}$ class category is necessary for ensuring that F admits greatest fixpoint in $\mathcal{C}$.

Theorem 6.8 *Let F be an endofunctor on a class category $\mathcal{C}$. Then F is naturally isomorphic to a functor whose unique fixpoint, together with the identity, is a final coalgebra.*

Proof. By the strengthening of Aczel's Final Coalgebra Theorem, F admits final coalgebra (Ω, α_Ω), where $\alpha_\Omega : \Omega \to F(\Omega)$ is an isomorphism. Let us define the functor G as follows. Let $G\Omega = \Omega$. For any $X \neq \Omega$, choose GX such that there exists $\alpha_X : GX \to FX$ bijective, but $GX \neq X$. Moreover, if $f : X \to Y$, let $Gf = (\alpha_Y)^{-1} \circ f \circ \alpha_X$.
One can easily check that G is naturally isomorphic to F, Ω is the unique fixpoint of G, and (Ω, id) is a final G-coalgebra. □

But using an analogous technique to that used to prove Theorem 6.8 above, one can also show that:

Theorem 6.9 *Let F be an endofunctor on a class category $\mathcal{C}$. Then F is naturally isomorphic to a functor which has no fixpoints.*

Finally, we list a number of remarks and problems arising from the investigation of this section.

- The hypothesis F uniform on maps in Aczel's Special Final Coalgebra Theorem gives a sufficient but not a necessary condition for an inclusion preserving functor to have the greatest fixpoint as final coalgebra. The functors F_1, F_2 previously considered are counterexamples to the necessity. Thus another approach is that of generalizing the definition of functor uniform on maps, in order to capture a wider collection of functors. Actually, by analyzing the proof of the Special Final Coalgebra Theorem, one can axiomatize the properties of the function operator $\widehat{\ }$, which allow to prove that the greatest fixpoint is a (weakly) final coalgebra. This leads us to introduce a notion of functor *generally uniform on maps*, where the function $\widehat{f}$ is substituted by f^*, for $(\ )^*$ a generic function operator which is "well-behaved", i.e. it satisfies the axiomatized properties. Then the greatest fixpoint of a functor generally uniform on maps is a (weakly) final coalgebra. This should allow us to comprise also the cases of the functors F_1, F_2 (and B_κ, for weakly final coalgebras). Here we omit the details.
- All non-uniform on maps functors considered in Section 6 miss the uniformity property because of their "strange" behaviour on non-injective functions, while on injective functions they are well-behaved. Thus the question naturally arises whether all inclusion preserving functors are well-behaved on injective functions, i.e. they are always uniform on maps w.r.t. injective functions.
- Moreover, all (non-uniform on maps) functors of Section 6 admit as final coalgebra a fixpoint (although not necessarily the greatest one), together with the identity. Therefore, another question which arises is whether this is always the case for inclusion preserving functors.
- The functor G considered in Section 6 behaves as the standard iterated powerset $\mathcal{P}^2$ on objects and on injective functions. Moreover, both G and $\mathcal{P}^2$ admit (V, id) as final coalgebra. We conjecture that this is an instance of a general situation,

where functors which coincide on objects and on injective functions have the same final coalgebras.

7 Final Remarks and Directions For Future Work

We conclude the paper with a list of final comments and lines for future work.

