A Condensed Goal-Independent Bottom-Up Fixpoint Semantics
Modeling the Behavior of *tccp*

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In this paper, we present a new compositional bottom-up semantics for the *Timed Concurrent Constraint Language* (*tccp* in short) which is defined for the full language. In particular, is able to deal with the non-monotonic characteristic of *tccp*, which constitutes a substantial additional technical difficulty w.r.t. other compositional denotational semantics present in literature (which do not tackle the full language).

The semantics is proved to be (correct and) fully abstract w.r.t. the full behavior of *tccp*, including infinite computations. This is particularly important since *tccp* has been defined to model reactive systems and they may not terminate with a purpose.

The overall of these features makes our proposal particularly suitable as the basis for the definition of semantic-based program manipulation tools (like analyzers, debuggers or verifiers), especially in the context of reactive systems.

Categories and Subject Descriptors: F.3.2 [LOGICS AND MEANINGS OF PROGRAMS]: Semantics of Programming Languages—Denotational semantics; Algebraic approaches to semantics; D.3.1 [PROGRAMMING LANGUAGES]: Formal Definitions and Theory—Semantics

General Terms: Languages, Theory

Additional Key Words and Phrases: Concurrent Constraint Programming

ACM Reference Format:

D01: http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION

The concurrent constraint paradigm (*ccp* in short; [Saraswat 1993]) is a simple but powerful model for concurrent systems. It is different from other programming paradigms mainly due to the notion of store-as-constraint that replaces the classical store-as-valuation model. In this way, the languages from this paradigm can easily deal with partial information: an underlying constraint system handles constraints on system variables.

Within the *ccp* family, [de Boer et al. 2000] introduced the *Timed Concurrent Constraint Language* (*tccp* in short) by adding to the original *ccp* model the notion of time and the ability to capture the absence of information. With these features, it is possible to specify—in a very natural way—behaviors typical of reactive systems such as

This work has been partially supported by the EU (FEDER) and the Spanish MEC/MICINN, ref. TIN 2010-21062-C02-02, and by Generalitat Valenciana, ref. PROMETEO2011/052.

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D01: http://dx.doi.org/10.1145/0000000.0000000

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYY.
timeouts or preemption actions. For instance, we can have more compact and realistic models since we do not need to explicitly model error signals: the absence of information from a given (sub)system can be enough to react to an erroneous situation. Thanks to all its features, tccp is certainly the most expressive dialect of the ccp family.

In the literature, much effort has been devoted to the development of appropriate denotational semantics for languages in the ccp paradigm (e.g. [de Boer and Palamidessi 1991; de Boer et al. 1995; Falaschi et al. 1997]). Compositionality and full abstraction are two highly desirable properties for a semantics, since they are needed for many purposes. A fully abstract model can be considered the semantics of a language [de Boer and Palamidessi 1991].

In [de Boer et al. 1995], the difficulties for handling nondeterminism and infinite behavior in the ccp paradigm was investigated. The authors showed that the presence of nondeterminism, local variables and synchronization require relatively complex structures for the denotational model of (non timed) ccp languages. In most ccp languages, nondeterminism is defined in terms of a global choice, which poses even more difficulties than a local-choice model [Falaschi et al. 1997].

Successively, [Nielsen et al. 2002a] showed that for timed concurrent constraint languages, the presence of timing constructs which handle negative information in addition to non-determinism and local variables significantly complicates the definition of compositional and fully abstract semantics. Moreover, infinite behaviors (which become natural in the timed extensions) are an additional nightmare [de Boer et al. 1995].

Presumably because of all these mentioned difficulties, for all languages of the ccp family, the proposals of compositional semantics in the literature have been given by introducing (quite) severe restrictions on the languages. Essentially, they all limit the use of negative information and non-determinism, that are the distinguishing features that enhance the expressiveness of the paradigm w.r.t. other traditional ones. For us, this is contradictory and certainly unsatisfactory. Thus, we strived to develop a semantics which is fully abstract for the full tccp language. This is particularly important when one is interested in applying the semantics to develop (semantics-based) fully automatic program manipulation tools (like debuggers, verifiers and analyzers).

We have a long experience [Comini et al. 1999; Comini 1998; Comini et al. 2003; Comini et al. 2001; Alpuente et al. 2003; Alpuente et al. 2005; Bacci and Comini 2011] in the development of semantics-based program manipulation tools for declarative languages via the Abstract Interpretation approach. Abstract Interpretation [Cousot and Cousot 1979] is a theory of approximation of discrete systems that allows one to formally specify provably correct approximation processes for the behavior of any computing system.

It is important to observe that both our direct experience and the not very satisfactory results in [Alpuente et al. 2003; Falaschi et al. 2007] showed that, besides the necessary requirement of full abstraction and the straightforward usefulness of compositionality, there are some other properties of the (concrete) semantics that are particularly relevant for having an effective and efficient implementation which computes a precise-as-possible (abstract) approximate semantics. We are confident that this statement applies also to other semantic-based debugging or verification techniques not based on abstract interpretation.

Let us point out these properties and discuss about their positive effects.

**Goal-independent.** A semantics has a goal-independent definition when the denotation of any compound (nested) expression is defined in terms of the denotations of most general calls. For instance, the semantics of an expression like $e := f(g(v_1, v_2), v_3)$ is obtained by suitable semantics operators which, considering
values \(v_1, v_2, v_3\) and the semantics of \(f(x, y)\) and \(g(w, z)\), can reconstruct the proper semantics of \(e\). Operational (top-down) semantics are rarely defined in this way since it is more natural (and easy) to give a (compositional) goal-dependent definition which produces the effects of the current expression that has to be evaluated. When one is interested in the results of the evaluation of a specific expression \(e\), it would make little sense to define a more complicated goal-independent semantics formalization that first evaluates all most general expressions and then tailors such evaluations on \(e\) to mimic the effects of a top-down goal-dependent resolution mechanism. In the tailoring process, many parts of the computed denotations will not be used and thus much computation effort would be wasted.

However, when we are no longer focused on determining the actual evolution of a specific expression but we are interested in determining the properties of a program for all possible executions, things change radically. In this case, we necessarily have to determine the semantic information regarding all possible expressions, and then it is more economical to have a goal-independent definition and compute just the semantics of most general calls (and, when is needed, reconstruct from these the semantics of specific instancies).

**Condensed.** A semantics is **condensed** when denotations contain only the minimal necessary number of semantic elements that are needed to characterize the classes of semantically equivalent syntactic objects (or, in other words, the minimal information needed to distinguish a syntactic object \(x\) from the other syntactic objects that are not semantically equivalent to \(x\)).

This may not seem a useful property for a concrete semantics, which—in general—would nevertheless contain infinite elements even when is condensed. However, this reduction could anyway frequently change some infinite denotations into finite ones and—most important—all the abstractions of a condensed concrete semantics will inherit this property by construction. Hence, by having minimal (abstract) denotations, one obtains by definition algorithms that compute just the minimal number of (abstract) semantic elements. This is definitely a stunning advantage over non condensed approaches which rarely can regain this efficiency in some other way. One could argue that it would be possible to live with a simpler non-condensed concrete semantics and then, for each abstraction of interest, work on the specific case to find out its condensed representation. We find more economical (especially in the long run) to do the effort once for the concrete semantics and then obtain, by construction, that all abstractions are condensed (with no additional effort).

**Bottom-up.** A **bottom-up** definition (in addition to the previous properties), has also an immediate direct benefit for abstract computations. With a bottom-up definition, at each iteration we have to collect the contributions of all rules. For each rule we will use the join operation of the abstract domain in parallel onto all components of the body of the rule. With a top-down definition instead, we have to expand one component of the goal expression at a time, necessarily using several subsequent applications of the join operation (of the abstract domain) over all components, rather than a unique simultaneous join of all the semantics of components. The reduced use of the join of a bottom-up formulation has a twofold benefit. On one side, it speeds up convergence of the abstract fixpoint computation. On the other side, it considerably improves precision.

Thus—with the application to program manipulation tools in mind—we have developed a new (small-step) compositional, bottom-up, goal-independent and condensed semantics which is (correct and) **fully abstract** w.r.t. the small-step behavior of **full tccp**.

To obtain this semantics the idea is to enrich the classical behavioral timed traces with
information about the essential conditions that the store must (or must not) satisfy in order to make the program proceed with one or another execution branch. Thus, we associate conditions to the store of each computation step and then we collect just the most general hypothetical computations. Since conditions are constructed by using only the information in the guards of a program, we obtain a condensed semantics which also deals with non-monotonicity, because into denotations we have the minimal information needed to exploit computations arising from absence of information.

Since tccp was originally defined to model reactive systems, which many times include systems that do not terminate with a purpose, we have developed our semantics to distinguish among terminating, suspending and non-terminating computations. This improves the original semantics for tccp defined in [de Boer et al. 2000] which identifies suspending and non-terminating computations. In particular, terminating computations are those that reach a point in which no agents are pending to be executed. Suspending computations are those that reach a point in which there are some agents pending to be executed, but there is not enough information in the store to entail the conditions that would make them evolve. We think it is essential to distinguish these two kinds of computations since, conceptually, a suspended computation has not completely finished its execution, and, in some cases, it could be a symptom of a system error.

To complete our proposal, we also define a big-step semantics (by abstraction of our small-step semantics) which tackles also outputs of infinite computations. We prove that its fragment for finite computations is (essentially) isomorphic to the traditional big-step semantics of [de Boer et al. 2000]. Moreover, we also formally prove that it is not possible to have a correct input-output semantics which is defined solely on the information provided by the input/output pairs.

Organization of the paper. The rest of the paper is organized as follows. Section 2 recalls the foundations of the tccp language. Section 3 introduces our small-step denotational semantics for tccp. It also includes illustrative examples for the main concepts. Then, Section 4 introduces our big-step semantics and formally relates it to the (original) one of [de Boer et al. 2000]. Section 5 discusses about applications. Finally, Sections 6 and 7 present related work and conclude.

To improve readability of the paper, the most technical results and the proofs (of all results) can be found in the electronic appendix.

2. PRELIMINARIES

The languages defined within the ccp paradigm (as extensions of the original model of Saraswat in [Saraswat and Rinard 1990]) are parametric w.r.t. a cylindric constraint system. The constraint system handles the data information of the program in terms of constraints.

2.1. Cylindric constraint systems

An elegant formalization of constraint systems, the cylindric constraint systems, was introduced in [Saraswat and Rinard 1990], where a hiding operator is defined in terms of a general notion of existential quantifier (to handle local variables). However in this work, since we are dealing with tccp, we use the formalization of [de Boer et al. 2000].

A cylindric constraint system is an algebraic structure $C = \langle C, \preceq, \lor, \land, \top, \bot, \forall, \exists \rangle$, where $\forall$ is a denumerable set of variables, such that:

1. $(C, \preceq, \lor, \land, \top, \bot)$ is a complete algebraic lattice where $\lor$ is the lub operator, $\land$ is the glb operator, $\top$ and $\bot$ are respectively the greatest and least element of $C$. 

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(2) For each $x \in \text{Var}$ there exist a cylindric operator $\exists_x : C \rightarrow C$ such that, for any $x, y \in \text{Var}$ and $c, d \in C$,

$$c \vdash \exists_x c \quad \quad c \vdash d \Rightarrow \exists_x c \vdash \exists_x d$$

$$\exists_x(\exists_y c) = \exists_y(\exists_x c) \quad \quad \exists_x(c \otimes \exists_x d) = \exists_x c \otimes \exists_x d$$

where $\vdash$, called entailment, is the inverse relation of $\rightarrow$.

In the sequel, we abuse of notation and, given $C \subseteq C$, write $\exists_x C$ for $\{\exists_x c \mid c \in C\}$. We can find in the literature several examples of cylindric constraint systems that are useful when modeling data structures, logic programs or other specific domains [Saraswat et al. 1991; de Boer et al. 1995; de Boer et al. 1997; Aristizábal et al. 2011].

In the illustrative examples throughout the paper we will use, for the sake of simplicity, the following classical cylindric constraint system $L$ of linear disequalities. The domain of constraints $L$ is formed by taking equivalence classes, modulo logical equivalence $\equiv$, of finite conjunctions of either linear disequalities (strict and not) or equalities over $\mathbb{Z}$ and $\text{Var} = \{x, y, \ldots\}$ (e.g. $x > 4, y \geq 10 \land w < -3, \ldots$). The entailment relation is implication $\Rightarrow$ (thus, the order of the lattice is $\equiv$). The lub is conjunction $\land$ and $\exists_x$ is the operation which removes (after information has been propagated within a constraint) all conjuncts referring to variable $x$ (e.g. $\exists_x(x = y \land x > 3) = y > 3$). It can be easily verified that $L := \langle C, \equiv, \land, \lor, \text{false}, \text{true}, \text{Var}, \exists \rangle$ is a cylindric constraint system.

### 2.2. Timed Concurrent Constraint Programming

The $tccp$ language, introduced in [de Boer et al. 2000], is particularly suitable to specify concurrent reactive systems. In $tccp$, the computation progresses as the concurrent and asynchronous activity of several agents that can (monotonically) accumulate information in a store or also query information from it. The notion of time is introduced by defining a discrete and global clock\(^1\) and progresses depending on agents that are executed as defined in the following.

Given a cylindric constraint system $C = \langle C, \equiv, \land, \lor, \text{false}, \text{true}, \text{Var}, \exists \rangle$ and a set of process symbols $P$, the syntax of agents is given by the following grammar:

$$A ::= \text{skip} \mid \text{tell}(c) \mid A \parallel A \mid \exists_x A \mid \sum_{i=1}^{n} \text{ask}(c_i) \rightarrow A \mid \text{now } c \text{ then } A \text{ else } A \mid p(x_1, \ldots, x_m)$$

where $c, c_1, \ldots, c_n$ are finite constraints in $C$; $p$ is a symbol in $P$ of arity $m$ (denoted as $p_{\{m \in \Pi\}}$) and $x, x_1, \ldots, x_m \in \text{Var}$. $A$ is a $tccp$ program $P$ is an object of the form $D . A$, where $A$ is an agent, called initial agent, and $D$ is a set of process declarations of the form $p(\hat{x}) ::= A$ (for some agent $A$), where $\hat{x}$ denotes a generic tuple of variables.

The following definition introduces the operational semantics of the language. It is slightly different from the original one in [de Boer et al. 2000]. In particular, we have introduced conditions in specific rules (namely Rules $\text{R2}, \text{R4}$ and $\text{R10}$) in order to detect when the store becomes $\text{ff}$. This modification is made to follow the original philosophy of $cctp$ computations defined in [Saraswat et al. 1991], where computations that reach an inconsistent store are considered failure computations. In [de Boer et al. 2000], this check is not explicitly done. In our context, we are interested in detecting when a computation reaches $\text{ff}$; however, once $\text{ff}$ is reached, no action can modify the store ($\text{ff}$ is the greatest element in the domain) and—after that moment—all guards in the program agents are always entailed, thus the computation from that instant has little interest. In particular, we do not want to distinguish computations which end in $\text{ff}$ from those which loop on store $\text{ff}$, contrarily to what [de Boer et al. 2000] does.

\(^1\)Differently from other languages where time is explicitly introduced by defining new timing agents.
The transition system for tccp.

\[
\begin{align*}
\langle \text{tell}(c), d \rangle \rightarrow \langle \text{skip}, c \otimes d \rangle & \quad d \neq \text{ff} \\
\langle \sum_{i=1}^{n} \text{ask}(c_i) \rightarrow A_i, d \rangle \rightarrow \langle A_j, d \rangle & \quad j \in [1, n], d \vdash c_j, d \neq \text{ff} \\
\langle A, d \rangle \rightarrow \langle A', d' \rangle & \quad d \vdash c \\
\langle \text{now } c \text{ then } A \text{ else } B, d \rangle \rightarrow \langle A', d' \rangle & \quad d \vdash c \\
\langle A, d \rangle & \rightarrow \langle A', d' \rangle \\
\langle \text{now } c \text{ then } A \text{ else } B, d \rangle \rightarrow \langle A, d \rangle & \quad d \vdash c, d \neq \text{ff} \\
\langle B, d \rangle \rightarrow \langle B', d' \rangle & \quad d \neq c \\
\langle \text{now } c \text{ then } A \text{ else } B, d \rangle \rightarrow \langle B, d \rangle & \quad d \neq c \\
\langle A \parallel B, d \rangle \rightarrow \langle A' \parallel B', d' \otimes c' \rangle & \quad \langle A, d \rangle \rightarrow \langle A' \parallel B', d' \rangle \\
\langle A \parallel B, d \rangle & \rightarrow \langle A', B', d' \rangle \\
\langle A, l \in \exists_x d \rightarrow \langle B, l' \rangle & \quad \langle \exists^l x A, d \rangle \rightarrow \langle \exists^l x B, d \otimes \exists_x l' \rangle \\
\langle p(\bar{x}), d \rangle & \rightarrow \langle A, d \rangle \quad \text{if } A \in D, d \neq \text{ff}
\end{align*}
\]

Fig. 1. The transition system for tccp.

Definition 2.1 (Operational semantics of tccp). The operational semantics of tccp is formally described by a transition system \( T = (\text{Conf}, \rightarrow) \). Configurations in \( \text{Conf} \) are pairs \( \langle A, c \rangle \) representing the agent \( A \) to be executed in the current global store \( c \). As usually done, we assume that the tccp syntax is closed under the usual structural equivalence relation where the parallelism operator is associative and commutative, and agents \( A \parallel \text{skip} \) and \( A \) are equivalent.

The transition relation \( \rightarrow \subseteq \text{Conf} \times \text{Conf} \) is the least relation satisfying the rules of Figure 1. Each transition step takes exactly one time-unit.

As can be seen from the rules, the \text{skip} agent represents the successful termination of the computation. The \text{tell}(c) \ agent adds the constraint \( c \) to the store and then stops. It takes one time-unit, thus the constraint \( c \) is visible to other agents from the following time instant. The store is updated by means of the \( \otimes \) operator of the constraint system.

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The choice agent \( \sum_{i=1}^{n} \text{ask}(c_i) \rightarrow A_i \) consults the store and non-deterministically executes (at the following time instant) one of the agents \( A_i \) whose corresponding guard \( c_i \) is entailed by the current store; otherwise, if no guard is entailed by the store, the agent suspends.

The conditional agent \( \text{now } c \text{ then } A \text{ else } B \) behaves in the current time instant like \( A \) (respectively \( B \)) if \( c \) (respectively not) is entailed by the store. Note that, because of the ability of tccp to handle partial information, \( d \neq c \) is not equivalent to \( d \vdash \neg c \). Thus, the else branch is taken not only when the condition is falsified, but also when...
there is not enough information to entail the condition. This characteristic is known in the literature as the ability to process “negative information” [Saraswat et al. 1994; Saraswat et al. 1996].

A || B models the parallel composition of A and B in terms of maximal parallelism (in contrast to the interleaving approach of ccp), i.e., all the enabled agents of A and B are executed at the same time.

The agent ∃xA makes variable x local to A. To this end, it uses the ∃ operator of the constraint system. More specifically, it behaves like A with x considered local, i.e., the information on x provided by the external environment is hidden to A, and the information on x produced by A is hidden to the external world. In [de Boer et al. 2000], an auxiliary construct ∃x is used to explicitly show the store local to A. In particular, in Rule R9, the store d in the agent ∃x A represents the store local to A. This auxiliary operator is linked to the hiding construct by setting the initial local store to tt, thus ∃x A := ∃x x A.

Finally, the agent p(¯x) takes from D a declaration of the form p(¯x) := A and then executes A at the following time instant. For the sake of simplicity, we assume that sets of declarations D are closed w.r.t. renaming of parameter names, i.e., if p(¯x) := A ∈ D then, for any ¯y ∈ Var, also p(¯y) := A{ ¯x/ ¯y} ∈ D.

3. MODELING THE SMALL-STEP OPERATIONAL BEHAVIOR OF TCCP

In this section, we introduce a new condensed, compositional, bottom-up denotational semantics which is (correct and) fully abstract w.r.t. the small-step (operational) behavior of tccp. Note that (as mentioned in the introduction), having a semantics which enjoys simultaneously all these (just mentioned) properties, besides the theoretical interest, it is also a matter of pragmatical relevance since they are the key factors for having an effective and efficient implementation of semantics-based program manipulation tools.

In order to introduce such semantics, we need first to define some (technical) notions. In the sequel, all definitions are parametric w.r.t. a cylindric constraint system \( C = (C, \leq, \emptyset, \otimes, \{ tt, \emptyset, \exists \}) \). We denote by \( A_{\Pi} \) the set of agents and \( D_{\Pi} \) the set of sets of process declarations built on signature \( \Pi \) and constraint system \( C \). By \( \epsilon \) we denote the empty sequence; by \( s_1 \cdot s_2 \) the concatenation of two sequences \( s_1, s_2 \). We also abuse notation and, given a set of sequences \( S \), by \( s_1 \cdot S \) we denote \( \{ s_1 \cdot s_2 \mid s_2 \in S \} \).

Let us formalize first the notion of behavior of a set \( D \) of process declarations in terms of the transition system described in Figure 1. It collects all the small-step computations associated to \( D \) as the set of (all the prefixes of) the sequences of computation steps (in terms of sequences of stores), for all possible initial agents and stores.

**Definition 3.1.** Let \( D \in D_{\Pi} \). Then the small-step (observable) behavior of \( D \) is defined as:

\[
B^s[D] := \bigcup_{\forall x \in C, \forall A \in A_{\Pi}} B^s[D . A]_c \quad \text{where}
\]

\[
B^s[D . A]_c := \{ c_0 . c_1 . . . . c_n \mid \langle A, c_0 \rangle \rightarrow \langle A_1, c_1 \rangle \rightarrow \ldots \rightarrow \langle A_n, c_n \rangle \} \cup \{ \epsilon \}
\]

(where \( \rightarrow \) is the transition relation given in Figure 1).