- In this paper, we have addressed the problem of the existence of final coalgebras and we have explored characterizations of (weakly) final coalgebras as fixpoints. Another problem which is relevant for the Semantics of Programming Languages is that of studying the equivalences induced by the final morphisms, i.e. morphisms into the final coalgebra. If the functor is "well-behaved", i.e. if it preserves weak pullbacks, then the equivalences induced by final morphisms can be characterized as *greatest coalgebraic bisimulations*. However, there are functors which do not preserve weak pullbacks, such as for example the functor B_2 studied in Section 6. It would be interesting to explore coinductive characterizations of final equivalences, possibly in terms of bisimulations which are not necessarily coalgebraic bisimulations, e.g. in the line of [2,12].
- We have seen that the analysis carried out in Section 3 on inclusion preserving functors, and the general continuity results discussed in Section 4 can be fruitfully put to use for proving many interesting properties of set functors, inter alia the (non-)existence of functorial extensions for various set operators. In a forthcoming paper, we use them to study the uniqueness (up-to natural isomorphism) of functorial extensions of various classes of set operators, extending previous works on DVO functors (i.e. functors Determined by their Value on Objects), see [6].
- In Section 5, we have given a contribution to the theory of non-existing functors, by considering a special system of cardinality inequations which cannot be satisfied by any functor. It would be interesting to explore in greater generality the systems of cardinality inequations which can be satisfied by a functor. In the literature, this has been done for systems of equations, see [16,19], but, as far as we know, a treatment of inequations is still missing.

References

[1] Aczel, P., "Non-Well Founded Sets", CSLI Lecture Notes **14**, Stanford, 1988.

[2] Aczel, P., and Mendler, N., *A Final Coalgebra Theorem*, Category Theory and Computer Science, Pitt, D., et al. eds., Springer LNCS **389** (1989), 357–365.

[3] Adamek, J., Milius, S., and Velebil, J., *On Coalgebra Based on Classes*, Theoretical Computer Science **316** (2004), 3–23.

[4] Adamek, J., Milius, S., and Velebil, J., *A General Final Coalgebra Theorem*, Mathematical Structures in Computer Science **15.3** (2005), 409–432.

[5] Adamek, J., and Trnkova, V., *Automata and Algebras in a Category*, Kluwer Publishing Company, Dordrecht, 1990.

[6] Barkhudaryan, A., El Bashir, R., and Trnkova, V., *Endofuncors of set and Cardinalities*, Cahiers de Topologie et Geometrie Differentielle Categorique **XLIV**(3) (2003), 117–239.

[7] Barr, M., *Terminal coalgebras in well-founded set theory*, Theoretical Computer Science **124** (1994), 189–192.

[8] Barwise, J., and Moss, L., "Vicious Circles", Lecture Notes **60**, CSLI Publications, 1996.

[9] Cancila, D., "Investigations in the Categorical Foundations and Applications of Coalgebras and Hypersets", Ph.D. thesis, Udine University, Italy, 2004.

[10] Danner, N., and Moss, L., *On the Foundations of Corecursion*, Logic Journal of the IGPL **5**(2) (1997), 231–257.

[11] Forti, M., and Honsell, F., *Set Theory with Free Construction Principles*, Classe di Scienze, Annali Scuola Normale Sup. di Pisa **10**(4) (1983), 493–522.

[12] Honsell, F., Lenisa, M., *Coalgebraic Coinduction in (Hyper)set-theoretic Categories*, CMCS'2000, Electronic Notes in Theoretical Computer Science **33** (2000).

[13] Jech, T., *Set Theory*, Academic Press, New York, 1978.

[14] Koubek, V., *Set functors*, Comment. Math. Univ. Carolinae **12** (1971), 175–195.

[15] Moss, L., *Parametric Corecursion*, Theoretical Computer Science (2000).

[16] Rhineghost, Y.,T., *The Emergence of Functors - A Continuation of "The Functor that Wouldn't be"*, Categorical Perspectives, Birkhauser Verlag, Trends in Mathematics, (2001), 37–46.

[17] Trnková, V., *Some Properties of Set Functors*, Comment. Math. Univ. Carolinae **10**(2) (1969), 323–352.

[18] Turi, D., and Rutten, J., *On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces*, Mathematical Structures in Computer Science **8**(5) (1998), 481–540.

[19] Zmrzlina, A., *Too Many Functors – A Continuation of "The Emergence of Functors"–*, Categorical Perpectives, Birkhauser Verlag, Trends in Mathematics, (2001), 47–62.