We call the sequences in \( B^s[D . A]_c \) behavioral timed traces or simply traces (when clear from the context).

We denote by \( \approx_{ss} \) the equivalence relation between process declarations induced by \( B^s \), namely for all \( D_1, D_2 \in D_{\Pi} \), \( D_1 \approx_{ss} D_2 \iff B^s[D_1] = B^s[D_2] \).

With this definition, we can formally state the requirement of full abstraction for semantics \( S \) as \( S[D_1] = S[D_2] \iff D_1 \approx_{ss} D_2. \)
To achieve a goal-independent semantics, a typical solution is to define denotations by using only the most general traces (in our case those for the weakest store) plus define a suitable semantic operator which can reconstruct the semantics of any expression (in our case agent) from such most general denotations. The wanted result can be achieved in this way only if the set of all traces for each expression is itself condensing (borrowing the terminology program analysis [Jacobs and Langen 1992; Marriott and Søndergaard 1993]), which in our case means that the set of all traces for an agent $A$ with initial store $c$ can be reconstructed from the set of all traces of $A$ with initial store $true$. The problem in following this approach in the $tccp$ case is that $B^{\cdot}$ is not condensing, since not all behavioral timed traces can be retrieved from the most general ones. This is due to the ask, now and hiding constructs. For instance, consider the agent $A := now x = 3$ then $tell(z = 0)$ else $tell(z = 1)$. Given the initial store $true$, we obtain the trace $true \cdot z = 1$, while for the stronger initial store $x = 3$ we obtain the trace $x = 3 \cdot (x = 3 \land z = 0)$, which is not comparable to the former (since $z = 0 \not\Rightarrow z = 1$ and $z = 1 \not\Rightarrow z = 0$). Hence, the latter trace cannot be obtained from the former, which has been generated for the most general store. Indeed—in general—in $tccp$, given $S := B^{\cdot}[D \cdot A]_c$ (the set of traces for an agent $A$ with initial store $c$), if we compute $B^{\cdot}[D \cdot A]_d$ with a stronger initial store $d (d \cdot c)$, then some traces of $S$ may disappear and, what is more critical, new traces, which are not instances of the ones in $S$, can appear. In the community of the $ccp$ paradigm [de Boer et al. 1995; Saraswat et al. 1996], this characteristic is known as “non-monotonicity of the language”.

Because of $tccp$’s non-monotonicity, $B^{\cdot}$ is also not compositional. For instance, consider the agents $A_1 := \text{tell}(x = 1)$ and $A_2 := \text{ask}(true) \rightarrow \text{now } (x = 1)$ then $\text{tell}(y = 0)$ else $\text{tell}(y = 1)$

For each $c$, $B^{\cdot}[\emptyset \cdot A_1]_c = \{c \cdot (x = 1 \land c)\}$. Moreover, for each $c$ that implies $^2 x = 1$, $B^{\cdot}[\emptyset \cdot A_2]_c = \{c \cdot c \cdot (y = 0 \land c)\}$ while, when $c \not\Rightarrow (x = 1)$, $B^{\cdot}[\emptyset \cdot A_2]_c = \{c \cdot c \cdot (y = 1 \land c)\}$. Now, for the parallel composition of these agents $A_1 \parallel A_2$, $B^{\cdot}[\emptyset \cdot A_1 \parallel A_2]_{true} = \{true \cdot (x = 1) \cdot (x = 1 \land y = 0)\}$ which cannot be computed by merging the traces of $A_1$ and $A_2$.

Thus, it does not come as a surprise that for all non-monotonic languages of the $ccp$ paradigm, the compositional semantics that have been written [Saraswat et al. 1991; Saraswat et al. 1994; de Boer et al. 1995; de Boer et al. 1997; Falaschi et al. 1997; Nielsen et al. 2002b; Falaschi et al. 2007; Olarte and Valencia 2008; Falaschi et al. 2009] are not defined for the full language, either because they avoid the constructs that cause non-monotonicity or because they restrict their use. Hence, the ability to handle non-monotonicity (and thus the full language without any limitation) is certainly one of the strengths of our proposal.

The example above shows why, due to the non-monotonicity of $tccp$, in order to obtain a compositional (and goal-independent) semantics for the full language it is not possible to follow the traditional strategy and collect in the semantics the traces associated to the weakest initial store. Actually, we have found the solution to the problem of compositionality by trying to solve another (related) problem. Since in a top-down (goal-dependent) approach the (initial) current store is propagated, then the decisions regarding a conditional or choice agent (where the computation evolves depending on the entailment of the guards in the current store) can be taken immediately. However, if we want to define a fixpoint semantics which builds the denotations bottom-up we have the problem that, while we are building the fixpoint, we do not know the current store yet. Thus, it is impossible to know which execution branch has to be taken in correspondence of a program’s guard.

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2We recall that in this exemplification cylindric constraint system, the entailment is logical implication.
To solve both problems our proposal is to enrich behavioral timed traces with information about the essential conditions that the store must (or must not) satisfy in order to make the program proceed with one or another execution branch. Thus, we associate conditions to the store of each computation step and then we collect (only) the most general hypothetical computations. These conditions are constructed by using the information in the guards of the ask and now constructs of a program.

We will formally show that this indeed solves both the problem of constructing bottom-up the semantics and of having a compositional and condensed semantics coping with non-monotonicity.

### 3.1. The semantic domain

Let us start by introducing the notion of condition, that is the base to build our denotations. Intuitively, we need “positive conditions” for branches related to the entailment of guards and “negative conditions” for non-entailment, i.e., for the branches where the current store does not entail the associated condition.

**Definition 3.2 (Conditions).** A condition \( \eta \), over Cylindric Constraint System C, is a pair \( \eta = (\eta^+, \eta^-) \) where

- \( \eta^+ \in C \) is called **positive condition**, and
- \( \eta^- \in \wp(C) \) is called **negative condition**.

A condition is **valid** when \( \eta^+ \neq \emptyset \), \( tt \notin \eta^- \) and \( \forall c \in \eta^- . \eta^+ \neq c \). We denote \( \Lambda_C \) the set of all conditions and \( \Delta_C \) the subset of valid ones.

The conjunction of two conditions \( \eta_1 = (\eta_1^+, \eta_1^-) \) and \( \eta_2 = (\eta_2^+, \eta_2^-) \) is defined (by abuse of notation) as \( \eta_1 \odot \eta_2 := (\eta_1^+ \odot \eta_2^+, \eta_1^- \cup \eta_2^-) \). Two conditions are called **incompatible** if their conjunction is not valid.

A store \( c \in C \) is **consistent** with \( \eta \), written \( c \gg \eta \), if \( \eta^+ \odot c \neq \emptyset \) and \( \forall h \in \eta^- . c \neq h \). Moreover, we say that \( c \) **satisfies** \( \eta \), written \( c \models \eta \), when \( c \models \eta^+ \) and \( \forall h \in \eta^- . c \neq h \).

We extend the \( \exists_x \) operator to conditions as \( \exists_x(\eta^+, \eta^-) := (\exists_x \eta^+, \exists_x \eta^-) \).

Due to the partial nature of the constraint system, for negative conditions we cannot use the \( \text{glb} \) (disjunction) \( \bigoplus_{i=1}^n c_i \) instead of set \( \{c_1, \ldots, c_n\} \) since we can have a store \( c \) such that \( c \models \bigoplus_{i=1}^n c_i \) while \( \forall i . c \neq c_i \). For instance, we can have two guards \( x > 2 \) and \( x \leq 2 \) and it may happen that the current store does not satisfy any of them, but their \( \text{glb} \) \( x > 2 \odot x \leq 2 \) (which is **true**) is entailed by any store.

Clearly, if a store—different from \( \emptyset \)—satisfies a condition, then it is also consistent with that condition. If two conditions are incompatible, then there exists no constraint \( c \in C \setminus \{\emptyset\} \) that entails simultaneously both conditions.

Now we are ready to enrich with conditions the notion of trace.

**Definition 3.3 (Conditional state).** A **conditional state**, over Cylindric Constraint System C, is one of the following constructs.

- **Conditional store.** A pair \( \eta \gg c \), for each \( \eta \in \Lambda_C \) and \( c \in C \).
- **Stuttering.** The construct \( \text{stutt}(C) \), for each finite \( C \subseteq C \setminus \{tt\} \).
- **End-of-process.** The construct \( \bigcirc \).

In a conditional store \( t = \eta \gg c \), the constraint \( c \) is the **store** of \( t \).

We say that \( \eta \gg c \) is **valid** if \( \eta \) is valid.

We extend \( \exists_x \) to conditional states as \( \exists_x((\eta^+, \eta^-) \gg c) := \exists_x(\eta^+, \eta^-) \gg \exists_x c \), \( \exists_x \text{stutt}(C) := \text{stutt}(\exists_x C) \) and \( \exists_x \bigcirc := \bigcirc \).

The conditional store \( \eta \gg c \) is used to represent a hypothetical computation step where \( \eta \) is the condition that the current store must satisfy in order to make the com-
putation proceed. Moreover, $c$ represents the information that is added by an agent to the global store up to the current time instant.

The stuttering $\text{stutt}(C)$ is needed to model the suspension of the computation due to an ask construct, i.e., it represents the fact that there is no guard in $C$ (the guards of a choice agent) entailed by the current store.

**Definition 3.4 (Conditional trace).** A conditional trace (over Cylindric Constraint System C) is a (possibly infinite) sequence $t_1, \ldots, t_n$ of valid conditional states (over C)—where $\emptyset$ can be used only as a terminator—that respects the following properties:

- **Monotonicity.** For each $t_i = \eta_i \Rightarrow c_i$ and $t_j = \eta_j \Rightarrow c_j$ such that $j \geq i$, $c_j \vdash c_i$.
- **Consistency.** For each $t_i = \eta_i \Rightarrow c_i$ and $t_{i+1}$ either of the form $(\eta_{i+1}, \eta_{i+1}) \Rightarrow c_{i+1}$ or $\text{stutt}(\eta_{i+1})$, we have that $\forall c^- \in \eta_{i+1}.c_i \not\vdash c^-$. We denote by $CT_C$ the set of all conditional traces, or simply CT when clear from the context.

The **limit store** of a (finite or infinite) trace $s$ is the lub of the stores (of the conditional states) of $s$.

A finite conditional trace that is ended with $\emptyset$ as well as an infinite conditional trace is said, respectively, failed or (finately) successful depending on whether its limit store $c$ is $\text{ff}$ or not. Such $c$ is called computed result.

Each conditional trace models a hypothetical $\text{tccp}$ computation where, for each time instant, we have a conditional state where each condition represents the information that the global store has to satisfy in order to proceed to the next time instant.

The **Monotonicity** property is needed since in $\text{tccp}$, as well as in $\text{ccp}$ but not in all its extensions, each store in a computation entails the previous ones. Note that because of this, for any finite conditional trace $t_1, \ldots, t_n$ whose sequence of stores (of the conditional stores) is $c_1, \ldots, c_m$ ($m \leq n$), the limit store $\bigcup_{i=1}^{m} c_i$ is just the last store $c_m$.

The **Consistency** property affirms that the store of a given conditional store cannot be in contradiction with the condition associated to the successive conditional state.

**Example 3.5.** It is easy to verify that the sequence $r_1 := (\text{true}, \emptyset) \Rightarrow y = 0 \cdot (x > 2, \emptyset) \Rightarrow y = 0 \land z = 3 \cdot \emptyset$ is a conditional trace. The first component of the trace states that in the first time instant the store $y = 0$ is computed in any case (the condition $(\text{true}, \emptyset)$ is always satisfied). The second component requires the constraint $x > 2$ to be satisfied by the (global) store in order to proceed by adding to the next state the information $z = 3$.

Instead, the sequence $r_2 := (\text{true}, \emptyset) \Rightarrow x = 0 \cdot (x = 0, \emptyset) \Rightarrow tt \cdot \emptyset$ is not a conditional trace since the **Monotonicity** property does not hold because $tt \not\vdash x = 0$. Also $r_3 := (\text{true}, \emptyset) \Rightarrow x = 0 \cdot \text{stutt}(\{x \geq 0\}) \cdot \emptyset$ is not a conditional trace: it does not satisfy the **Consistency** property since $x = 0$ implies the (only) negative condition in the successive conditional state ($x \geq 0$).

Note that finite conditional traces not ending in $\emptyset$ are partial traces that can still evolve and thus they are always a prefix of a longer conditional trace.

**Definition 3.6 (Semantic domain).** A set $R \subseteq \text{CT}$ is closed by prefix if for each $r \in R$, all the prefixes $p$ of $r$ (denoted as $p \leq_{\text{prefix}} r$) are also in $R$.

We denote the domain of non-empty sets of conditional traces that are closed by prefix as $\mathbb{P}$ (i.e., $\mathbb{P} := \{ R \subseteq \text{CT} \mid R \neq \emptyset, r \in R \Rightarrow \forall p \leq_{\text{prefix}} r, p \in R \}$).

We order elements in $\mathbb{P}$ by set inclusion $\subseteq$.

It is worth noting that $(\mathbb{P}, \subseteq, \cup, \cap, \text{CT}, \{\epsilon\})$ is a complete lattice.

This conceptual representation is pretty simple, especially to understand the lattice structure, considered the fact that we admit infinite traces. However, each prefix-
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of iccp A:11

closed set contains a lot of redundant traces, which are quite inconvenient for technical definitions. Thus, we will use an equivalent representation obtained by considering the crown of prefix-closed sets. Namely, given \( P \in \mathbb{P} \), we remove all the prefixes of a trace in the set with the function \( \text{maximal}(P) = \{ r \in P \mid \exists p \in P \setminus \{ r \}. r \preceq \text{pref} p \} \). Let \( M := \text{maximal}(\text{CT}) \), \( M := \{ \text{maximal}(P) \mid P \in \mathbb{P} \} \) and call maximal conditional trace sets the elements of \( M \). The inverse of map maximal is, for each \( M \in M \),

\[
\text{prefix}(M) := \{ p \in \text{CT} \mid p \leq \text{pref} r, r \in M \} \tag{3.1}
\]

The order of \( M \) is induced from the one in \( \mathbb{P} \) as \( M_1 \subseteq M_2 \iff \text{prefix}(M_1) \subseteq \text{prefix}(M_2) \) which is equivalent to say that \( M_1 \subseteq M_2 \iff \forall r_1 \in M_1 \exists r_2 \in M_2.r_1 \preceq \text{pref} r_2 \). We define the \( \text{lb} \) and the \( \text{glb} \) of \( M \) analogously. It is straightforward to prove that \( (\mathbb{P}, \subseteq) \xrightarrow{\text{maximal}} (M, \subseteq) \) is an order-preserving isomorphism, so \( (M, \subseteq, \cup, \cap, M, \{ \epsilon \}) \) is also a complete lattice.

Although this second representation is very convenient for technical definitions, it is not suited for examples. For instance, different maximal traces have frequently (significant) common prefixes; hence, some parts have to be written many times and, more important, it can be difficult to visualize the repetition (obfuscating the comprehension). Thus, in our examples we will use another equivalent representation in terms of prefix trees. Namely, we will use trees with (non root) nodes labeled with conditional states. Given \( P \in \mathbb{P} \), \( \text{tree}(P) \) builds the prefix tree of \( P \), obtained by combining all the sequences that have a prefix in common in the same path. Let \( T := \{ \text{tree}(P) \mid P \in \mathbb{P} \} \). The inverse of \( \text{tree} \) is the function \( \text{path}: T \rightarrow \mathbb{P} \) which returns the set of all possible paths starting from the root. Let \( \preceq \) be the order on \( T \) induced by the order on \( \mathbb{P} \), i.e., \( T_1 \preceq T_2 \iff \text{path}(T_1) \subseteq \text{path}(T_2) \). We define the \( \text{lb} \) and \( \text{glb} \) of \( T \) in a similar way. It is straightforward to prove that \( (\mathbb{P}, \preceq) \xrightarrow{\text{tree}} (T, \preceq) \) is an order-preserving isomorphism, so also \( (T, \preceq) \) is a complete lattice. In the sequel we will use the representation which is most convenient in each case.

3.2. Fixpoint Denotations of Programs

The technical core of our semantics definition is the agent semantics evaluation function (Definition 3.15, page 14) which, given an agent \( A \) and an interpretation \( \mathcal{I} \) (for the process symbols of \( A \)), builds the maximal conditional traces associated to \( A \). To define it, we need first to introduce some auxiliary semantic functions.

**Definition 3.7 (Propagation Operator).** Let \( r, c \in \mathbb{C} \) and \( c \in \mathbb{C} \). We define the propagation of \( c \) in \( r \), written \( r \downarrow_c \), by structural induction as \( \otimes c = \otimes, \epsilon_c = \epsilon \) and

\[
((\eta^+, \eta^-) \mapsto d \cdot r') \downarrow_c = \begin{cases} 
(\eta^+ \otimes c, \eta^-) \mapsto d \otimes c \cdot (r') \downarrow_c & \text{if } c \gg (\eta^+, \eta^-), d \otimes c \neq \text{ff} \\
(\eta^+ \otimes c, \eta^-) \mapsto \text{ff} \cdot \otimes & \text{if } c \gg (\eta^+, \eta^-), d \otimes c = \text{ff} 
\end{cases}
\]

\[
(\text{stutt}(\eta^-) \cdot r') \downarrow_c = \text{stutt}(\eta^-) \cdot (r') \downarrow_c \quad \text{if } \forall c^- \in \eta^-, c \neq c^- 
\]

We abuse notation and denote by \( R \downarrow_c \) the point-wise extension of \( \downarrow_c \) to sets of conditional traces: \( R \downarrow_c := \{ r \downarrow_c \mid r \in R \) and \( r \downarrow_c \) is defined \).

This operator is used in the definition of the semantics of constructs that add new information to traces. By definition, the propagation operator \( \downarrow \) is a partial function \( \mathbb{M} \times \mathbb{C} \rightarrow \mathbb{M} \) that instantiates a conditional trace with a given constraint and checks the consistency of the new information with the conditional states in the trace. This information needs to be propagated also to the successive (i.e., future) conditional states in order to maintain the monotonicity of the store.
Example 3.8. Given the conditional trace \( r := (\text{true}, \emptyset) \Rightarrow x > 10 \cdot (\text{true}, \emptyset) \Rightarrow x > 20 \cdot \emptyset \), the propagation of \( y > 2 \) in \( r \mid_{y \geq 2} \) is \( (y > 2, \emptyset) \Rightarrow x > 10 \land y > 2 \Rightarrow (y > 2, \emptyset) \Rightarrow x > 20 \land y > 2 \cdot \emptyset \).

For \( r' := (\text{true}, \{ y > 0 \}) \Rightarrow true \cdot \emptyset \) the propagation \( r' \mid_{y \geq 2} \) is not defined since \( y > 2 \nRightarrow (true, \{ y > 0 \}) \).

Finally, given the conditional trace \( r'' := (\text{true}, \emptyset) \Rightarrow y < 0 \cdot \emptyset \), the propagation \( r'' \mid_{y \geq 2} \) produces the conditional trace \( (y > 2, \emptyset) \Rightarrow false \cdot \emptyset \) since \( y > 2 \nRightarrow (true, \emptyset) \) and \( y < 0 \land y > 2 = false \).

Note that the consecutive propagation of two constraints \( r \mid_{(\emptyset, \emptyset, \emptyset)} \) is equivalent to \( r \mid_{(\emptyset, \emptyset, \emptyset)} \) (as stated formally in Lemma A.2 of the electronic appendix).

The following parallel composition auxiliary operator is used in the definition of the semantics of the parallel construct. Intuitively, this operator combines (with maximal parallelism) the information coming from two conditional traces. It checks the satisfiability of the conditions and the consistency of the resulting stores.

Definition 3.9 (Parallel composition). The parallel composition partial operator \( \parallel : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{M} \) is the commutative closure of the following partial operation defined by structural induction as: \( r \parallel \epsilon := r \), \( r \parallel \parallel := r \) and

\[
\left( \text{stutt}(\eta_1) \cdot r_1 \right) \parallel \left( \text{stutt}(\eta_2) \cdot r_2 \right) := \text{stutt}(\eta_1 \cup \eta_2) \cdot (r_1 \parallel r_2)
\]

Moreover, if \( \eta_1 \otimes \eta_2 \) is valid, then

\[
\left( \eta_1 \Rightarrow c_1 \cdot r_1 \right) \parallel \left( \eta_2 \Rightarrow c_2 \cdot r_2 \right) := \begin{cases} 
\eta_1 \otimes \eta_2 \Rightarrow c_1 \otimes c_2 \cdot ((r_1 \downarrow_{c_1}) \parallel (r_2 \downarrow_{c_1})) & \text{if } c_1 \otimes c_2 \neq \text{ff} \\
\eta_1 \otimes \eta_2 \Rightarrow \text{ff} \cdot \emptyset & \text{if } c_1 \otimes c_2 = \text{ff},
\end{cases}
\]

Finally, if \( \forall c^* \in \eta_2 \cdot \eta_1 \neq c^* \), then

\[
\left( \eta_1 \cdot r_1 \right) \parallel \left( \text{stutt}(\eta_2) \cdot r_2 \right) := \eta_1 \cdot (r_1 \parallel (r_2 \downarrow_{c_1}))
\]

Clearly, by definition, \( \parallel \) is commutative. Moreover, because of \( \otimes \) associativity, \( \parallel \) is also associative. It is worth noting that, if the propagated constraint is in contradiction with a condition into a trace \( r \) then the parallel composition is not defined on that \( r \).

Example 3.10. Consider \( r_1 := (\text{true}, \emptyset) \Rightarrow y > 2 \cdot (y > 2, \emptyset) \Rightarrow y > 2 \cdot \emptyset \) and \( r_2 := (z = 1, \emptyset) \Rightarrow z = 1 \cdot \emptyset \). Since \( r_1 \) and \( r_2 \) do not share variables, the compatibility checks always succeed and then \( r_1 \parallel r_2 = (z = 1, \emptyset) \Rightarrow y > 2 \land z = 1 \cdot (y > 2 \land z = 1, \emptyset) \Rightarrow y > 2 \land z = 1 \cdot \emptyset \).

Consider now \( r_3 := \text{stutt}(\{ y > 0 \}) \cdot (y > 0, \emptyset) \Rightarrow y > 0 \land z \neq 3 \cdot \emptyset \). Traces \( r_1 \) and \( r_3 \) share the variable \( y \) and it can be seen that the information regarding \( y \) in the two traces is consistent, thus \( r_1 \parallel r_3 = (\text{true}, \{ y > 0 \}) \Rightarrow y > 2 \cdot (y > 2, \emptyset) \Rightarrow y > 2 \land z = 3 \cdot \emptyset \).

Finally, consider \( r_4 := (\text{true}, \emptyset) \Rightarrow true \cdot (\text{true}, \{ y > 0 \}) \Rightarrow true \cdot \emptyset \). This trace, in the second time instant, requires that the constraint \( y > 0 \) cannot be entailed by the current store. However, the trace \( r_1 \) states, at the same time instant, that \( y > 2 \). This is the reason because \( r_1 \parallel r_4 \) is not defined.

Note that \( \downarrow \) distributes over \( \parallel \), in the sense that \( (r_1 \parallel r_2) \downarrow_c = (r_1 \downarrow_c) \parallel (r_2 \downarrow_c) \) (as stated formally in Lemma A.3 of the electronic appendix).

The last auxiliary operator that we need is the hiding operator \( \exists : \text{Var} \times \mathbb{M} \rightarrow \mathbb{M} \) which, intuitively, hides the information regarding a given variable in a conditional trace.

Definition 3.11 (Hiding operator). Given \( r \in \mathbb{M} \) and \( x \in \text{Var} \), we define the hiding of \( x \) in \( r \), written \( \exists x r \), by structural induction as \( \exists x \epsilon := \epsilon \), \( \exists x \emptyset := \emptyset \),

\[
\exists x (\eta^*, \eta^-) \Rightarrow c \cdot r' := \exists x ((\eta^*, \eta^-) \Rightarrow c) \cdot \exists x r'
\]

\[
\exists x (\text{stutt}(\eta^-) \cdot r') := \exists x \text{stutt}(\eta^-) \cdot \exists x r'
\]
We distinguish two special classes of conditional traces.

Definition 3.12 (Self-sufficient and $x$-self-sufficient conditional trace). A maximal trace $r \in M$ is said to be self-sufficient if the first condition is $(t, \emptyset)$ and, for each $t_i = \eta_i \Rightarrow c_i$ and $\forall r_{i+1} \in C_{i+1}, c_i \models \eta_{i+1}$ (each store satisfies the successive condition). Moreover, $r$ is self-sufficient w.r.t. $x \in \mathcal{V}$ ($x$-self-sufficient) if $\exists \mathcal{V}_{\text{Var} \setminus \{x\}} r$ is self-sufficient.

Definition 3.12 is stronger than Definition 3.4 since the latter does not require satisfiability but just consistency of the store w.r.t. conditions. Informally, this new definition demands that for self-sufficient conditional traces, no additional information (from other agents) is needed in order to complete the computation. In an $x$-self-sufficient conditional trace the same happens but only considering information about variable $x$.

Example 3.13. The conditional trace $r_1$ of Example 3.5 is not self-sufficient since $y \neq 0 \models x > 2$.

Now consider a variation where we add the information $x = 4$ to the stores, namely $r_2 := (\text{true}, \emptyset) \Rightarrow y = 0 \land x = 4 \cdot (x > 2, \emptyset) \Rightarrow y = 0 \land x = 3 \land x = 4 \cdot \emptyset$. It is easy to see that $r_2$ is a self-sufficient conditional trace, essentially because we add enough information in the first store to satisfy the second condition, i.e., $y = 0 \land x = 4 \models (x > 2, \emptyset)$.

Moreover, $r_2$ is also $x$-self-sufficient since $\exists \mathcal{V}_{\text{Var} \setminus \{x\}} r_2 = (\text{true}, \emptyset) \Rightarrow x = 4 \cdot (x > 2, \emptyset) \Rightarrow x = 4 \cdot \emptyset$, which is a self-sufficient trace.

3.2.1. Interpretations. Now we introduce the notion of interpretation, which is used to give meaning to process calls by associating to each process symbol a set of (maximal) conditional traces “modulo variance”.

Definition 3.14 (Interpretations). Let $\mathbb{PC}_\Pi := \{ p(\bar{x}) | p \in \Pi, \bar{x} \text{ are distinct variables} \}$ (or simply $\mathbb{PC}$ when clear from the context).

Two functions $I, J : \mathbb{PC} \to M$ are variants, denoted by $I \simeq J$, if for each $\pi \in \mathbb{PC}$ there exists a variable renaming $\rho$ such that $(I(\pi))\rho = J(\pi)\rho$.

An interpretation is a function $I : \mathbb{PC} \to M$ modulo variance.

The semantic domain $\mathbb{I}_\Pi$ (or simply $\mathbb{I}$ when clear from the context) is the set of all interpretations ordered by the pointwise extension of $\models$ (which by an abuse of notation we also denote by $\models$).

The partial order on $\mathbb{I}$ formalizes the evolution of the computation process. $(\mathbb{I}, \models)$ is a complete lattice and its least upper bound and greatest lower bound are the pointwise extension of $\cup$ and $\cap$, respectively. In the sequel we abuse the notations of $M$ for $\mathbb{I}$ as well. The bottom element is $1_\Pi := \lambda \pi. \{ \}$. Essentially, we define the semantics of each predicate in $\Pi$ over formal parameters whose names are actually irrelevant. It is important to note that $\mathbb{PC}_\Pi$ (modulo variance) has the same cardinality of $\Pi$ (and is thus finite) and therefore each interpretation is a finite collection (of possibly infinite elements). Hence, in the sequel, we explicitly write interpretations by cases, like

$$I := \begin{cases} \pi_1 \mapsto T_1 \\ \pi_n \mapsto T_n \end{cases} \quad \text{representing} \quad I(\pi_1) := T_1 \quad I(\pi_n) := T_n$$

In the following, any $I \in \mathbb{I}$ is implicitly considered as an arbitrary function $\mathbb{PC} \to M$ obtained by choosing an arbitrary representative of the elements of $I$ generated by $\models$. Actually, all the operators that we use on $\mathbb{I}_\Pi$ are also independent of the choice of the

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3i.e., a family of elements of $M$ indexed by $\mathbb{PC}$ modulo variance.
representative. Therefore, we can define any operator on \( \mathbb{I} \) in terms of its counterpart defined on functions \( \mathbb{P} \mathbb{C} \to \mathbb{M} \).

Moreover, we also implicitly assume that the application of an interpretation \( \mathcal{I} \) to a process call \( \pi \), denoted by \( \mathcal{I}(\pi) \), is the application \( I(\pi) \) of any representative \( I \) of \( \mathcal{I} \) which is defined exactly on \( \pi \). For example, if \( \mathcal{I} = (\lambda p(x,y).\{(true,\emptyset) \mapsto x = y\}) / \_ \), then \( \mathcal{I}(p(u,v)) = \{(true,\emptyset) \mapsto u = v\} \).

3.2.2. Semantics Evaluation Function of Agents. We are finally ready to define the evaluation function of an agent \( A \) w.r.t. an interpretation \( \mathcal{I} \), which computes the set of (maximal) conditional traces associated to the agent \( A \). It is important to note that the computation does not depend on an initial store. Instead, the weakest (most general) condition for each agent is (computed and) accumulated in the conditional traces.

**Definition 3.15 (Semantics Evaluation Function for Agents).** Given \( A \in A_{\mathbb{I}} \) and \( \mathcal{I} \in \mathbb{I}_{\mathcal{I}} \), we define the **semantics evaluation** \( A[\mathcal{I}] \in \mathcal{M} \) by structural induction as follows.

\[
A[\mathcal{I}] := \{\emptyset\} \quad (3.2)
\]

\[
A[\mathcal{I}] := \{\{(tt,\emptyset) \mapsto c \cdot \emptyset\} \mid c \in A[\mathcal{I}]\} \quad (3.3)
\]

\[
A[\mathcal{I}] := \{\{r_A \mid r_A \in A[\mathcal{I}], r_B \in A[B][\mathcal{I}]\} \quad (3.4)
\]

\[
A[\mathcal{I}] := \{\{r \mid r \in A[\mathcal{I}], r \text{ is } x\text{-self-sufficient}\} \quad (3.5)
\]

\[
A[\mathcal{I}] := (tt,\emptyset) \mapsto tt \cdot \mathcal{I}(p(x)) \quad (3.6)
\]

\[
\sum_{i=1}^{n} \text{ask}(c_i) \to A[\mathcal{I}] := \text{lfp}_M A R. \left( \{ \text{stutt}(\{c_1,\ldots,c_n\}) \cdot R \mid \right) \quad (3.7)
\]

\[
A[\mathcal{I}] := \{\{(c,\emptyset) \mapsto c \cdot (r_{i,c}) \mid 1 \leq i \leq n, r \in A[\mathcal{I}]\} \quad (3.8a)
\]

\[
\{(\eta^+ \cdot c,\eta^-) \mapsto d \cdot c \cdot (r_{i,c}) \mid (\eta^+,\eta^-) \mapsto d \cdot r \in A[\mathcal{I}], d \cdot c \neq \emptyset, \forall c \in \eta^- \cdot \eta^+ \cdot c \neq c^- \} \quad (3.8b)
\]

\[
\{(\eta^+ \cdot c,\eta^-) \mapsto f \cdot c \mid (\eta^+,\eta^-) \mapsto d \cdot r \in A[\mathcal{I}], d \cdot c \neq \emptyset, \forall c \in \eta^- \cdot \eta^+ \cdot c \neq c^- \} \quad (3.8c)
\]

\[
\{(c,\eta^-) \mapsto c \cdot (r_{i,c}) \mid \text{stutt}(\eta^-) \cdot r \in A[\mathcal{I}], \forall c \in \eta^- \cdot c \neq c^- \} \quad (3.8d)
\]

\[
\{(tt,\emptyset) \mapsto tt \cdot \emptyset \mid \emptyset \in A[\mathcal{I}]\} \quad (3.8e)
\]

\[
\{(\eta^+,\eta^- \cup \{c\}) \mapsto d \cdot r \mid (\eta^+,\eta^-) \mapsto d \cdot r \in A[B][\mathcal{I}], \eta^+ \neq c \} \quad (3.8f)
\]

\[
\{(tt,\emptyset \cup \{c\}) \mapsto tt \cdot r \mid \text{stutt}(\eta^-) \cdot r \in A[B][\mathcal{I}]\} \quad (3.8g)
\]

By \( \text{lfp}(F) \) we denote the least fixed point of any monotonic function \( F: \mathcal{L} \to \mathcal{L} \), over some lattice \( \mathcal{L} \).

We now explain in detail each case of the definition.

(3.2). The semantics of the skip agent contains just the trace composed of the end-of-process construct that marks the end of the computation.

\[\text{Recall that by } s_1 \cdot S \text{ we denote } \{s_1 \cdot s_2 \mid s_2 \in S\}.\]
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \textit{tccp} A:15

(3.3). For the \texttt{tell}(c) agent we have condition \((\texttt{tt}, \varnothing)\) since \(c\) must be added to the store in any case (in the next time instant). Next, the computation terminates (with the end-of-process symbol \(\varnothing\)).

(3.4). The semantics for the parallel composition of two agents is defined in terms of the auxiliary operator \(\parallel\), explained in Definition 3.9.

(3.5). The hiding construct must hide the information about \(x\) from all traces that cannot be altered by the presence of external information about \(x\), thus the hiding operation is applied just to \(x\)-self-sufficient conditional traces (Definition 3.12), that are those for which no additional information about variable \(x\) is needed (from other agents) in order to complete the computation.

(3.6). The semantics of process call \(p(\bar{x})\) simply delays by one time instant the traces for \(p(\bar{x})\) in interpretation \(I\) by prefixing them with \((\texttt{tt}, \varnothing) \rightarrow \texttt{tt}\).

(3.7). The semantics for the non-deterministic choice collects, for each guard \(c_i\), a conditional trace of the form \((c_i, \varnothing) \rightarrow c_i \cdot (r \downarrow c_i)\). This trace requires that \(c_i\) has to be satisfied by the current store (positive part of the condition in the first state). Then, the constraint \(c_i\) is propagated to the trace \(r\) (the continuation of the computation, which belongs to the semantics of \(A_i\)).

Furthermore, we collect the stuttering traces, which correspond to the case when the computation suspends. Traces representing this situation are of the form \(\text{stutt}(\{c_1, \ldots, c_n\}) \cdot r\) where \(r\) is, recursively, an element of the semantics of the choice agent.

(3.8). The definition for the conditional agent \texttt{now} \(c\) then \(A\) else \(B\) is in principle similar to the previous case. However, since the \texttt{now} construct must be instantaneous, in order to correctly model the timing of the agent we have seven cases depending on the possible forms of the first conditional state of the semantics of \(A\) (respectively \(B\)), on the value of the resulting store (\(\text{ff}\) or not) and on the fact that the guard \(c\) is satisfied or not in the current time instant.

(3.8a)–(3.8d) represent the case in which the guard \(c\) is satisfied by the current store. In this case, the agent \texttt{now} must behave instantaneously as \(A\). For this reason, we distinguish four different cases corresponding to the possible form of conditional traces associated to \(A\). In particular, (3.8a) corresponds to the case when the computation of \(A\) ends, thus also the computation of the conditional agent must end. In (3.8b), the information added (in one step) by \(A\) is compatible with the condition and with the rest of the computation and, moreover, does not produce \(\text{ff}\) when merged—by using \(\varnothing\)—with the current store \(d\). (3.8c) stops the conditional trace since the information produced by \(A\) added to the current store produces the inconsistent store \(\text{ff}\). Finally, (3.8d) corresponds to the case when \(A\) suspends.

(3.8e)–(3.8g) consider the cases when \(c\) is not entailed by the current store. In this situation, the agent \texttt{now} must behave instantaneously as \(B\), and the definition follows the same reasoning as for (3.8a), (3.8b) and (3.8d). The main difference is that, instead of adding \(c\) to the positive condition in the first conditional state, we add \(\{c\}\) to the negative condition.

In the sequel, we use a standard notation for the iterates of the computation of the least fixedpoint of a monotonic function \(F: \mathcal{L} \rightarrow \mathcal{L}\) over lattice \(\mathcal{L}\) whose bottom is \(\bot\) and \(\text{lub}\) is \(\uparrow\). Namely, \(F^k\uparrow\) denotes, for each \(k \in \mathbb{N}\), \(F^k(\bot)\) and \(F^\omega\) denotes \(\uparrow\{F^k(\bot) \mid k \in \mathbb{N}\}\).

Recall that, for a continuos \(F\), \(lfp(F) = F^\omega\).

\textbf{Example 3.16}. Let us evaluate the semantics for the \textit{tccp} agent \(A_1 \coloneqq A_2 \parallel A_3\) where

\(A_2 \coloneqq \text{tell}(y = 2) \parallel \text{tell}(x = y)\)

\(A_3 \coloneqq \text{ask}(\text{true}) \rightarrow \text{now} (x = 0) \text{ then tell}(z > 0) \text{ else } A_4\)

\(A_4 \coloneqq \text{ask}(y \geq 0) \rightarrow \text{tell}(z \leq 0)\)
Since there are no process calls, the interpretation $\mathcal{I}$ is irrelevant for the result. We start by computing the semantics for $A_4$, i.e., $A[\mathbb{A}_4]_{\mathcal{I}} = \text{lfp}_{\mathsf{bd}}(F)$ where

$$F(R) := \{ r \} \cup \text{stutt}\{(y \geq 0)\} \cdot R \quad \text{and} \quad r := (y \geq 0, \varnothing) \Rightarrow y \geq 0 \cdot (y \geq 0, \varnothing) \Rightarrow y \geq 0 \land z \leq 0 \cdot \mathcal{R}$$

The iterates of $F$ are:

$$F^1 = F(\{ \epsilon \}) = \{ r, \text{stutt}\{(y \geq 0)\}\}$$

$$F^2 = F(F^1) = \{ r, \text{stutt}\{(y \geq 0)\} \cdot r, \text{stutt}\{(y \geq 0)\} \cdot \text{stutt}\{(y \geq 0)\}\}$$

$$\text{lfp}_{\mathsf{bd}}(F) = \{( \text{stutt}\{(y \geq 0)\})^n \cdot r | n \in \mathbb{N}\} \cup \{ \text{stutt}\{(y \geq 0)\} \ldots \text{stutt}\{(y \geq 0)\} \ldots \}$$

Figure 2 graphically represents $A[\mathbb{A}_4]_{\mathcal{I}}$, which consists of a trace for the case in which the guard is satisfied, and a set of traces for the case in which it suspends. As it can be observed, the tree in Figure 2 consists of an infinite replication of the same pattern. We can depict such infinite trees as finite graphs, as in Figure 3. The back-loop arc is just a graphical shortcut which represents the (infinite) tree that is obtained by unrolling the loop. It is important to note that nodes reached by a path of length 2 (via the back-loop arc) have to be considered as a single arc, thus corresponding just to a one time instant delay.

With the semantics of $A_4$, we compute $A[\mathbb{A}_4]_{\mathcal{I}} = \{ r_1, r_2 \} \cup R$ where

$$r_1 := (\text{true}, \varnothing) \Rightarrow \text{true} \cdot (x = 0, \varnothing) \Rightarrow x = 0 \land z > 0 \cdot \mathcal{R}$$

$$r_2 := (\text{true}, \varnothing) \Rightarrow \text{true} \cdot (y \geq 0, \{ x = 0 \}) \Rightarrow y \geq 0 \cdot (y \geq 0, \varnothing) \Rightarrow y \geq 0 \land z \leq 0 \cdot \mathcal{R}$$

$$R := (\text{true}, \varnothing) \Rightarrow \text{true} \cdot (\text{true}, \{ y \geq 0, x = 0 \}) \Rightarrow \text{true} \cdot A[\mathbb{A}_4]_{\mathcal{I}}$$
All the traces of \( A \parallel A^3 \parallel I \) start with the conditional store \((true, \emptyset) \rightarrow true\) corresponding to the ask agent with guard \(true\). The trace \( r_1 \) corresponds to the case when (in the current time instant) the guard \( x = 0 \) is satisfied; the trace \( r_2 \) corresponds to \( x = 0 \) not satisfied and \( y \geq 2\) satisfied; while we have \( R \) when none is satisfied and \( A_1 \) is executed.

Now we can compute the semantics for \( A_1 \) by parallel composition of \( A[\parallel A^3 \parallel I] \) with \( A[\parallel A^2 \parallel I] \) which does not produce contributes since the constraint \( y = 2\), when propagated to the second component of \( r_1 \), is in contradiction with the positive part of the condition \((y = 2 \land x = y \land x = 0 \equiv false)\). Indeed, \((true, \emptyset) \rightarrow (y = 2 \land x = y) \cdot ((x = 0, \emptyset) \rightarrow x = 0 \land z > 0 \cdot \Box) \downarrow (y = 2 \land x = y) = (true, \emptyset) \rightarrow (y = 2 \land x = y) \cdot (false, \emptyset) \rightarrow false \cdot \Box\) is not a trace since \((false, \emptyset)\) is not a valid condition.

The combination of the set of traces \( R \) (corresponding to the suspension of the agent \( A_1 \)) and the \( \text{tell} (y = 2) \) agent also produces no trace. Definition 3.15 prescribes to compute \((true, \emptyset) \rightarrow y = 2 \land x = y \cup \{((true, \{y \geq 0, x = 0\}) \rightarrow true \cdot r') \downarrow (y = 2 \land x = y) \mid r' \in A[\parallel A^3 \parallel I]\}, which is empty, since \( y = 2 \land x = y \implies (true, \{y \geq 0, x = 0\}) \) because \( y = 2 \land x = y \implies y \geq 0\). These traces would correspond to the suspension of the agent \( A_1 \), and this can happen only when \( y \geq 2 \) is not satisfied, but the first component of the parallel agent tells \( y = 2 \) (thus \( y \geq 0 \) is satisfied). Therefore, only the combination of the trace \( r_2 \) in \( A[\parallel A^3 \parallel I] \) and the trace of \( A[\parallel A^2 \parallel I] \) produces a trace. Namely
\[
\begin{align*}
A[\parallel A^3 \parallel I] & = \{ (true, \emptyset) \rightarrow (y = 2 \land x = y) \cdot (y = 2 \land x = y, \{x = 0\}) \rightarrow (y = 2 \land x = y). \\
& \quad (y = 2 \land x = y, \emptyset) \rightarrow (y = 2 \land x = y \land z \leq 0) \cdot \Box\}
\end{align*}
\]

Due to the partial nature of the constraint system, the combination of the hiding operator with non-determinism can make the language behavior non-monotonic. As already mentioned, this is the reason because for all non-monotonic languages of the \( tccp \) paradigm, the compositional semantics that have been written either avoid non-monotonic constructs or restrict their use. Let us show now that we are able to handle the following example, which is an adaptation to \( tccp \) of the one used in [de Boer et al. 1997; Nielsen et al. 2002b] to illustrate the non-monotonicity problem.

**Example 3.17.** Consider the non-monotonic agent

\[ A := \text{ask}(x = 1) \rightarrow \text{tell}(true) + \text{ask}(true) \rightarrow \text{tell}(y = 2). \]

It is easy to see that for the initial store \(true\) just the second branch can be taken, whereas for the (greater) initial store \( x = 1 \), the two branches can be executed.

Since there are no process calls, for any interpretation \( I \), \( A[I] = \{r_1, r_2\} \), where
\[
\begin{align*}
r_1 & := (x = 1, \emptyset) \rightarrow x = 1 \cdot (x = 1, \emptyset) \rightarrow x = 1 \cdot \Box \\
r_2 & := (true, \emptyset) \rightarrow true \cdot (true, \emptyset) \rightarrow y = 2 \cdot \Box
\end{align*}
\]

We have two possible traces depending on whether the initial store is strong enough to entail \( x = 1 \) or not.

[de Boer et al. 1997; Nielsen et al. 2002b] show that within their semantics they do not collect all possible evaluations for agent \( A' := \text{tell}(x = 1) \parallel \exists x A \). On the contrary, in our case, since
\[
\begin{align*}
\exists_{\text{var}\setminus\{x\}} r_1 & = (x = 1, \emptyset) \rightarrow x = 1 \cdot (x = 1, \emptyset) \rightarrow x = 1 \cdot \Box \\
\exists_{\text{var}\setminus\{x\}} r_2 & = (true, \emptyset) \rightarrow true \cdot (true, \emptyset) \rightarrow true \cdot \Box
\end{align*}
\]

only \( r_2 \) is \( x \)-self-sufficient and, by Definition 3.15,
\[
A[\exists x A] = \{ (true, \emptyset) \rightarrow true \cdot (true, \emptyset) \rightarrow y = 2 \cdot \Box \}.
\]
By composing we have
\[ A[\sigma']_D = \{ (true, \emptyset) \rightarrow x = 1 \cdot (x = 1, \emptyset) \rightarrow y = 2 \land x = 1 \cdot \emptyset \}. \]

It is easy to see that the information on the variable \( x \) added by the tell agent does not affect the internal execution of the agent \( A \), as expected.

There are some technical decisions that ensure the correctness of the defined semantics. One can note that in the definition of the propagation operator (Definition 3.7), the propagated information is added not only to the store of the state, but also to the (positive part of the) condition. This means that the positive part of the conditions in a trace contains not only the information that had to be satisfied up to that computation step, but also the constraints that have been added during computation in the previous time instants. From the computations in the examples above, it may seem that the propagation of the accumulated information in the conditions of the states could be redundant. However, it is necessary in order to have full abstraction w.r.t. the behavior, otherwise we would distinguish agents whose behavior is actually the same, as shown in the following example.

Example 3.18. Consider the following two (very similar) agents:
\[ A_1 := \text{ask}(x > 2) \rightarrow \text{tell}(y = 1) \quad \quad \quad A_2 := \text{ask}(x > 4) \rightarrow \text{tell}(y = 1) \]

We have similar but different semantics. Namely,
\[ A[A_1]_D = \{ (\text{stutt}(\{x > 2\}) \}^n \cdot r_1 \mid n \in \mathbb{N} \} \cup \{ \text{stutt}(\{x > 2\}) \} \cdot \text{stutt}(\{x > 2\}) \} \]
\[ r_1 = (x > 2, \emptyset) \rightarrow true \cdot (x > 2, \emptyset) \rightarrow y = 1 \cdot \emptyset \]
\[ A[A_2]_D = \{ (\text{stutt}(\{x > 4\}) \}^n \cdot r_2 \mid n \in \mathbb{N} \} \cup \{ \text{stutt}(\{x > 4\}) \} \cdot \text{stutt}(\{x > 4\}) \} \]
\[ r_2 = (x > 4, \emptyset) \rightarrow true \cdot (x > 4, \emptyset) \rightarrow y = 1 \cdot \emptyset \]

However, consider now the following two agents, which embed \( A_1 \) and \( A_2 \) in the same context:
\[ A_1' := \text{tell}(x = 7) \parallel \text{ask}(true) \rightarrow A_1 \quad \quad \quad A_2' := \text{tell}(x = 7) \parallel \text{ask}(true) \rightarrow A_2 \]

Then, the two traces corresponding to the satisfaction of the guards are, respectively:
\[ r_3 = (true, \emptyset) \rightarrow x = 7 \cdot r_1(x = 7) \quad \quad \quad r_4 = (true, \emptyset) \rightarrow x = 7 \cdot r_2(x = 7) \]

Since the propagated constraint is stronger than the guards in both the agents, the resulting compositions are the same. In fact, thanks to the accumulation of the store in the condition, we do not distinguish them:
\[ r_1(x = 7) = r_2(x = 7) = (true, \emptyset) \rightarrow x = 7 \cdot r_1(x = 7) \rightarrow x = 7 \cdot (x = 7, \emptyset) \rightarrow y = 1 \cdot \emptyset \]

If the constraint \( x = 7 \) were not added to the condition, but only to the store of the state, then we would have two different conditional traces for these two agents.

3.2.3. Fixpoint Denotations of Process Declarations. Now we can finally define the semantics for a set of process declarations \( D \).

Definition 3.19 (Fixpoint semantics). Given \( D \in \mathcal{D}^I \), we define \( D[D] : \mathbb{I} \rightarrow \mathbb{I} \), for each \( p \in P \), as
\[ D[D]_p \omega(p(x)) := \bigcup \{ A[A]_D \mid p(x) := A \in D \}. \]

The fixpoint denotation of \( D \) is \( \mathcal{F}[D] := \text{lfp}(\mathcal{F}[D]) = D[D] \uparrow \omega \).

We denote with \( \approx_\mathcal{F} \) the equivalence relation on \( \mathcal{D}^I \) induced by \( \mathcal{F} \). Namely, \( D_1 \approx_\mathcal{F} D_2 \iff \mathcal{F}[D_1] = \mathcal{F}[D_2] \).

The semantics of a tcp program \( D . A \) is \( \mathcal{P}[D . A ] := A[A]_{\mathcal{F}[D]}. \)
The limit represented in Figure 4.

\[ (x > 2, \emptyset) \rightarrow x > 2 \land y < 0 \]
\[ (true, \{ x > 2 \}) \rightarrow true \]
\[ \mathcal{I}(q(x,y)) \]

Fig. 4. Tree representation for \( A[\mathcal{I}]_x \) of Example 3.20.

\[ (x > 2, \emptyset) \rightarrow x > 2 \land y < 0 \]
\[ (true, \{ x > 2 \}) \rightarrow true \]

Fig. 5. Graph representation of the fixpoint \( \mathcal{F}[D](q(x,y)) \) of Example 3.20.

\( \mathcal{F}[D] \) is well defined since \( D[D] \) is continuous (as stated formally in Lemma A.5 of the electronic appendix).

Let us show how the semantics for a set of process declarations is computed by means of some examples.

**Example 3.20.** Let \( D := \{ q(x,y) ::= A \} \) where
\[ A ::= \text{now}(x > 2) \text{ then tell}(y < 0) \text{ else } q(x,y). \]

First we need to compute, for each \( I \in \mathbb{I} \), the evaluation of the body of the process declaration. Namely,
\[ A[\mathcal{I}]_x = \{ \overline{r} \} \cup \{ (true, \{ x > 2 \}) \rightarrow true \cdot s \mid s \in \mathcal{I}(q(x,y)) \} \]
where \( \overline{r} := (x > 2, \emptyset) \rightarrow x > 2 \land y < 0 \cdot \emptyset \). Intuitively, the trace \( \overline{r} \) corresponds to the then branch of the conditional agent, whereas the else branch is represented by a set of traces, one for each trace in the interpretation of the process call. \( A[\mathcal{I}]_x \) is graphically represented in Figure 4.

The iterates of \( D[D] \) are
\[ D[D]^1 = \{ q(x,y) \rightarrow \{ \overline{r}, (true, \{ x > 2 \}) \rightarrow true \} \]
\[ D[D]^2 = \{ q(x,y) \rightarrow \{ \overline{r}, (true, \{ x > 2 \}) \rightarrow true \cdot \overline{r}, (true, \{ x > 2 \}) \rightarrow true \cdot (true, \{ x > 2 \}) \rightarrow true \} \]
\[ \vdots \]
\[ D[D]^\omega = \{ q(x,y) \rightarrow \{ (true, \{ x > 2 \}) \rightarrow true \}^{\mathbb{N}} \cdot \overline{r} \mid n \in \mathbb{N} \}
\[ \cup \{ (true, \{ x > 2 \}) \rightarrow true \cdot (true, \{ x > 2 \}) \rightarrow true \cdot \ldots (true, \{ x > 2 \}) \rightarrow true \ldots \} \]

The limit \( \mathcal{F}(D)(q(x,y)) = (D[D]^\omega)(q(x,y)) \) is graphically represented in Figure 5.

**Example 3.21.** Let \( D := \{ p(x) ::= A \} \) where \( A ::= \text{ask}(x = 4) \rightarrow p(x) \). First we need to compute, for each \( I \in \mathbb{I} \), the evaluation of the body of the process declaration. Namely,
\[ A[\mathcal{I}]_x = \{ (\text{stutt}(\{ x = 4 \}))^{\mathbb{N}} \cdot \overline{r} \cdot s \mid n \in \mathbb{N}, s \in \mathcal{I}(p(x)) \} \]

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYY.
Fig. 6. Graph representation for $A[A]_X$ of Example 3.21.

Fig. 7. Graph representation of the fixpoint $F[D](p(x))$ for the Example 3.21.

$$\{\text{stutt}(\{x = 4\}) \ldots \text{stutt}(\{x = 4\}) \ldots\}$$

where $\tilde{r} : (x = 4, \emptyset) \Rightarrow x = 4 \cdot (x = 4, \emptyset) \Rightarrow x = 4$. It is worth noticing that the second conditional state of $\tilde{r}$ corresponds to the delay that is introduced each time that a process call is run. $A[A]_X$ is graphically represented in Figure 6.

The iterates of $D[D]$ are

$$D[D]^{\uparrow 1} = \left\{ p(x) \mapsto \left\{ (\text{stutt}(\{x = 4\}))^{\cdot n} \cdot \tilde{r} \mid n \in \mathbb{N}\right\} \cup \right\} \text{stutt}(\{x = 4\}) \ldots \text{stutt}(\{x = 4\}) \ldots \right\}$$

$$D[D]^{\uparrow 2} = \left\{ p(x) \mapsto \left\{ (\text{stutt}(\{x = 4\}))^{\cdot n} \cdot \tilde{r} \cdot \tilde{r} \mid n \in \mathbb{N}\right\} \cup \right\} \text{stutt}(\{x = 4\}) \ldots \text{stutt}(\{x = 4\}) \ldots \right\}$$

$$\vdots$$

$$F[D] = \left\{ p(x) \mapsto \left\{ (\text{stutt}(\{x = 4\}))^{\cdot n} \cdot \tilde{r} \cdot \tilde{r} \ldots \mid n \in \mathbb{N}\right\} \cup \right\} \text{stutt}(\{x = 4\}) \ldots \text{stutt}(\{x = 4\}) \ldots \right\}$$

$F[D](p(x))$ is graphically represented in Figure 7. Note that the application of the propagation operator to the previous iterates removes all the stuttering sequences, and this is the reason because just the first stuttering sequence remains.

**Example 3.22.** Let $D := \{ p(x, y) := A \}$ where

$A := \text{ask}(y > x) \rightarrow p(x + 1, y) + \text{ask}(y \leq x) \rightarrow \text{skip}$
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \textit{tcp} A:21

\[
\begin{align*}
(y > x, \emptyset) & \rightarrow y > x \\
(y > x, \emptyset) & \rightarrow y > x \\
(y \leq x, \emptyset) & \rightarrow y \leq x
\end{align*}
\]

\[\mathcal{I}(p(x + 1))_{\downarrow_{y>x}}\]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig8.png}
\caption{Graph representation for $\mathcal{A}[A]_I$ in Example 3.22.}
\end{figure}

As usually done in the \textit{tcp} community, we assume that we can use expressions of the form $x + 1$ directly in the arguments of a process call. We can simulate this behavior by writing $\exists x' (\text{tell}(x' = x + 1) \parallel p(x', y))$ instead of $p(x + 1, y)$ (but introducing a delay of one time unit). We have

\[\mathcal{A}[A]_I = \{(y > x, \emptyset) \rightarrow y > x \cdot (y > x, \emptyset) \rightarrow y > x \cdot r_{y \geq x} \mid r \in \mathcal{I}(p(x + 1, y))\} \cup\]
\[
\{(y \leq x, \emptyset) \rightarrow y \leq x \cdot \emptyset\} \cup\]
\[
\{(\mathit{stutt}((y > x, y \leq x)))^{n} \cdot (y > x, \emptyset) \rightarrow y > x\}
\]
\[
\{(y > x, \emptyset) \rightarrow y > x \cdot r_{y \geq x} \mid n \in \mathbb{N}, r \in \mathcal{I}(p(x + 1, y))\} \cup\]
\[
\{(\mathit{stutt}((y > x, y \leq x)))^{n} \cdot (y \leq x, \emptyset) \rightarrow y \leq x \cdot \emptyset \mid n \in \mathbb{N}\} \cup\]
\[
\{\mathit{stutt}((y > x, y \leq x)) \ldots \mathit{stutt}((y > x, y \leq x)) \ldots\}
\]

which is graphically shown in Figure 8. For this agent, we have three branches, one for each condition of the choice and one corresponding to the stuttering possibility.

The first iteration of $D[D]$ is

\[
D[D]^{\uparrow 1} = \begin{cases} 
 p(x, y) \rightarrow \{(y > x, \emptyset) \rightarrow y > x \cdot (y > x, \emptyset) \rightarrow y > x\} \cup \\
 \{(y \leq x, \emptyset) \rightarrow y \leq x \cdot \emptyset\} \cup \\
 \{(\mathit{stutt}((y > x, y \leq x)))^{n} \cdot (y > x, \emptyset) \rightarrow y > x\}
\end{cases}
\]
\[
\begin{align*}
(y > x, \emptyset) & \rightarrow y > x \mid n \in \mathbb{N} \} \cup \\
\{(\mathit{stutt}((y > x, y \leq x)))^{n} \cdot (y \leq x, \emptyset) \rightarrow y \leq x \cdot \emptyset \mid n \in \mathbb{N}\} \cup \\
\{\mathit{stutt}((y > x, y \leq x)) \ldots \mathit{stutt}((y > x, y \leq x)) \ldots\}
\end{align*}
\]

which is graphically represented in Figure 9. Figure 10 represents the second iteration $D[D]^{\uparrow 2}(p(x, y))$, whereas Figure 11 is the graphical representation of $\mathcal{F}[D](p(x, y))$. By looking at the semantics, it can be observed that the process stops in one time instant when $y \leq x$ and in $1 + 2(y - x)$ time instants otherwise.

\textit{Example 3.23.} Consider the following process declaration, presented in [Falaschi and Villanueva 2006], which models a subsystem of a microwave controller. The underlying constraint system is the (well-known) Herbrand constraint system [de Boer...
et al. 1995].

\[
\text{microwave}(\text{Door}, \text{Button}, \text{Error}) ::= \exists D \exists B \exists E \\
(\text{tell}(\text{Error} = [1 \mid E]) \parallel \text{tell}(\text{Door} = [1 \mid D]) \parallel \text{tell}(\text{Button} = [1 \mid B]) \\
\parallel \text{now}(\text{Door} = [\text{open} \mid D] \land \text{Button} = [\text{on} \mid B]) \\
\text{then } (\exists E1 \text{tell}(E = [1 \mid E1]) \parallel \exists B1 \text{tell}(B = [\text{off} \mid B1])) \\
\text{else } \exists E1 \text{tell}(E = [0 \mid E1]) \\
\parallel \text{microwave}(D, B, E))
\]

This process declaration detects if the door is open while the microwave is turned on. In that case, it forces that in the next time instant the microwave is turned-off and it emits an error signal (value 1); otherwise, the agent emits a signal of no error (value 0). Due to the monotonicity of the store, streams are used to model imperative-style variables [de Boer et al. 2000]. In the example, the streams Error, Door and Button store the values that the simulated modifiable variables get along the computation. The first three tell agents link the future values of the streams with the future streams E, D and B. Then, when it is detected a possible risk (characterized by the guard of the now agent), the microwave is turned off and an error signal is emitted (by the then
branch of the conditional agent). The final recursive call restarts the same control at the next time instant.

The fixpoint semantics $\mathcal{F}(\text{microwave}(D, B, E))$ is graphically represented in Figure 12, where:

$$risk_k := \exists_D \exists_B (\text{Door} = [\text{open} | \ldots | D] \wedge \text{Button} = [\text{on} | \text{off} | \text{on} | \ldots | B])$$
Proposition 3.27. Correctness and full abstraction is a direct consequence of Theorems 3.25 and 3.26 and

The following theorem is the key result to prove full abstraction of \( F \) (semantics), in the sense that, for any \( c \), shows that the small-step behavior of a program \( c \) instantiates the hypothetical states of a conditional trace \( r \) producing the corresponding (real) behavioral timed trace. Intuitively, this operator works by consistently adding to each conditional state the information given by the initial store \( c \), discarding those sequences which falsify conditions.

Definition 3.24 (Instantiation operator). The instantiation operator \( \downarrow: M \times C \rightarrow C^{*} \) is a partial function defined by structural induction as: \( \epsilon \downarrow_{c} := c \); otherwise \( r \downarrow_{ff} := ff \); otherwise \( r \downarrow_{c} := r \downarrow_{c \circ d} \) if \( c \models \eta \) and \( (c \circ d) \neq ff \).

We abuse notation by denoting with \( R \downarrow_{c} \) the extension of \( \downarrow_{c} \) to \( M \): \( R \downarrow_{c} := \{ r \downarrow_{c} \mid r \in R \) and \( r \downarrow_{c} \) is defined \( \} \).

The instantiation operator is consistent w.r.t. the propagation operator (Definition 3.7), in the sense that, for any \( c' \) that entails \( c \), \( r \downarrow_{c} = (r \downarrow_{c'}) \downarrow_{c} \) (as stated formally in Lemma A.6 of the electronic appendix). Moreover, the instantiation operator \( \downarrow \) “distributes” over the parallel composition operator \( \parallel \) (Definition 3.9) (as stated formally in Lemma A.8 of the electronic appendix).

The key result to prove correctness of \( F \) w.r.t. \( \approx_{ss} \) is the following theorem which shows that the small-step behavior of a program \( P \) can be determined by instantiation of the semantics \( P[\parallel P] \).

Theorem 3.25. For each program \( P \) and each \( c \in C \), prefix(\( P[\parallel P] \parallel_{c} \)) = \( B^{ss}[P] \parallel_{c} \).

The following theorem is the key result to prove full abstraction of \( F \) w.r.t. \( \approx_{ss} \).

Theorem 3.26. Let \( P_1, P_2 \) be two programs. Then \( P[\parallel P_1] = P[\parallel P_2] \) if and only if \( B^{ss}[P_1] = B^{ss}[P_2] \).

Proposition 3.27. Let \( D_1, D_2 \in D_C^{II} \). Then \( D_1 \approx_{F} D_2 \) if and only if \( \forall A \in A_C^{II}. P[D_1 \emph{ } ] \parallel_{A} = P[D_2 \emph{ } ] \parallel_{A} \).

Correctness and full abstraction is a direct consequence of Theorems 3.25 and 3.26 and Proposition 3.27.

Corollary 3.28 (Correctness and Full Abstraction of \( F \)). Let \( D_1, D_2 \in D_C^{II} \). Then \( D_1 \approx_{ss} D_2 \) if and only if \( D_1 \approx_{F} D_2 \).
4. BIG-STEP SEMANTICS

A small-step behavior contains all the details of the computation. However typically only some parts of the execution are considered relevant. So frequently it is better to reason only about a specific abstraction of the small-step behavior, instead of dealing with all execution details. In the literature, many authors (like [de Boer et al. 2000]) call observations all the abstractions of the small-step behavior of a specific program\(^5\) (including the small-step behavior itself as the degenerate identity abstraction). Moreover, they typically use this same name for the collection of all observables of a set of declarations.

Many other authors use the term observable property (or simply observable) for an abstraction function \(\phi\) which, when applied to the set of traces of a program, delivers the observations of interest. Then the observation, or observable behavior, of program \(P\) is just the application of \(\phi\) to the traces of \(P\).

We prefer to use the latter nomenclature and, in the sequel, we call observable behavior of a program \(Q\) w.r.t. observable \(\phi\) (or simply \(\phi\)-observable behavior of \(Q\)) the image \(\phi(B^\omega[Q])\) and we denote it by \(B^\omega[\phi](Q)\).

The observable property which is usually considered in papers dealing with semantics of ccp languages (e.g. see [de Boer and Palamidessi 1991]) is the one that collects the input/output pairs of terminating computations, including deadlocked ones. Indeed, using the (original version of the) transition system of Definition 2.1, [de Boer et al. 2000] defines the notion of input-output observable behavior as \(O^\omega(A) := \{ \{c_0, c_n\} \mid (A_0, c_0) \leadsto^* (A_n, c_n) \uparrow \}\). In this definition, there is an implicit reference to a set of declarations \(D\). Since in the sequel we need to state some formal results for two (different) sets of declarations simultaneously, we use the explicit notation \(O^\omega[D . A]\) instead of \(O^\omega(A)\).

As we already mentioned, in tccp also infinite computations must be considered, for example when we are modeling reactive systems. However, we nevertheless want to be able to distinguish if an input-output pair refers to a finite or infinite computation. Thus, we use input-output pairs with associated termination mode of the form \((c_0, \text{mode}(c_n)),\) where \(c_0 \in C\) is the input store of the computation, \(c_n \in C\) is the output store (which is the lab of the stores of the computation) and \(\text{mode}\) is either \(\text{fin}\) or \(\text{inf}\) for finite or infinite computations, respectively.

**Definition 4.1.** Given \(c, c' \in C\) such that \(c' \Rightarrow c\), an input-output pair with termination mode is either \((c, \text{fin}(c'))\) or \((c, \text{inf}(c'))\).

We denote by \(\text{IO}\) the set of input-output pairs with termination mode and by \(\mathcal{I}_\text{O}\) the domain \(\mathcal{P}(\text{IO})\), ordered by set inclusion.

Clearly, \((\mathcal{I}_\text{O}, \subseteq, \cup, \cap, \text{IO}, \emptyset)\) is a complete lattice.

**Definition 4.2 (Input-output behavior of programs).** The input-output observable is defined as

\[ \text{io}(T) := \{ \{c_0, \text{fin}(c_n)\} \mid c_0 \cdots c_n \in T \} \cup \{ \{c_0, \text{inf}(\emptyset \geq \{c_i\})\} \mid c_0 \cdots c_n \in T \}. \]

For each \(D \in D^\Pi_C\) and \(A \in A^\Pi_C\), the induced input-output behavior \(B^\omega[D . A]\) is defined as \(\text{io}(B^\omega[D . A])\). We denote by \(\equiv_{\text{io}}\) the equivalence relation between process declarations induced by \(B^\omega\), namely \(D_1 \equiv_{\text{io}} D_2 \iff \forall A \in A^\Pi_C. B^\omega[D_1 . A] = B^\omega[D_2 . A]\).

We denote by \(\pi_F\) the projection which selects just the pairs whose mode is \(\text{fin}\) and by \(\mathcal{I}_\text{O}_F\) we denote \(\pi_F(\mathcal{I}_\text{O})\). Moreover, we denote by \(B^\omega_F[D . A]\) the finite fragment of \(B^\omega[D . A]\) i.e., \(\pi_F(B^\omega[D . A])\).

---

\(^5\)Notice that a tccp program is the syntactic correspondent of a (program's) expression of a generic language, while a \(tccp\) set of declarations is the syntactic correspondent of a program.
Note that, by Definitions 3.1 and 4.2, 
\[ B^{\alpha}[D . A_0] = \{ \langle c_0, \text{fin}(c_0) \rangle \mid c_0 \in C, \langle A_0, c_0 \rangle \to^* \langle A_n, c_n \rangle \not\in \} \cup \]
\[ \{ \langle c_0, \text{inf}(\oplus c_0) \rangle \mid c_0 \in C, \langle A_0, c_0 \rangle \to \to^* \langle A_i, c_i \rangle \to \cdots \} \]

In the sequel, we define an abstract interpretation ([Cousot and Cousot 1979]) of the small-step semantics \( \mathcal{P}[D . A] \) (Definition 3.19) which gives \( B^{\alpha}[D . A] \). Then we prove that the finite fragment of this abstraction (i.e., \( B^{\alpha}_F[D . A] \)) is essentially isomorphic to \( O^{\alpha}[D . A] \). Actually, there is a negligible difference between \( B^{\alpha}_F[D . A] \) and \( O^{\alpha}[D . A] \) due to the change we made in the definition of the small-step operational semantics. We will state the formal result in Subsection 4.2.

To define the semantics modeling the input-output observable as suggested by the abstract interpretation approach, we proceed as described in the following. We assume familiarity with basic results of abstract interpretation. However, this familiarity is needed only to be confident about the soundness of the results that we obtain by standard results of abstract interpretation theory. To understand the definitions it is sufficient to know that, given two complete lattices \((C, \sqsubseteq)\) and \((A, \leq)\), a Galois Insertion of \((A, \leq)\) into \((C, \sqsubseteq)\)—denoted by \((C, \sqsubseteq) \xrightarrow{\gamma} (A, \leq)\)—is a pair of maps \(\alpha : C \to A\) and \(\gamma : A \to C\) that satisfy certain properties (see [Cousot and Cousot 1979]). \((C, \sqsubseteq)\) and \((A, \leq)\) are the concrete and abstract domains, while \(\alpha\) and \(\gamma\) are the abstraction and concretization maps.

First, we formalize program properties of interest (in this particular case the input-output behavior) as a Galois Insertion \((M, \sqsubseteq) \xrightarrow{\gamma} (\mathbb{I}O, \preceq)\) and then we lift it over interpretations \(\mathcal{I} \xrightarrow{\gamma} \mathbb{P}C \to \mathbb{I}O\) by function composition as \(\hat{\alpha}(f) = \alpha \circ f\). The best correct (optimal) abstract version of the semantics \(\mathcal{D}[D]\) is simply obtained as \(\mathcal{D}^\alpha[D] := \hat{\alpha} \circ \mathcal{D}[D] \circ \hat{\gamma}\). Abstract interpretation theory assures that \(\mathcal{F}^\alpha[D] := \mathcal{I}f(\mathcal{D}^\alpha[D])\) is the best correct approximation of \(\mathcal{F}[D]\). Correct because \(\alpha(\mathcal{F}[D]) \in \mathcal{F}^\alpha[D]\) and best because it is the minimum (w.r.t. \(\preceq\)) of all correct approximations.

4.1. Input-output semantics with infinite outcomes

Now we formally define the Galois Insertion which abstracts conditional traces to input-output pairs with termination mode. In the sequel, we denote by \(\text{last}(s)\) the partial function that, for a non-empty finite sequence \(s\), gives its last element and is otherwise undefined.

**Definition 4.3 (Input-Output abstraction).** Given any \(M \in M\), we define
\[
\alpha_{io}(M) := \{ \langle c_0, \text{fin}(c_0) \rangle \mid c_0 \in C, r \in M, \text{last}(r \downarrow c_0) = c_n \} \cup \]
\[
\{ \langle c_0, \text{inf}(\oplus c_0) \rangle \mid c_0 \in C, r \in M, r \downarrow c_0 = c_0 \ldots c_i \ldots \} \]
\[
\gamma_{io}(P) := \bigcup \{ r \in M \mid \langle c_0, \text{fin}(c_0) \rangle \in P, r \downarrow c_0 = c_n \} \cup \]
\[
\bigcup \{ r \in M \mid \langle c_0, \text{inf}(c) \rangle \in P, r \downarrow c_0 = c_0 \ldots c_i \ldots, c = \oplus c_0 c_i \} \]

We abuse notation and denote with the same symbols the lifting to interpretations, i.e., \(\alpha_{io}(\mathcal{I}) := \alpha_{io} \circ \mathcal{I}\), \(\gamma_{io}(\mathcal{I}^\alpha) := \gamma_{io} \circ \mathcal{I}^\alpha\).

\((M, \sqsubseteq, \cup, \cap, M, \{\epsilon\}) \xrightarrow{\gamma_{io}} (\mathbb{I}O, \preceq, \cup, \cap, \mathbb{I}O, \emptyset)\) is a Galois insertion (as stated formally in Lemma A.9 of the electronic appendix).

The input-output behavior of a program is indeed obtainable by abstraction of its (concrete) semantics.

**Proposition 4.4.** Let \(D \in D_C^H\) and \(A \in A_C^H\). Then, \(\alpha_{io}(\mathcal{P}[D . A]) = B^{\alpha}[D . A]\).
Now (as anticipated), following the (classical) abstract interpretation approach, we
define the optimal abstract version of $D$ as $D^{\alpha_\circ} := \alpha_\circ \circ D \circ \gamma_\circ$, and thus the best (possible)
correct approximation w.r.t. $\alpha_\circ$ of the semantic function $F$ is the least fixpoint of $D^{\alpha_\circ}$, i.e., $F^{\alpha_\circ}[D] := \mathfrak{fp}(D^{\alpha_\circ}[D])$. Unfortunately, $F^{\alpha_\circ}[D]$ turns out to be very imprecise,
mainly because the information contained in the input-output pairs is not enough to
keep the synchronization between parallel processes. Indeed, the declarations equivalence induced by $F^{\alpha_\circ}$ is not correct w.r.t. $\alpha_\circ$ (Definition 4.2), since we can have two
programs with the same $F^{\alpha_\circ}$ that have different $B^{\alpha_\circ}$, as shown by the following example.

**Example 4.5.** Consider the two sets of declarations $D_1 := \{d_1, d_2\}$ and $D_2 := \{d_1, d_3\}$
where
- $d_1 := p(x, y) :: q(x) \parallel \text{ask}(\text{true}) \rightarrow \text{now } x = 2 \text{ then tell}(y = 0) \text{ else tell}(y = 1)$
- $d_2 := q(x) :: \text{tell}(x = 2)$
- $d_3 := q(x) :: \text{ask}(\text{true}) \rightarrow \text{tell}(x = 2)$

Clearly, $D_2$ differs from $D_1$ just because of the delay in adding the constraint $x = 2$ to
the store. This difference shows up in the input-output behavior of $p(x, y)$. Indeed,

\[
\alpha_\circ[D_1, p(x, y)] = \{\{c, \text{fin}(c \land x = 2 \land y = 0)\} \mid c \in L\}
\]

\[
\alpha_\circ[D_2, p(x, y)] = \{\{c, \text{fin}(c \land y = 0)\} \mid c \in L, c \Rightarrow x = 2\} \cup \{\{c, \text{fin}(c \land x = 2 \land y = 1)\} \mid c \in L, c \Rightarrow x = 2\}
\]

and then (by Proposition 4.4) $D_1 \not\equiv_\circ D_2$. However, the abstract fixpoint semantics $F^{\alpha_\circ}$
does not distinguish $D_1$ from $D_2$. Indeed,

\[
F^{\alpha_\circ}[D_1] = F^{\alpha_\circ}[D_2] = \left\{\begin{array}{l}
q(x) \rightarrow \{c, \text{fin}(c \land x = 2)\} \mid c \in L
\end{array}\right\}
\]

Given that $F^{\alpha_\circ}$ is the best possible approximation, this also formally proves that it
is not possible to have a correct input-output semantics defined solely on the information
provided by the input/output pairs (some more information in denotations is necessarily needed to be correct).

This also formally justifies (a posteriori) why [de Boer et al. 2000] defined $O^{\alpha_\circ}(A)$ as
a filter of a more concrete semantics instead of using a direct definition.

### 4.2. Modeling the input-output semantics of [de Boer et al. 2000]

In this section, we formally show that the original input-output semantics of $tccp$
$O^{\alpha_\circ}[\mathcal{P} \cdot A]$ (defined in [de Boer et al. 2000]) is essentially isomorphic to $B^{\alpha_\circ}[\mathcal{P} \cdot A]$
(the finite fragment of the semantics introduced in the previous section).

**Theorem 4.6.** Let $P_1$ and $P_2$ be two $tccp$ programs such that no trace in
$\mathcal{P}[P_1] \cup \mathcal{P}[P_2]$ is a failed conditional trace. Then, $O^{\alpha_\circ}[P_1] = O^{\alpha_\circ}[P_2]$ if and only if
$B^{\alpha_\circ}[P_1] = B^{\alpha_\circ}[P_2]$.

This theorem does not hold for any pair of $tccp$ programs. When none of the programs
reaches store $ff$ (along some execution path), we actually have the same input-output
pairs (except for the tag $\text{fin}$). However, when the store $ff$ is reached during a com-
putation, this is no longer necessarily true, as shown by the following example. This

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6Although possible, a direct (expanded) definition of $D^{\alpha_\circ}$ is not relevant for our present purposes.
explains why we qualify as “essentially isomorphic” the relation between $O^{io}[D \cdot A]$ and $B^{io}_{\alpha}[D \cdot A]$.

**Example 4.7.** Let $P_1 := D.\text{loop}$ and $P_2 := D.\text{tell}(false)$, where $D := \{\text{loop}::-\text{tell}(false) \parallel \text{loop}\}$. We have that $B^{io}_{\alpha}[P_1] = B^{io}_{\alpha}[P_2] = \{(c, \text{fin}(false))\}$ while $O^{io}[P_1] = \emptyset \neq O^{io}[P_2] = \{(c, \text{false})\}$.

The difference is due to the change we made in the definition of the small-step operational semantics. More specifically, in the operational semantics that we use (Definition 2.1), when the store $ff$ is reached, we cannot have further transitions. We devised $\rightarrow$ in this way to be conform with the original rationale of the ccp paradigm. As a consequence, when a sequence computes $ff$, it is considered as a failed computation with output $ff$. In contrast, in the operational semantics of [de Boer et al. 2000], the transition relation $\rightarrow$ does not consider the $ff$ store as a special case and then it is possible to execute an agent on the $ff$ store.

Note that, if one is interested, it is straightforward to modify Definition 4.3 to compute exactly $O^{io}[P]$.

To conclude, it is interesting to note that $B^{io}_{\alpha}[P]$ can be equivalently obtained by first appropriately filtering the conditional traces and then applying the abstraction $\alpha_{io}$. Formally, given $M \in M$, let $\pi_M^F(M) := \{r \in M \mid r \text{ ends with } \emptyset \text{ or it contains a stuttering}\}$ and let $M_F := \pi_M^F(M)$. Note that this domain contains only traces such that the application of the $\parallel$ operator produces only finite sequences of stores. It is straightforward to prove that the following diagram commutes

$$
\begin{array}{c}
(M, \emptyset) \xrightarrow{\pi_M^F} (M_F, \emptyset) \\
\downarrow \quad \downarrow \\
(\emptyset_\alpha, \emptyset) \xrightarrow{\alpha_{io}} (\emptyset_\alpha, \emptyset)
\end{array}
$$

5. APPLICATION OF OUR SEMANTICS

In [Comini et al. 2011] we have used (a preliminary version of) our semantics to develop an Abstract Diagnosis framework for tccp (following the approach for Logic Programming of [Comini et al. 1999; Comini 1998]). Since we could use a condensed semantics, we were able to formulate an efficacious parametric debugging methodology for tccp based on approximating the $D[D]$ operator by means of an abstract $D^\alpha[D]$ operator obtained by abstract interpretation.

We showed that, given the intended abstract specification $S^\alpha$ of the semantics of a set of declarations $D$, we can determine all the rules which are wrong w.r.t. $S^\alpha$ by a single application of $D^\alpha[D]$ to $S^\alpha$. Thus, for suitable abstract domains, we obtain an effective static check which is able to identify the exact sources of errors.

There are some good features of this application that show up because of the properties of the concrete semantics we have proposed in this paper. Namely,

— it can be used with partial specifications,
— it can be used with partial sets of declarations.

Obviously, one cannot detect errors in rules involving processes which have not been specified; but for the rules that involve only processes that have a specification, the check can be made, even if the whole set of declarations has not been written yet. To
the best of our knowledge, this is the only method for debugging that can work with incomplete programs.

Comparing our approach to other abstract diagnosis approaches for the ccp paradigm, thanks to our semantics, we have some advantages too. [Falaschi et al. 2007] proposes a first approach to the declarative debugging of a language of the ccp family (utcc), which (like in the case of [Comini et al. 2011]) is also based on the abstract diagnosis approach of [Comini et al. 1999]. However, that work does not cover the particular extra difficulty of non-monotonicity, common to (most of) the other timed concurrent constraint languages. We have already justified through the paper why we consider essential to model also that aspect of concurrent constraint programs. We recall that this ability is crucial in order to model specific behaviors of reactive systems, such as timeouts or preemption actions. This is also the main reason why our abstract (and concrete) semantics are significantly different from [Falaschi et al. 2007] and from formalizations for other declarative languages.

Moreover, the abstract diagnosis instance on the depth\((k)\) domain presented in [Falaschi et al. 2007] is not effective, unless one considers a finite constraint system (situation that limits enormously the applicability of the proposal). The reason is due to the fact that the underlying semantics is not condensed and then, in general, even if abstract denotations contain only traces that are bounded in length, they need to maintain an infinite number of traces, one for each possible initial constraint. By applying the same abstraction technique on our condensed semantics, one obtains instead finite denotations.

One would argue that it is theoretically possible to find a suitable transformation of the depth\((k)\) proposal of [Falaschi et al. 2007] to make it effective, but in the end this would boil down to regain the condensation at the abstract semantics level. However, given the equality between depth\((k)\) traces and the initial part of concrete traces, essentially this would be the same of giving a condensed version of the concrete semantics.

Recently, we have also developed another abstraction of our semantics over a domain of formulas expressed in an extension with constraints of a linear temporal logic (csLTL). In that work, we provide an alternative automatic decision method to check whether a given property specified in csLTL is valid w.r.t. a ccp program. Most of the classical automatic program verification approaches are based on browsing the structure of some form of model (which represents the behavior of the program) to check if a given specification is valid. This implies that a subset of the model has to be built, and sometimes the needed fragment is quite huge. This is known as the state explosion problem. Our proposal, unlike other automatic program verification techniques does not require to build a model at all. The compact definition of the semantics evaluation functions over csLTL that we have obtained is due to the formulation of our concrete semantics definition.

6. RELATED WORK

As we have said in the introduction, for timed concurrent constraint languages, the presence of

— non-determinism,
— local variables and
— timing constructs which are able to handle negative information

significantly complicates the definition of a fully abstract compositional semantics. In this section, we briefly show the impact of these difficulties when defining appropriate denotational semantics for other (timed) concurrent constraint languages. Most of the defined semantics are inspired in that of ccp, and characterize the finite input-output
or strongest postcondition observable behaviors. The strongest postcondition observable collects the pairs of input-output stores such that the program does not produce additional information, i.e., the input coincides with the output.

Soon in [de Boer et al. 1995], the difficulties for handling nondeterminism and infinite behavior in the ccp paradigm was investigated. The authors showed that the presence of nondeterminism and synchronization requires relatively complex structures for the denotational model of (non timed) ccp languages. Moreover, infinite behaviors (which become natural in the timed extensions) are an additional nightmare. Traditionally, solutions to these difficulties have been based on the introduction of restrictions on the language. In [Saraswat et al. 1991] are given the basic ideas for the definition of appropriate semantics for ccp languages and—more specifically— is given a model based on observing the resting points of (finite) ccp processes. The defined semantics is fully abstract for the determinate fragment of ccp (i.e., choice agents have always a single branch). For (finite) nondeterminate processes that are monotonic in nature, a fully abstract semantics is given basing on the observation of ask/tell interactions. In [Falaschi et al. 1997], a simple denotational semantics fully abstract w.r.t. the upward-closed observable behavior is defined for confluent ccp, which is the subclass of ccp programs whose observable behavior does not depend on the chosen (non-deterministic) branch. They also define a correct semantics characterizing the input-output relation of (finite) processes for the restricted-choice ccp, which is a confluent sublanguage of ccp (syntactically restricted to choice agents where either all the branches have the same guard or the guards are all mutually exclusive). As the basis for a method to prove (partial) correctness of ccp programs, in [de Boer et al. 1997] a denotational semantics which characterizes the strongest postcondition is given. The semantics is fully abstract for confluent ccp. It is also shown that the strongest postcondition semantics is not compositional w.r.t. the hiding agent.

The introduction of time in the ccp paradigm raises even more difficulties, in all different timed languages that have been proposed. Based on the deterministic fragment of ccp, in [Saraswat et al. 1994] the authors defined the tcc language and its semantics which is fully abstract just for hiding free processes. This restriction allows one to avoid the problem of non-monotonic behaviors. As we have shown in Example 3.17, due to the partial nature of the constraint system, the combination of the hiding operator with non-determinism can make the language behavior non-monotonic [de Boer et al. 1995; Nielsen et al. 2002b].

The ntcc language extends tcc with non-determinism [Nielsen et al. 2002a] and, inspired by the elegant model for ccp based on closure operators of [Saraswat et al. 1991], a denotational semantics for the strongest postcondition is defined. The semantics is fully abstract for locally-independent processes, i.e., processes in which the non-monotonic agents do not contain bounded variables (i.e., local variables via the hiding construct). More recently, [Falaschi et al. 2007] proposed a denotational semantics of the fragment of ntcc that excludes the non-monotonic construct unless.

The Default tcc language [Saraswat et al. 1996] is an extension of tcc that makes use of default values in order to model strong preemption. It adds to tcc language a limited form of negative information handling, with a construct that has to be used under so called stable assumptions for the negative information in order to avoid chaotic behaviors (notion borrowed from reactive languages like ESTEREL [Berry and Gonthier 1992]). This aids to overcome the problem of the non-monotonic behavior since, in some sense, defaults force to have the Monotonicity property of Definition 3.4. The proposed compositional semantics is fully abstract for agents which satisfy stable assumptions. Their denotational model associates a condition to the computation which plays a similar role to the first positive condition of our conditional traces (but we can allot more behaviors thanks to the others positive and negative conditions along the trace).
Default tcc language however has a limited expressive power compared w.r.t. tcpp since it is deterministic and does not have process calls (and is thus not Turing complete).

The most recent dialect of timed ccp we know, the utcc language, was introduced in [Olarte and Valencia 2008] as an extension of tcc for modeling mobility (communication of private names, typically used in security protocols or mobile systems). In [Falaschi et al. 2009], a denotational model for utcc processes based on a simple domain is defined for data-flow analysis. This semantics is fully abstract only for the monotonic fragment of the language. For the same language, [Olarte and Valencia 2008] defines a denotational semantics characterizing the input-output behavior of processes. This semantics is fully abstract for the monotonic fragment of utcc and is based on temporal formulas.

Thus—to conclude—to our knowledge ours is the only proposal which defines a fully abstract semantics for a full dialect of timed ccp with “negative” constructs (having so a non-monotonic behavior), except the one of default tcc. However, tcpp is non-deterministic and turing complete whereas default tcc is deterministic and not turing complete.

7. CONCLUSIONS

In this work, we have presented a small-step semantics that is fully abstract w.r.t. the tcpp language behavior and that is suitable to be used as the basis of semantics-based program manipulation techniques such as abstract diagnosis. The task of defining a compositional fully-abstract semantics for the language has shown to be difficult due to the non-monotonic nature of the language, which is a characteristic shared with other concurrent languages of the ccp family.

To our knowledge, this is the first fully abstract compositional denotational semantics for a non-deterministic language in the ccp family that covers the whole language (including the non-monotonic behavior).

We have also defined a big-step semantics for tcpp as an abstraction of the small-step one. This semantics collects the limit stores of (finite and infinite) computations. We have proven that its fragment for finite computations is precise enough to recover the original input-output semantics of the language [de Boer et al. 2000]. Moreover, we also have proven that it is not possible to have a correct input-output semantics which is defined solely on the information provided by the input/output pairs.

As future work, we plan to investigate further on applications of our semantics to obtain novel analysis and verification methods. Moreover, we plan to adapt the ideas presented here to define appropriate fully-abstract semantics for other concurrent languages of the ccp family, such as ntcc, utcc and tcc. Thanks to this we will be able to straightforwardly adapt the abstract diagnosis methodology to such languages. These adaptations of the semantics are not immediate, since these languages have significant differences w.r.t. tcpp, but (given the richness of tcpp w.r.t. the other languages) we are confident that the required effort will be reasonable.

ELECTRONIC APPENDIX

The electronic appendix for this article can be accessed in the ACM Digital Library.

REFERENCES


A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of $tccp$ A:33


Received; revised; accepted
In the sequel, to avoid a proliferation of parenthesis, we assume that $\downarrow_c$ and $\uparrow_c$ have priority over $\cdot$ and $\|$.

**Proofs of Section 3**

By construction, we can see that the conditional traces computed by $A$ always satisfy that the store in a given time instant entails the positive condition. Formally,

**Property A.1** Let $A \in \mathcal{A}_C^1$, $I \in \mathbb{I}_I$ and $r \in A[I]$. For each conditional tuple $(\eta^+, \eta^-) \mapsto a$ occurring in $r$, $a \mapsto \eta^-$.

**Proof.** This property is directly verified by (3.7) and (3.8) of Definition 3.15: when a guard is added to the positive condition, it is also added to the correspondent store, and propagated to the subsequent trace. □

There exists a relation between the propagation operator $\downarrow$ and the lub $\otimes$ of the constraint system: the consecutive propagation of two constraints $(r \downarrow_c) \downarrow_c$ is equivalent to $r \downarrow_{(c \otimes c')}$.

**Lemma A.2.** Let $c, c' \in C$ and $r \in M$ such that $(r \downarrow_c) \downarrow_c$ is defined. Then $r \downarrow_{(c \otimes c')}$ is defined and $(r \downarrow_c) \downarrow_c = r \downarrow_{(c \otimes c')}$.

**Proof.** We proceed by structural induction on $r$.

$r = e$ and $r = \otimes_1$. Straightforward.

$r = (\eta^+, \eta^-) \mapsto d \cdot r'$. By hypothesis, $(r \downarrow_c) \downarrow_c$ is defined, thus, $c \gg (\eta^+ \otimes c', \eta^-)$ and $(r' \downarrow_c) \downarrow_c$ is defined. It follows directly that $c \otimes c' \gg (\eta^+, \eta^-)$ and, by inductive hypothesis, $(r' \downarrow_{(c \otimes c')})$ is defined. Thus, $(r \downarrow_{(c \otimes c')})$ is defined too.

$\begin{align*}
(r \downarrow_c) \downarrow_c &= (((\eta^+, \eta^-) \mapsto d \cdot r')) \downarrow_c \\
&= ((c' \otimes \eta^+, \eta^-) \mapsto c' \otimes d \cdot r' \downarrow_c) \downarrow_c \\
&= (c \otimes c' \gg (\eta^+, \eta^-)) \mapsto c \otimes c' \otimes d \cdot (r \downarrow_{(c \otimes c')}) \\
&= (c \otimes c' \otimes \eta^+, \eta^-) \mapsto c \otimes c' \otimes d \cdot r \downarrow_{(c \otimes c')}. \\
&= r \downarrow_{(c \otimes c')}.
\end{align*}$
defined too.

\[(r \downarrow_{c'}) \downarrow_{c} = ((\text{stutt}(\eta^\prime) \cdot r') \downarrow_{c'}) \downarrow_{c} \]

[ by Definition 3.7]

= \text{stutt}(\eta^\prime) \cdot (r' \downarrow_{c'}) \downarrow_{c} \]

[ by Inductive Hypothesis]

= \text{stutt}(\eta^\prime) \cdot r' \downarrow_{c \oplus c'} \]

[ by Definition 3.7]

= r \downarrow_{c \oplus c'}

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There exists a relation between the parallel composition and the operator of propagation as stated by the following lemma.

**Lemma A.3.** Let \( r_1, r_2 \in M \) and \( c \in C \) such that \( r_1 \parallel r_2 \mid_c \) is defined. Then \( (r_1 \parallel r_2) \downarrow_{c} \)

is defined and \( r_1 \parallel r_2 \mid_c = (r_1 \parallel r_2) \downarrow_{c} \).

**Proof.**

We proceed by structural induction on \( r_1 \). Note that, since \( r_1 \parallel r_2 \mid_c \) is defined, it follows that \( r_1 \parallel r_2 \mid_c \) are defined as well.

\( r_1 = c \) (or \( r_1 = \varepsilon \)) and any \( r_2 \mid_{1} \) The statement follows directly from Definitions 3.7 and 3.9.

\( r_1 = (\eta^\prime_1, \eta_1) \mapsto d_1 \cdot r'_1 \) and \( r_2 = (\eta^\prime_2, \eta_2) \mapsto d_2 \cdot r'_2 \mid_{1} \) Since \( r_1 \parallel r_2 \mid_c \) and \( r_1 \parallel r_2 \mid_c \) are defined, it follows that \( c \) is consistent with both \( \eta_1 = (\eta^\prime_1, \eta_1) \) and \( \eta_2 = (\eta^\prime_2, \eta_2) \), and thus, with \( \eta \odot \eta_2 \). We have to distinguish two cases.

\( c \odot d_1 = ff \) and \( c \odot d_2 = ff \mid_{1} \) By inductive hypothesis, \( (r'_1 \parallel r'_2) \downarrow_{c} \) is defined and, since \( c \gg \eta_1 \odot \eta_2 \), \( (r_1 \parallel r_2) \downarrow_{c} \) is defined as well.

\( r_1 \parallel r_2 \mid_c = ((\eta^\prime_1, \eta_1) \mapsto d_1 \cdot r'_1) \parallel ((\eta^\prime_2, \eta_2) \mapsto d_2 \cdot r'_2) \downarrow_{c} \)

[ by Definition 3.7]

= ((\eta^\prime_1 \odot c, \eta_1) \mapsto d_1 \odot c \cdot r'_1) \parallel ((\eta^\prime_2 \odot c, \eta_2) \mapsto d_2 \odot c \cdot r'_2) \downarrow_{c} \]

[ by Definition 3.9]

= ((\eta^\prime_1 \odot c \odot c, \eta_1 \cup \eta_2) \mapsto d_1 \odot d_2 \odot c \cdot (r'_1 \parallel r'_2) \downarrow_{c} \]

[ by Inductive Hypothesis]

= ((\eta^\prime_1 \odot \eta^\prime_2 \odot c, \eta_1 \cup \eta_2) \mapsto d_1 \odot d_2 \odot c \cdot (r'_1 \parallel r'_2) \downarrow_{c} \]

[ by Definition 3.7]

= (r_1 \parallel r_2) \downarrow_{c}

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A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \textit{tccpApp}–3

\[ (\eta^+_1 \otimes \eta^+_2 \otimes c, \eta^-_1 \cup \eta^-_2) \Rightarrow \mathcal{E} \]

[ by Definition 3.7 and Definition 3.9 ]

\[ (r_1 \parallel r_2)_c \]

\[ r_1 = (\eta^+_1, \eta^-_1) \Rightarrow d_1 \cdot r'_1 \text{ and } r_2 = \text{stutt}(\eta^-_2) \cdot r'_2 \]

Since \( r_1 \parallel r_2 \) and \( r_2 \parallel r_2 \) are defined, we have that \( c \gg \eta^-_1 \) and \( c \not\gg \eta^-_2 \). Therefore, \( c \gg (\eta^+_1, \eta^-_1 \cup \eta^-_2) \). By inductive hypothesis, \( (r'_1 \parallel r'_2)_c \) is defined, thus also \( (r_1 \parallel r_2)_c \) is defined.

\[ r_1 \parallel r_2 \parallel r_2 = ((\eta^+_1, \eta^-_1) \Rightarrow d_1 \cdot r'_1)_c \parallel (\text{stutt}(\eta^-_2) \cdot r'_2)_c \]

[ by Definition 3.7 ]

\[ = ((\eta^+_1 \otimes c, \eta^-_1) \Rightarrow d_1 \otimes c \cdot r'_1)_c \parallel (\text{stutt}(\eta^-_2) \cdot r'_2)_c \]

[ by Definition 3.9 ]

\[ = (\eta^+_1 \otimes c, \eta^-_1 \cup \eta^-_2) \Rightarrow d_1 \otimes c \cdot (r'_1 \parallel r'_2)_c \]

[ by Inductive Hypothesis ]

\[ = (\eta^+_1 \otimes c, \eta^-_1 \cup \eta^-_2) \Rightarrow d_1 \otimes c \cdot (r'_1 \parallel r'_2)_c \]

[ by Definition 3.7 ]

\[ = (r_1 \parallel r_2)_c \]

An important technical result states that the evaluation function for agents \( A \) is closed under context embedding. A context \( C[ ] \) consists in a \textit{tccp} agent with a \textit{hole}, which means that \( C[A] \) represents the result of replacing the hole in \( C[ ] \) with the agent \( A \).

**Lemma A.4.** \( \text{Let } A_1, A_2 \in A \parallel \text{ and } \mathcal{I} \in \parallel. Then } A \parallel A_1 \parallel \mathcal{I} = A \parallel A_2 \parallel \mathcal{I} \text{ if and only if, for all context } C[ ], A[C[A_1]] \parallel \mathcal{I} = A[C[A_2]] \parallel \mathcal{I}. \)

**Proof.**

\( \Leftarrow \) Directly holds.

\( \Rightarrow \) This implication follows from Definition 3.15. The evaluation function \( A \) is defined by composition of the semantics of its subagents. In particular, the semantics of both,
\[ A \subseteq B \] and \[ C \subseteq D \], is computed from the semantics of \( A \) and \( B \), respectively. Since \( A \) and \( B \) are equivalent, then also the semantics of \( A \) and \( C \) coincide.

\[ \square \]

**Lemma A.5.** For each \( A \in \mathcal{A} \) and each \( D \in \mathcal{D} \), \( A \subseteq A \) and \( D \subseteq D \) are continuous.

**Proof.** Consider \( A \in \mathcal{A} \) and \( D \in \mathcal{D} \). To prove the continuity of \( A \subseteq A \), we have to verify two properties: monotonicity and finiteness. The continuity of \( D \subseteq D \) follows directly from the continuity of \( A \subseteq A \) and from Definition 3.19.

**Monotonicity.** It is sufficient to show that for each \( \mathcal{I}_1, \mathcal{I}_2 \in \mathcal{I} \), and for each \( A \in \mathcal{A} \), \( \mathcal{I}_1 \subseteq \mathcal{I}_2 \Rightarrow \mathcal{A}[A]_{\mathcal{I}_1} \subseteq \mathcal{A}[A]_{\mathcal{I}_2} \). Observe that the only case in which \( A \) depends on the interpretation is the case of the process call. By definition of \( \subseteq \), \( \mathcal{I}_1(p(x)) \subseteq \mathcal{I}_2(p(x)) \), thus:

\[
\mathcal{A}[p(x)]_{\mathcal{I}_1} = \bigcup_{r \in \mathcal{I}_1}(p(x)) = \bigcup_{r \in \mathcal{I}_1}(p(x)) = \mathcal{A}[p(x)]_{\mathcal{I}_2}
\]

**Finitarity.** Again, it is sufficient to consider the evaluation function \( A \) for the case of the process call. \( \mathcal{A}[A]_{\mathcal{I}} \) depends on a finitary subset of \( \mathcal{I} \), in particular on the subset regarding \( p(x) \) which is a finitary set of conditional traces closed by prefix.

\[ \square \]

**Lemma A.6.** Let \( r \in M \) and \( c, c' \in C \) such that \( c \vdash c' \) and \( r \downarrow c \) is defined. Then \( (r \downarrow c) \downarrow c \) is defined and \( r \downarrow c = (r \downarrow c) \downarrow c \).

**Proof.** By hypothesis, \( r \downarrow c \) is defined, thus \( c \) is compatible with all the conditions occurring in \( r \). Since \( c \vdash c' \), it is easy to notice that also \( c' \) is compatible with all the conditions occurring in \( r \), thus \( r \downarrow c' \) is defined. Then, \( (r \downarrow c) \downarrow c \) is defined as well. If \( c = \mathcal{J} \), by Definition 3.24, \( r \downarrow c = \mathcal{J} = (r \downarrow c) \downarrow c \). Otherwise, if \( c \neq \mathcal{J} \), we proceed by induction on the structure of \( r \).

\[ r = c \text{ and } r = \mathcal{J} \] The statement follows directly from Definitions 3.7 and 3.24.

\[ r = (\eta^+, \eta^-) \Rightarrow d \cdot r' \] We distinguish three sub-cases.

1. **\( d \otimes c \neq \mathcal{J} \)** Since \( c \vdash c' \), it follows that \( d \otimes c' \neq \mathcal{J} \), thus:

\[
(r \downarrow c) \downarrow c = ((\eta^+ \otimes \eta^-) \Rightarrow d \cdot r') \downarrow c
\]

\[ \text{[by Definition 3.7]} \]

\[
= c \cdot r' \downarrow c \dot{\otimes} c
\]

\[ \text{[by Inductive Hypothesis]} \]

\[
= c \cdot r' \downarrow c \cdot d \otimes c'
\]

2. **\( d \otimes c = \mathcal{J} \) and \( d \otimes c' \neq \mathcal{J} \)** We have that:

\[
(r \downarrow c) \downarrow c = ((\eta^+ \otimes \eta^-) \Rightarrow d \cdot r') \downarrow c
\]

\[ \text{[by Definition 3.7]} \]

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYYY.
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of tccpApp–5

\[ (\eta^+ \otimes c', \eta^-) \Rightarrow d \otimes c'. r_1 \downarrow_{c'} \downarrow_c \]

[by Definition 3.24]

\[ = c \cdot \overline{\text{ff}} \]

By Definition 3.24, \( r \downarrow_c = (\eta^+ \otimes c', \eta^-) \Rightarrow d \cdot r' \downarrow_c = c \cdot \overline{\text{ff}} \), thus \( r \downarrow_c = (r \downarrow_{c'}) \downarrow_c \).

\[ d \otimes c' = \overline{\text{ff}} \]

Since \( c \vdash c' \), it follows that \( d \otimes c = \overline{\text{ff}} \), thus:

\[ (r \downarrow_{c'}) \downarrow_c = (((\eta^+ \otimes c', \eta^-) \Rightarrow d \cdot r') \downarrow_{c'}) \downarrow_c \]

[by Definition 3.7]

\[ = (\eta^+ \otimes c', \eta^-) \Rightarrow \overline{\text{ff}} \cdot \overline{\text{ff}} \downarrow_c \]

[by Definition 3.24]

\[ = c \cdot \overline{\text{ff}} \]

By Definition 3.24, it follows that \( r \downarrow_c = c \cdot \overline{\text{ff}} = (r \downarrow_{c'}) \downarrow_c \).

\[ r = \text{stutt}(\eta^-) \cdot r' \]

By Definition 3.24, it follows that:

\[ (r \downarrow_{c'}) \downarrow_c = ((\text{stutt}(\eta^-) \cdot r') \downarrow_{c'}) \downarrow_c \]

[by Definition 3.7]

\[ = (\text{stutt}(\eta^-) \cdot r') \downarrow_{c'} \]

[by Definition 3.24]

\[ = c \]

By Definition 3.24, \( r \downarrow_c = (\text{stutt}(\eta^-) \cdot r') \downarrow_c = c \), thus \( r \downarrow_c = (r \downarrow_{c'}) \downarrow_c \).

\[ \square \]

In order to formulate the following Lemma A.8, we need to introduce the counterpart of \( \parallel \) on behavioral timed traces.

**Definition A.7.** Let \( s, s_1, s_2 \in C^* \). \( \parallel : C^* \times C^* \rightarrow C^* \) is defined by structural induction as:

\[ s \parallel e := s \]

\[ c \parallel s := s \] (A.1a)

\[ (c_1 \cdot s_1) \parallel (c_2 \cdot s_2) := \begin{cases} (c_1 \otimes c_2) \cdot (c_2 \otimes s_1) \parallel c_1 \otimes s_2 & \text{if } c_1 \otimes c_2 \neq \overline{\text{ff}} \\ \overline{\text{ff}} & \text{if } c_1 \otimes c_2 = \overline{\text{ff}} \end{cases} \] (A.1b)

where, by abusing notation, \( c \otimes (c_1 \cdot c_n) \) denotes \((c \otimes c_1) \cdots (c \otimes c_n)\).

We extend this operator to sets of behavioral timed traces as \( S_1 \parallel S_2 = \{ s_1 \parallel s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2 \} \).

**Lemma A.8.** Let \( c \in C; A_1, A_2 \in \mathcal{A}^I_\mathcal{L}; I \in \parallel \); \( r_1, r_2 \in A[A_1]_I \) and \( r_2 \in A[A_2]_I \) such that \( r_1 \parallel r_2, r_1 \parallel c \) and \( r_2 \parallel c \) are defined. Then, \( (r_1 \parallel r_2) \parallel c \) is defined and \( r_1 \parallel r_2 \parallel c = (r_1 \parallel r_2) \parallel c \).

**Proof.** Since both \( r_1 \parallel c \) and \( r_2 \parallel c \) are defined, \( c \) satisfies all the conditions in \( r_1 \) and \( r_2 \). It is easy to notice from Definition 3.9 that \( c \) satisfies also the conditions of \( r_1 \parallel r_2 \), thus, \( (r_1 \parallel r_2) \parallel c \) is defined as well.

We proceed to prove that \( r_1 \parallel c \parallel r_2 \parallel c = (r_1 \parallel r_2) \parallel c \) by induction on the structure of \( r_1 \).

\[ r_1 = e \text{ and any } r_2 \]

By Definition 3.9, \( (r_1 \parallel r_2) \parallel c = (e \parallel r_2) \parallel c = r_2 \parallel c \). By Definition 3.24 and by Equation (A.1a), we obtain: \( r_1 \parallel c \parallel r_2 \parallel c = e \parallel r_2 \parallel c = r_2 \parallel c \). Thus, \( r_1 \parallel c \parallel r_2 \parallel c = (r_1 \parallel r_2) \parallel c \).
r_1 = \textit{\& and any } r_2 \neq \epsilon_1 \text{ By Definition 3.9, } (r_1 \parallel r_2)_c = (\otimes r_2)_c = r_2 \parallel c. \text{ By Definition 3.24 and by Equation (A.1b), } r_1 \parallel r_2 \parallel c = c \parallel r_2 \parallel c = r_2 \parallel c, \text{ since } r_2 \neq c. \text{ Thus, } r_1 \parallel c \parallel r_2 \parallel c = (r_1 \parallel r_2)_c \parallel c.

\[ d_1 \otimes d_2 = \text{ff,} \]

\[ (r_1 \parallel r_2)_c = ((\eta_1 \Rightarrow d_1 \cdot r'_1) \parallel (\eta_2 \Rightarrow d_2 \cdot r'_2))_c \]

[by Definition 3.9]

\[ = (\eta_1 \otimes \eta_2 \Rightarrow d_1 \otimes d_2 \cdot (r'_1 \downarrow d_1 \parallel r'_2 \downarrow d_1))_c \]

[by Definition 3.24]

\[ = c \cdot (r'_1 \downarrow d_1 \parallel r'_2 \downarrow d_1)_{c \otimes d_1 \otimes d_2} \]

[by Inductive Hypothesis]

\[ = c \cdot (r'_1 \downarrow d_1 \parallel r'_2 \downarrow d_1 \parallel r'_1 \downarrow d_1 \parallel r'_2 \downarrow d_1)_{c \otimes d_1 \otimes d_2} \]

[by Lemma A.6]

\[ = c \cdot (r'_1 \downarrow d_1 \parallel r'_2 \downarrow d_1)_{c \otimes d_1 \otimes d_2} \]

[by Equation (A.1b)]

\[ = (c \cdot r'_1 \downarrow c \otimes d_1) \parallel (c \cdot r'_2 \downarrow c \otimes d_2) \]

By Definition 3.24, \( r_1 \parallel c \parallel r_2 \parallel c = (c \cdot r'_1 \downarrow c \otimes d_1) \parallel (c \cdot r'_2 \downarrow c \otimes d_2); \) therefore, we conclude \( r_1 \parallel c \parallel r_2 \parallel c = (r_1 \parallel r_2)_c \parallel c. \)

\[ d_1 \otimes d_2 = \text{ff,} \]

\[ (r_1 \parallel r_2)_c = ((\eta_1 \Rightarrow d_1 \cdot r'_1) \parallel (\eta_2 \Rightarrow d_2 \cdot r'_2))_c \]

[by Definition 3.9]

\[ = (\eta_1 \otimes \eta_2 \Rightarrow \text{ff} \cdot \otimes)_c \]

[by Definition 3.24]

\[ = c \cdot \text{ff} \]

By Definition 3.24 and by Equation (A.1b), \( r_1 \parallel c \parallel r_2 \parallel c = c \cdot \text{ff,} \text{ thus } r_1 \parallel c \parallel r_2 \parallel c = (r_1 \parallel r_2)_c \parallel c. \)

\[ r_1 = \eta_1 \Rightarrow d_1 \cdot r'_1 \text{ and } r_2 = \text{stutt}(\eta'_2) \cdot r'_2 \text{,} \]

\[ (r_1 \parallel r_2)_c = ((\eta_1 \Rightarrow d_1 \cdot r'_1) \parallel \text{stutt}(\eta'_2) \cdot r'_2)_c \]

[by Definition 3.9]

\[ = ((\eta'_1 \cdot \eta'_1 \cup \eta'_2) \Rightarrow d_1 \cdot (r'_1 \parallel r'_2 \downarrow d_1))_c \]

[by Definition 3.24]

\[ = c \cdot (r'_1 \parallel r'_2 \downarrow d_1)_{c \otimes d_1} \]

[by Inductive Hypothesis]

\[ = c \cdot (r'_1 \downarrow c \otimes d_1 \parallel r'_2 \downarrow c \otimes d_1) \]

[by Lemma A.6]
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \(tc\text{cpApp–7}\)

\[
= c \cdot (r_1 \downarrow_{c \otimes d} \uparrow_{c \otimes d})(d) \\
\text{[by Equation (A.1b)]} \\
= (c \cdot r_1 \downarrow_{c \otimes d}) (d) \\
\text{[by Definition 3.24, \(r_1 \downarrow_{c} \uparrow_{c} = (c \cdot r_1 \downarrow_{c \otimes d}) (d)\), thus \(r_1 \downarrow_{c} \uparrow_{c} = (r_1 \downarrow_{c} \uparrow_{c})\).} \\
\]

By Definition 3.24, \(r_1 \downarrow_{c} \uparrow_{c} = (c \cdot r_1 \downarrow_{c \otimes d}) (d)\), and \(r_2 = stutt(\eta_2) \cdot r'_2\),

\[
(r_1 \downarrow_{c} \uparrow_{c} \downarrow_{c} = (stutt(\eta_2) \cdot r_2 \downarrow_{c} \uparrow_{c} \downarrow_{c} = (stutt(\eta_2) \cdot r_2 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \\
\text{[by Definition 3.9]} \\
= (stutt(\eta_2) \cdot r_2 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \\
\text{[by Definition 3.24]} \\
= c \cdot (r_2 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \\
\text{[by Inductive Hypothesis]} \\
= c \cdot (r_2 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \\
\text{[by Equation (A.1b)]} \\
= (c \cdot r_2 \downarrow_{c} \uparrow_{c}) \\
\]

By Definition 3.24, \(r_1 \downarrow_{c} \uparrow_{c} = (c \cdot r_1 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \uparrow_{c} = (r_1 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \uparrow_{c} = (r_1 \downarrow_{c} \uparrow_{c}) \downarrow_{c} \uparrow_{c} \downarrow_{c} \).

\[\square\]

Proof. [of Theorem 3.25] Let \(d \in C\) and \(P = DA\) with \(D \in D^{\Pi}_C\) and \(A \in A^{\Pi}_C\), we proceed by structural induction on \(A\).

**skip.** The proof in this case is straightforward.

prefix(\(A[\text{skip}]_x[D]\))_d = prefix(\(\{\varnothing\}\))_d = \{c, d\} = B^{\Pi}[D \cdot \text{skip}]_d

**tell(c).**

prefix(\(A[\text{tell}(c)]_x[D]\))_d = prefix(\(\{\varnothing, \varnothing\} \rightarrow c \cdot \varnothing\))_d

\[= \text{prefix}(d \cdot (d \otimes c)) \downarrow d \]

\[= B^{\Pi}[D \cdot \text{tell}(c)]_d \]

\(A=\sum_{i=1}^{n} \text{ask}(c_i) \rightarrow A_{i1}\) We prove the two directions separately.

\(\exists\) We show that, given a conditional trace \(r \in A[\{A\}_x[D]],\) it holds that \(\forall d \in C, \text{prefix}(r \downarrow_d) \in B^{\Pi}[D \cdot A]_d\). We have to distinguish two cases.

- \(r = (c_j, \varnothing) \rightarrow c_j \cdot r_j \downarrow_{c_j} \) with \(1 \leq j \leq n\). By (3.7) it follows that \(r_j \in A[\{A_j\}_x[D]].\)

In case \(r_j \downarrow_d\) is not defined (i.e., \(d \nvdash c_j\)), \(\text{prefix}(r_j \downarrow_d) = \varnothing \in B^{\Pi}[D \cdot A]_d\). Otherwise, if \(r_j \downarrow_d\) is defined, we have that \(d \vdash c_j\) and \(r_j \downarrow_{c_j} \downarrow_{d \otimes c_j}\) is defined too. We distinguish two sub-cases.

- \(d \neq \text{ff}\). In this case we have:

\[\text{prefix}(r \downarrow_d) \]

\[= \text{prefix}(\{((c_j, \varnothing) \rightarrow c_j \cdot r_j \downarrow_{c_j}) \downarrow_d \mid 1 \leq j \leq n, r_j \in A[\{A_j\}_x[D]]\}) \\
\text{[by Definition 3.24]} \]

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYY.
For each \( J \) that prefix \( r \) of \( J \) then, the small-step behavior is \( \leq 1 \) conditional trace \( t \) of \( A \). We denote by \( \Sigma \) the set of all conditional traces computed by the semantics evaluation function \( A \). There exists \( d \) such that \( \Sigma \) does not satisfy any guard \( A \). Moreover, by Definition 3.24 and Definition 3.1, it follows that \( \Sigma \) does not satisfy any guard \( A \). The element \( \epsilon \) directly belongs to \( B^{ss}[D \cdot A] \). Since \( d \) does not satisfy any guard \( A \), then, the small-step behavior is \( \leq 1 \) conditional trace \( t \) of \( A \). There are three cases to be considered.

1. \( d \) does not satisfy any guard. This means that for all \( 1 \leq j \leq n \), \( d \) does not satisfy any guard \( A \); then, the small-step behavior is \( B^{ss}[D \cdot A] = \{ \epsilon, d \} \). Thus, it exists a conditional trace \( r \in A[D] \) such that \( \text{prefix}(r) \) is not defined for the configuration \( \langle A, r \rangle \).

2. \( d \) does not satisfy any guard. This means that for all \( 1 \leq j \leq n \), \( d \) does not satisfy any guard \( A \); then, the small-step behavior is \( B^{ss}[D \cdot A] = \{ \epsilon, d \} \). Thus, it exists a conditional trace \( r \in A[D] \) such that \( \text{prefix}(r) \) is not defined for the configuration \( \langle A, r \rangle \).

3. \( d \) does not satisfy any guard. This means that for all \( 1 \leq j \leq n \), \( d \) does not satisfy any guard \( A \); then, the small-step behavior is \( B^{ss}[D \cdot A] = \{ \epsilon, d \} \). Thus, it exists a conditional trace \( r \in A[D] \) such that \( \text{prefix}(r) \) is not defined for the configuration \( \langle A, r \rangle \).
\[ d = \mathit{ff} \] In this case we have:

\[
\text{prefix}(r \downarrow g) = \text{prefix}(((c_j, \varnothing) \mapsto c_j \cdot r_j) \downarrow g)
\]

[ by Definition 3.24 ]

\[ = \{\epsilon, \mathit{ff}\} \]

[ by Definition 3.1 ]

\[ \subseteq B^s [D . A]_g \]

Therefore, we can conclude that \(\text{prefix}(A[A] \cdot x[D] \downarrow_d) = B^s [D . A]_d\). 

**now c then A₁ else A₂** We prove the two directions independently. We abbreviate the conditional agent and call it \(A (A := \text{now } c \text{ then } A_1 \text{ else } A_2)\).

\[ \subseteq \text{We show that } \forall d \in C. \text{prefix}(A[A] \cdot x[D] \downarrow_d) \subseteq B^s [D . A]_d \text{. now } c \text{ then } A_1 \text{ else } A_2_d. \] There are seven possible cases, one for each type of trace \(r\) in (3.8).

\[ r = (c, \varnothing) \mapsto c . \mathit{ff} \]

By (3.8) we have that \(\varnothing \subseteq A[A] \cdot x[D]\), which means, by Definition 3.15, that \(A_1 = \text{skip}\). We consider now the three possible cases:

\[ d \not\leftrightarrow c \text{ and } d \not\rightarrow \mathit{ff} \] It is straightforward that \(\text{prefix}(r \downarrow_d) = \text{prefix}(d \cdot d) = \{\epsilon, d, d \cdot d\}\). On the behavioral part, we know from Rule R4 that the observable of \(A\) is the set of all prefixes of \(d \cdot d\), so we can conclude \(\text{prefix}(r \downarrow_d) \subseteq B^s [D . A]_d\).

\[ d = \mathit{ff} \]

The small-step behavior is \(B^s [D . A]_g = \{\epsilon, \mathit{ff}\}\). Since \(\mathit{ff} \not\rightarrow c\) it is straightforward that \(\text{prefix}(r \downarrow_g) = \{\epsilon, \mathit{ff}\} \subseteq B^s [D . A]_g\). 

\[ d \not\leftrightarrow c \]

Then the application of \(\downarrow_d\) to the agent semantics does not compute any behavioral timed trace. Therefore, \(\text{prefix}(r \downarrow_d) = \varnothing \subseteq B^s [D . A]_d\). 

\[ r = (\eta^+ \otimes c, \eta^-) \mapsto a \otimes c . r' \cdot c \]

From (3.8) it follows that \((\eta^+, \eta^-) \mapsto a \cdot r' \in A[A_1] \cdot x[D]\), \(d\) being compatible with all the conditions occurring in \(r'\), \(a \otimes c \not\leftrightarrow \mathit{ff}\) and \(\forall h \in \eta^- . \eta^+ \otimes c \not\leftrightarrow h\).

In case \(r \downarrow_d\) is not defined (i.e., \(d \not\leftrightarrow (\eta^+ \otimes c, \eta^-)\) or when \(d\) is not compatible with some condition occurring in \(r'\)), we have that \(\text{prefix}(r \downarrow_d) = \varnothing\) which is directly included in \(B^s [D . A]_d\).

Otherwise, if \(r \downarrow_d\) is defined, it follows that \(d \not\leftrightarrow (\eta^+ \otimes c, \eta^-)\). This implies that \(d \not\leftrightarrow c\) since \(c\) belongs to the positive condition. Under these circumstances, we have:

\[
\text{prefix}(r \downarrow_d) = \text{prefix}(( (\eta^+ \otimes c, \eta^-) \mapsto a \otimes c . r' \cdot c ) \downarrow_d) ( (\eta^+, \eta^-) \mapsto a \cdot r' \in A[A_1] \cdot x[D] )
\]

[ by Definition 3.24 ]

\[ = \text{prefix}( (d \cdot (r' \cdot c)) \downarrow_d ) \]

[ by Lemma A.6 since \(d \leftrightarrow c\) ]

\[ = \text{prefix}(d \cdot (r' \cdot c) \downarrow_d ) \]

[ by Inductive Hypothesis ]

\[ \subseteq B^s [D . A]_d \]

[ by Rule R3 ]

\[ \subseteq B^s [D . A]_d \]

\[ r = (\eta^+ \otimes c, \eta^-) \mapsto \mathit{ff} \cdot c \]

We consider two possible cases:
\(d \models (\eta^+ \otimes c, \eta^-)\). This implies that \(d \vdash c\). Under these conditions, we get:

\[
\text{prefix}(r|_d) = \text{prefix}(((\eta^+ \otimes c, \eta^-) \Rightarrow ff \cdot \otimes)|_d)
\]

[ by Definition 3.24 ]

\[
= \{ \epsilon, ff \}
\]

[ by Definition 3.1 ]

\[
\subseteq B^s[D \cdot A]|_d
\]

\(d \not\models (\eta^+ \otimes c, \eta^-)\). In this case \(\text{prefix}(r|_d) = \emptyset\) which is directly included in \(B^s[D \cdot A]|_d\).

\(r = (c, \eta^-) \Rightarrow c \cdot r'\). In case \(r|_d\) is not defined (i.e., \(d \not\models (c, \eta^-)\)) or also when \(d\) is not compatible with some condition occurring in \(r'\) we have that \(\text{prefix}(r|_d) = \emptyset \subseteq B^s[D \cdot A]|_d\). Otherwise, by (3.8), \(\text{stutt}(\eta^-) \cdot r' \in A[A_1]|_x[D]\) and \(d\) is compatible with all the conditions occurring in \(r'\). We have to consider two sub-cases.

\(d \not\models ff\) In this case we have:

\[
\text{prefix}(r|_d) = \text{prefix}({(c, \eta^-) \Rightarrow c \cdot r'}|_d) \quad \text{stutt}(\eta^-) \cdot r' \in A[A_1]|_x[D]
\]

[ by Definition 3.24 ]

\[
= \text{prefix}({d \cdot r'|_{d \not\models c}} \quad \text{stutt}(\eta^-) \cdot r' \in A[A_1]|_x[D]
\]

[ since \(d \vdash c\) ]

\[
= \text{prefix}({d \cdot r'|_d} \quad \text{stutt}(\eta^-) \cdot r' \in A[A_1]|_x[D]
\]

[ by Definition 3.15 ]

\[
= \text{prefix}({d \cdot r'|_d} \quad r' \in A[A_1]|_x[D]
\]

[ by Equation (3.1) ]

\[
= \{ \epsilon, d \} \cup \{ d \cdot s \mid s \in \text{prefix}(A[A_1]|_x[D]|_d) \}
\]

[ by Inductive Hypothesis ]

\[
\subseteq \{ \epsilon, d \} \cup \{ d \cdot s \mid s \in B^s[D \cdot A]|_d \}
\]

[ by Rule R4 ]

\[
\subseteq B^s[D \cdot A]|_d
\]

The fourth step follows from the definition of the semantics \(A\) (Definition 3.15). The construct \(\text{stutt}\) is introduced only by an ask agent. Thus, we know that \(A_1\) is an ask agent. The Equation (3.8), states that \(\text{stutt}(\eta^-)\) is always followed by a conditional trace which belongs to the semantics of the ask, which can be reduced to say that \(r'\) belongs to \(A[A_1]|_x[D]\).

\(d = ff\) In this case we have that \(\text{prefix}(r|_d) = \{ \epsilon, ff \}\) which corresponds to the behavior \(B^s[D \cdot A]|_d\) since the transition relation \(\rightarrow\) is not defined for the agent \(A\) starting with store \(ff\).

\(r = (tt, c) \Rightarrow tt \cdot \otimes\) By (3.8), \(\otimes \in A[A_2]|_x\). By Definition 3.15, it follows that \(A_2\) is a skip agent. We consider two sub-cases.
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \texttt{tcpApp–11}

\[ d \vdash c \] It is straightforward that \( \text{prefix}(r\|d) = \text{prefix}(d \cdot d) = \{ e, d, d \cdot d \} \).

From Rule \textbf{R6}, we know that the observable of the agent \( A \) consists of the set of all prefixes of \( d \cdot d \). Therefore, \( \text{prefix}(r\|d) \subseteq B^{\mathit{ss}}[D \cdot A]_d \).

\[ d \vdash c \] In this case \( r \| d \) does not compute any trace because \( d \) does not satisfy the condition, thus \( \text{prefix}(r\|d) = \emptyset \subseteq B^{\mathit{ss}}[D \cdot A]_d \).

\[ r = (\eta^*, \eta^- \cup \{ c \}) \Rightarrow c' \cdot r' \] In case \( r\|d \) is not defined (i.e., \( d \not\models (\eta^*, \eta^- \cup \{ c \}) \)), \( \text{prefix}(r\|d) = \emptyset \), which is directly contained in \( B^{\mathit{ss}}[D \cdot A]_d \).

Otherwise, if \( r\|d \) it follows that \( (\eta^*, \eta^-) \Rightarrow c' \cdot r' \in A[A_2]_{\mathcal{F}[D]} \) and \( c' \not\vdash c \). If \( d \models (\eta^*, \eta^- \cup \{ c \}) \), we know also that \( d \vdash c \). Under these conditions, we have:

\[
\text{prefix}(r\|d) = \text{prefix}(\{ ((\eta^*, \eta^- \cup \{ c \}) \Rightarrow c' \cdot r') \}
\]

\[
= \text{prefix}(\{ d' \cdot r' \mid \{ d' \cdot r' \} \cdot \emptyset \subseteq A[A_2]_{\mathcal{F}[D]} \}
\]

\[
= \text{prefix}(A[A_2]_{\mathcal{F}[D]} \| d)
\]

\[ \text{by Inductive Hypothesis} \]

\[
\subseteq B^{\mathit{ss}}[D \cdot A]_d
\]

\[ r = (tt, \eta^- \cup \{ c \}) \Rightarrow tt \cdot r' \] By (3.8), we have that \( \text{stutt}(\eta^-) \cdot r' \in A[A_2]_{\mathcal{F}[D]} \).

In case \( r\|d \) is not defined (i.e., \( d \not\models (tt, \eta^- \cup \{ c \}) \)), \( \text{prefix}(r\|d) = \emptyset \), which is directly contained in \( B^{\mathit{ss}}[D \cdot A]_d \).

Otherwise, if \( r\|d \) is defined it follows that \( d \models (tt, \eta^- \cup \{ c \}) \). Then, we have:

\[
\text{prefix}(r\|d) = \text{prefix}(\{ (tt, \eta^- \cup \{ c \}) \Rightarrow tt \cdot r') \}
\]

\[
= \text{prefix}(\{ tt \cdot r' \mid \{ tt \cdot r' \} \cdot \emptyset \subseteq A[A_2]_{\mathcal{F}[D]} \}
\]

\[ \text{by Definition 3.24} \]

\[
= \text{prefix}(A[A_2]_{\mathcal{F}[D]} \| d)
\]

\[ \text{by Inductive Hypothesis} \]

\[
\subseteq B^{\mathit{ss}}[D \cdot A]_d
\]

The third step can be done since each construct \( \text{stutt}(\eta^-) \) is introduced by a choice agent, and Equation (3.7) states that it is always followed by a conditional trace \( r' \) belonging recursively to the semantics of \( A_2 \).

\[ \exists \text{ We have four cases, one for each rule defining the operational semantics for the conditional agent in Figure 1.} \]

\textbf{Rule R3} Let us recall the conditions to apply Rule \textbf{R3}: it must occur \( \langle A_1, d \rangle \rightarrow \langle A'_1, d' \rangle \) and \( d \vdash c \). In this case, we have that \( B^{\mathit{ss}}[D \cdot A]_d = \)}
There are two cases in which it may happen that $B_{ss} \subseteq D \cdot A \mid_d$. By inductive hypothesis, we know that $prefix(A[A_{1}]_{X[D]} \downarrow_d) \supseteq B_{ss} \subseteq D \cdot A_{1} \downarrow_d$, thus also $prefix(A[A_{1}]_{X[D]} \downarrow_d) \supseteq B_{ss} \subseteq D \cdot A \downarrow_d$. Next, we prove the inclusion $prefix(A[A_{1}]_{X[D]} \downarrow_d) \supseteq prefix(A[A_{1}]_{X[D]} \downarrow_d)$. We proceed by induction on the structure of a generic $r_1 \in A[A_{1}]_{X[D]}$ in order to find $r \in A[A]_{X[D]}$ such that $prefix(r \mid_d) \subseteq prefix(r \mid_d)$. 

\[ r_1 = \emptyset, \quad By (3.8), \quad r = (c, \emptyset) \Rightarrow c \cdot \emptyset \in A[A]_{X[D]} \]. We know that $\emptyset \downarrow_d = d$ and $r \downarrow_d = d \cdot (d \circ c) = d \cdot d$, since $d + c$. It is easy to see that the prefixes of $d$ are all included in the prefixes of $d \cdot d$.

\[ r_1 = (\eta^+ \cdot \eta^-) \Rightarrow c' \cdot r' \downarrow_d \quad By definition, \quad r = (\eta^+ \cdot \eta^-) \Rightarrow c' \cdot c \cdot r' \downarrow_d \in A[A]_{X[D]} \]. If $d \equiv (\eta^+ \cdot \eta^-)$, then $r \downarrow_d = d \cdot r' \downarrow_d \downarrow_d$, and, since $d + c$ by the initial assumptions, $r \downarrow_d = d \cdot r' \downarrow_d \downarrow_d = d \cdot r' \downarrow_d \downarrow_d = r_1 \downarrow_d$, thus the inclusion of the prefixes directly holds. Otherwise, if $d \not\equiv (\eta^+ \cdot \eta^-)$, then the operator $\downarrow_d$ is undefined in both cases.

\[ r_1 = \text{st}(\eta^-) \cdot r' \downarrow_d \quad By definition, \quad r = (c, \eta^-) \Rightarrow c \cdot r' \downarrow_d \in A[A]_{X[D]} \]. If for all $h^- \in \eta^-$, $d \not\equiv h^-$, then $r_1 \downarrow_d = d$ and it holds that its prefixes are all included in the prefixes of $r \downarrow_d = d \cdot r' \downarrow_d \downarrow_d$. Otherwise, if it exists $h^- \in \eta^-$ such that $d \not\equiv h^-$, $h^-$, then the $\downarrow_d$ operator is undefined in both cases.

**Rule R4**. The conditions to apply this rule are $(A_1, d) \not\equiv$, $d \equiv c$ and $d \not\equiv$, in which case the small-step behavior is defined as $B_{ss} \subseteq D \cdot A \mid_d = prefix(d \cdot d)$. There are two cases in which it may happen that $(A_1, d) \not\equiv$:

\[ A_1 = \text{skip}, \quad By (3.2), \quad \emptyset \in A[A_{1}]_{X[D]} \quad and \quad r = (c, \emptyset) \Rightarrow c \cdot \emptyset \in A[A]_{X[D]} \].

We now have that $r \downarrow_d = d \cdot (d \circ c) = d \cdot d$, whose prefixes coincide with $B_{ss} \subseteq D \cdot A \mid_d$.

\[ A_1 = \sum_{i \leq n} \text{ask}(c_i) \rightarrow B_i \quad and \quad \forall 1 \leq i \leq n \not\equiv c_i, \quad By (3.7), \quad \text{st}(\{c_1, \ldots, c_n\} \cdot r' \in A[A]_{X[D]} \quad and, \quad as, \quad consequence, \quad r = (c, \{c_1, \ldots, c_n\}) \Rightarrow c \cdot r' \downarrow_d \quad \text{belongs to} \quad A[A]_{X[D]} \}. \]

Now we compute $r \downarrow_d$ and we get the trace $d \cdot r' \downarrow_d \downarrow_d = d \cdot r' \downarrow_d$. By definition of the evaluation function $A$, $r'$ is different from the empty conditional trace $\epsilon$ (by (3.7) a $\text{st}(\cdot)$ construct is always followed by another conditional state). Therefore, $r' \downarrow_d = d \cdot d \cdot s$ for some behavioral trace $s$. As a consequence, the behavior of the agent $B_{ss} \subseteq D \cdot A \mid_d = d \cdot d$ is included in the set of prefixes of $r \downarrow_d = d \cdot d \cdot s$.

In case $d \not\equiv$, we are not allowed to apply any rule in Figure 1, so the small-step behavior is $B_{ss} \subseteq D \cdot A \mid_d = \{\epsilon, d\}$. In this case, $A_1 = \text{skip}$ since $d$ is strong enough to entail any guard of a generic agent $\sum_{i \leq n} \text{ask}(c_i) \rightarrow B_i$. As explained above, $\emptyset \in A[A_{1}]_{X[D]}$ and $r = (c, \emptyset) \Rightarrow c \cdot \emptyset \in A[A]_{X[D]}$, thus $r \downarrow_d = \emptyset$, and it is easy to note that $B_{ss} \subseteq D \cdot A \mid_d = prefix(\emptyset)$.

**Rule R5**. This case is analogous to the case for Rule R3 but, instead of executing the then branch $(A_1)$, the else branch of the conditional agent $(A_2)$ is taken, under the condition that $d \not\equiv c$. More specifically, the conditions imposed for the application of the rule are $(A_2, d) \not\equiv (A_2', d')$ and $d \not\equiv c$, in which case $B_{ss} \subseteq D \cdot A \mid_d = B_{ss} \subseteq D \cdot A_2 \mid_d$. By inductive hypothesis, we know that $prefix(A[A_{2}]_{X[D]} \downarrow_d) \supseteq B_{ss} \subseteq D \cdot A \mid_d$, thus also $prefix(A[A_{2}]_{X[D]} \downarrow_d) \supseteq B_{ss} \subseteq D \cdot A \mid_d$. In the following, we prove that $prefix(A[A]_{X[D]} \downarrow_d) \supseteq prefix(A[A_{2}]_{X[D]} \downarrow_d)$ when $d \not\equiv c$. We proceed by induction on the structure of a generic $r_2 \in A[A_{2}]_{X[D]}$ in order to find a conditional trace $r \in A[A]_{X[D]}$ such that $prefix(r \mid_d) \subseteq prefix(r \mid_d)$. 

\[ r_2 = \emptyset, \quad In this case, \quad r = (tt, \{c\}) \Rightarrow tt \cdot \emptyset \in A[A]_{X[D]} \]. We have $\emptyset \downarrow_d = d$, whose prefixes are included in those of $r \downarrow_d = d \cdot d$.
$r_2 = (\eta^+, \eta^-) \Rightarrow c' \cdot r'$. In this case, $r = (\eta^+, \eta^- \cup \{c\}) \Rightarrow c' \cdot r' \in A[\mathcal{F}_{\mathcal{D}}]$. Let us now assume that $d \models (\eta^+, \eta^-)$; then, $r_2|_d = d \cdot r'|_{d \circ c'}$. In addition, since by the initial assumptions $d \not\models c$, $r_2|_d = d \cdot r'|_{d \circ c'}$, the inclusion of the prefixes directly holds. Otherwise, if $d \not\models (\eta^+, \eta^-)$, then the operator $\parallel_{d}$ is undefined in both cases.

$r_2 = \text{stutt}(\eta^-) \cdot r'$ By definition, $r = (tt, \eta^- \cup \{c\}) \Rightarrow tt \cdot r' \in A[\mathcal{F}_{\mathcal{D}}]$. Assume that for all $h^- \in \eta^-$, $d \not\models h^-$. Then, $r_2|_d = d$, and its prefixes are all included in the prefixes of $r|_d = d \cdot r'|_d$. Otherwise, if it exists $h^- \in \eta^-$ such that $d \models h^-$, then the $\parallel_{d}$ operator is undefined in both cases.

**Rule R6** This case is analogous to the case for Rule R4. Now, the conditions to apply the rule are that $\langle A_2, d \rangle \not\models \not\models c$. In this case, the small-step behavior is $B^s[\mathcal{D} \cdot A]|_d = \text{prefix}(d \cdot d)$. There are two cases in which it may happen that $\langle A_2, d \rangle \not\models c$:

\[ A_2 = \text{skip} \] By (3.2), $\otimes \in A[\mathcal{A}_2]_{\mathcal{F}_{\mathcal{D}}}$ and $r = (tt, \{c\}) \Rightarrow tt \cdot \otimes \in A[\mathcal{F}_{\mathcal{D}}]$. Then, since $d \not\models c$, we have that $r|_d = d$, which coincides with $B^s[\mathcal{D} \cdot A]|_d$.

\[ A_2 = \sum_{i=1}^{n} \text{ask}(c_i) \Rightarrow B_i \text{ and } \forall 1 \leq i \leq n \not\models c_i \] By (3.7), $\text{stutt}((c_1, \ldots, c_n)) \cdot r' \in A[\mathcal{F}_{\mathcal{D}}]$ and, as a consequence, $r = (c, \{c_1, \ldots, c_n\}) \Rightarrow c \cdot r'$ belongs to $A[\mathcal{F}_{\mathcal{D}}]$. Now, we compute $r|_d$ and we get as result the trace $d \cdot r'|_d$. Since, by definition of the semantics evaluation function $A$, a stutt is always followed by another conditional tuple, then $r'$ is different from the empty trace. Therefore, $r|_d = d \cdot s$ for some trace $s$. As a consequence, the behavior of the agent $B^s[\mathcal{D} \cdot A]|_d = d \cdot d$ is included in the set of prefixes of $r|_d = d \cdot d \cdot s$.

$A_1 \parallel A_2$. We prove the two directions separately.

$\subseteq$ We distinguish five different cases. Let $r := r_1 \parallel r_2 \in A[A_1 \parallel A_2]_{\mathcal{F}_{\mathcal{D}}}$ such that $r_1 \in A[A_1]_{\mathcal{F}_{\mathcal{D}}}$ and $r_2 \in A[A_2]_{\mathcal{F}_{\mathcal{D}}}$.

$\parallel = r_1, r_2$ By Definition 3.9, $r_1$ is a generic conditional trace and $r_2 = \otimes$ (or $r_2 = c$). In other words, $r_2$ is associated to an agent that adds no information. We have:

\[
\text{prefix}((r_1 \parallel r_2)|_d) = \text{prefix}(r_1|_d)
\]

\[= \mathcal{A}[A_1]_{\mathcal{F}_{\mathcal{D}}} \downarrow d \]

\[= \mathcal{A}[A_2]_{\mathcal{F}_{\mathcal{D}}} \downarrow d \]

\[
[\text{by Inductive Hypothesis}]
\]

\[\subseteq B^s[\mathcal{D} \cdot A_1]|_d
\]

\[= B^s[\mathcal{D} \cdot A_1 \parallel A_2]|_d
\]

Since $A_2$ does not modify the store, we can conclude that the two behaviors $B^s[\mathcal{D} \cdot A_1]|_d$ and $B^s[\mathcal{D} \cdot A_1 \parallel A_2]|_d$ coincide.

$\parallel = \text{stutt}(\eta^- \cup \eta^-) \cdot r'$ In case $d \not\models h^-$ and $d \not\models h^-$, we have that $\text{prefix}(r|_d) = \text{prefix}(d) = \{\epsilon, d\} \in B^s[\mathcal{D} \cdot A_1]|_d$. Otherwise, $r|_d$ is not defined, thus, the set $\text{prefix}(r|_d)$ is empty and the inclusion directly holds.

$\parallel = (\eta \cdot \delta) \Rightarrow c_1 \otimes c_2 \cdot (r'_1 \otimes c_1 \parallel r'_2 \otimes c_1)$ By Definition 3.9, $r_1 = \eta \Rightarrow c_1 \cdot r'_1 \in A[A_1]_{\mathcal{F}_{\mathcal{D}}}$, $r_2 = \eta \Rightarrow c_2 \cdot r'_2 \in A[A_2]_{\mathcal{F}_{\mathcal{D}}}$ and $(c_1 \otimes c_2) \not\models \text{ff}$. Let us distinguish three sub-cases.

$\parallel \models (\eta \cdot \delta)$ and $d \not\models \text{ff}$ Due to the form of $r_1$ and $r_2$, we know that there exist two agents $A'_1$ and $A_2$ such that $\langle A_1, d \rangle \rightarrow \langle A'_1, d \otimes c_1 \rangle$ and $\langle A_2, d \rangle \rightarrow \langle A_2, d \rangle$.
\( \langle A'_2, d \otimes c_2 \rangle \), respectively. Then,

\[
\text{prefix}(r_d) = \text{prefix}(\{ d \cdot (r'_1 \downarrow c_2 \parallel r'_2 \downarrow c_1) \downarrow_{d \otimes c_1 \otimes c_2} \mid r'_1 \in A[A'_1]_{x[D]}, r'_2 \in A[A'_2]_{x[D]} \})
\]

[by Lemma A.3]

\[
= \text{prefix}(\{ d \cdot (r'_1 \downarrow_{c_1 \otimes c_2} \parallel r'_2 \downarrow_{c_1 \otimes c_2}) \downarrow_{d \otimes c_1 \otimes c_2} \mid r'_1 \in A[A'_1]_{x[D]} \text{ and } r'_2 \in A[A'_2]_{x[D]} \})
\]

[by Lemma A.6]

\[
= \text{prefix}(\{ d \cdot (r'_1 \parallel r'_2) \downarrow_{d \otimes c_1 \otimes c_2} \mid r'_1 \in A[A'_1]_{x[D]} \text{ and } r'_2 \in A[A'_2]_{x[D]} \})
\]

[by Lemma A.8]

\[
= \text{prefix}(\{ d \cdot (r'_1 \parallel r'_2) \downarrow_{d \otimes c_1 \otimes c_2} \mid r'_1 \in A[A'_1]_{x[D]} \text{ and } r'_2 \in A[A'_2]_{x[D]} \})
\]

[by Equation (3.1)]

\[
= \{ \epsilon, d \} \cup \{ \epsilon \cdot d \mid s'_1 \parallel s'_2 \mid s'_1 \in \text{prefix}(A[A'_1]_{x[D]} \downarrow_{d \otimes c_1 \otimes c_2}), s'_2 \in \text{prefix}(A[A'_2]_{x[D]} \downarrow_{d \otimes c_1 \otimes c_2}) \}
\]

[by Inductive Hypothesis]

\[
\subseteq \{ \epsilon, d \} \cup \{ \epsilon \cdot d \mid s \in B^s[D \cdot A'_1 \parallel A'_2]_{d \otimes c_1 \otimes c_2} \}
\]

[by Rule R7]

\[
\subseteq B^s[D \cdot A_1 \parallel A_2]_d
\]

\[
d = \text{ff}.
\]

We have that \( \langle A_1, \text{ff} \rangle \not\models \) and \( \langle A_2, \text{ff} \rangle \not\models \), thus \( B^s[D \cdot A_1 \parallel A_2]_\text{ff} = \{ \epsilon, \text{ff} \} \). Since \( d \models (\eta \otimes \delta) \), we have that \( \text{prefix}(r\mid_d) = \{ \epsilon, \text{ff} \} \), which corresponds to the small-step behavior \( B^s[D \cdot A_1 \parallel A_2]_{\text{ff}} \).

\[
\not\models (\eta \otimes \delta) \] In this case the set \( \text{prefix}(r\mid_d) \) is empty since \( \downarrow_d \) is not defined under these conditions, thus it is directly included in \( B^s[D \cdot A_1 \parallel A_2]_d \).

\[
r = (\eta \otimes \delta) \rightarrow \text{ff} \cdot \text{ff} \] By Definition 3.9 we have that \( r_1 = \eta \Rightarrow c_1 \cdot r'_1, r_2 = \delta \Rightarrow c_2 \cdot r'_2 \) and \( c_1 \otimes c_2 = \text{ff} \). We have to consider three cases:

\[
d \models (\eta \otimes \delta) \text{ and } d \neq \text{ff}
\]

\[
\text{prefix}(r\mid_d) = \text{prefix}(d \cdot c_1 \otimes c_2)
\]

\[
= \text{prefix}(d \cdot \text{ff})
\]

\[
= \{ \epsilon \cdot s \mid s \in B^s[D \cdot A'_1 \parallel A'_2]_{\text{ff}} \}
\]

[by Rule R7]

\[
\subseteq B^s[D \cdot A_1 \parallel A_2]_d
\]

In fact, also the second component of the behavior is the store \( \text{ff} \). This case represents the situation in which the contribution of the two conditional traces results in an inconsistent conditional trace.

\[
d = \text{ff} \] We have that \( \langle A_1, \text{ff} \rangle \not\models \) and \( \langle A_2, \text{ff} \rangle \not\models \), thus \( B^s[D \cdot A_1 \parallel A_2]_\text{ff} = \{ \epsilon, \text{ff} \} \). Since \( d \models (\eta \otimes \delta) \), we have that \( \text{prefix}(r\mid_d) = \{ \epsilon, \text{ff} \} \), which corresponds to the small-step behavior \( B^s[D \cdot A_1 \parallel A_2]_{\text{ff}} \).
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \texttt{tcpApp–15}

\[
d\#(\eta \otimes \delta), \text{In this case, } r_d \text{ is undefined, thus we have that } \varnothing \subseteq B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d.
\]

\[
r = (\eta^* \cup \delta^*) \Rightarrow c_1 \cdot (r_1 \parallel r_2) \text{. By Definition 3.9, } r_1 = \eta \Rightarrow c_1 \cdot r_1' \in \mathcal{A}[A_1]_{xF[D]} \\text{and } r_2 = \text{stutt}(\delta^*) \cdot r_2' \in \mathcal{A}[A_2]_{xF[D]} \text{ with } r_2' \text{ that recursively belongs to } \mathcal{A}[A_2]_{xF[D]} \text{ and for all } h \in \delta^*, \eta^* \cup h^*. \text{ Let us distinguish three sub-cases.}
\]

\[
d\#(\eta^* \cup \delta^*) \text{. Then,}
\]

\[
\text{prefix}(r_d) = \text{prefix}(\{d \cdot (r_1' \parallel r_2') \downarrow d \otimes c_1 \mid r_1' \in \mathcal{A}[A_1']_{xF[D]}, r_2' \in \mathcal{A}[A_2']_{xF[D]}\})
\]

\[
\text{[by Lemma A.6]}
\]

\[
= \text{prefix}(\{d \cdot (r_1' \parallel r_2') \downarrow d \otimes c_1 \mid r_1' \in \mathcal{A}[A_1']_{xF[D]}, r_2' \in \mathcal{A}[A_2']_{xF[D]}\})
\]

\[
\text{[by Lemma A.8]}
\]

\[
\text{[by Equation (3.1)]}
\]

\[
\subseteq \{\epsilon, d\} \cup \{d \cdot (s_1' \parallel s_2') \mid s_1' \in \text{prefix}(A[A_1']_{xF[D]} \downarrow d \otimes c_1), s_2' \in \text{prefix}(A[A_2']_{xF[D]} \downarrow d \otimes c_1)\}
\]

\[
\text{[by Inductive Hypothesis]}
\]

\[
\subseteq \{\epsilon, d\} \cup \{d \cdot s \mid s \in B^{ss}\lsbracket D . A_1' \parallel A_2\rsbracket_{d \otimes c_1}\}
\]

\[
\text{[by Rule R8]}
\]

\[
\subseteq B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d
\]

\[
d = ff \text{. We have that } (A_1, \text{ff}) \not\in \text{ and } (A_2, \text{ff}) \not\in \text{, thus } B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d = \{\epsilon, ff\}. \text{ Since } d \#(\eta \otimes \delta), \text{ we have that } \text{prefix}(r_d) = \{\epsilon, ff\}, \text{ which corresponds to the small-step behavior } B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d.
\]

\[
d\#(\eta^* \cup \delta^*) \text{. In this case, we have that } \text{prefix}(r_d) = \varnothing \subseteq B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d.
\]

2 In the following, we show that if \( s \in B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d \), then \( s \in \text{prefix}(A[A_1 \parallel A_2]_{xF[D]} \downarrow d) \), i.e., we can find a conditional trace \( r \in \mathcal{A}[A_1 \parallel A_2]_{xF[D]} \) such that \( s \in \text{prefix}(r_d) \). We have four possible cases, depending on the rules defining the operational semantics for the agent.

(1) If \( (A_1, d) \rightarrow (A_1', d_1') \) and \( (A_2, d) \rightarrow (A_2', d_2') \), the behavior of the parallel composition is \( B^{ss}\lsbracket D . A_1 \parallel A_2\rsbracket_d = \{d \cdot s' \mid s' \in B^{ss}\lsbracket D . A_1' \parallel A_2\rsbracket_{d_1 \otimes d_2}\} \). Let \( s \) be an element of that set. By inductive hypothesis, we know that there exist \( r_1 \in \mathcal{A}[A_1]_{xF[D]} \) and \( r_2 \in \mathcal{A}[A_2]_{xF[D]} \) such that \( d \cdot s_1' \in \text{prefix}(r_1_d) \) and \( d \cdot s_2' \in \text{prefix}(r_2_d) \), with \( s_1' \in B^{ss}\lsbracket D . A_1\rsbracket_d \) and \( s_2' \in B^{ss}\lsbracket D . A_2\rsbracket_d \). Now, consider \( r = r_1 \parallel r_2 \); this conditional trace belongs to \( \mathcal{A}[A_1 \parallel A_2]_{xF[D]} \) whenever \( r_1 \) and \( r_2 \) are compatible via parallel composition (i.e., \( r_1 \parallel r_2 \) is a valid conditional trace). We show that \( s \in \text{prefix}((r_1 \parallel r_2)_d) \).

\[
\text{prefix}((r_1 \parallel r_2)_d)
\]

\[
\text{[by Lemma A.8]}
\]
= \text{prefix}(r_1 \parallel_d r_2 \parallel_d)

[by Definition 3.24]

= \{\epsilon, d\} \cup \{(d \cdot s_1') \parallel (d \cdot s_2') \mid s_1' \in B^s[[D \cdot A_1']_{d_1'} \text{ and } s_2' \in B^s[[D \cdot A_2']_{d_2'}\}

[by Definition A.7]

= \{\epsilon, d\} \cup \{(d \cdot s_1' \parallel s_2') \mid s_1' \in B^s[[D \cdot A_1']_{d_1'} \text{ and } s_2' \in B^s[[D \cdot A_2']_{d_2'}\}

[by Definition A.7 and by Definition 3.1]

= \{\epsilon, d\} \cup \{d \cdot s' \mid s' \in B^s[[D \cdot A_1']_{d_1'} \parallel B^s[[D \cdot A_2']_{d_2'}\}

[by Rule R7 and Equation (A.1)]

= \{\epsilon, d\} \cup \{d \cdot s' \mid s' \in B^s[[D \cdot A_1' \parallel A_2']_{d_1'}\}

It follows directly that \( s \in \text{prefix}((r_1 \parallel r_2) \parallel_d). \)

(2) If \( \langle A_1, d \rangle \rightarrow \langle A_1', d_1' \rangle \) and \( \langle A_2, d \rangle \not\rightarrow \), then Rule R8 is applied and we have that \( B^s [[D \cdot A_1 \parallel A_2]_d = \{d \cdot s' \mid s' \in B^s[[D \cdot A_1']_{d_1'} \parallel A_2']_d\}. \) Let \( s \) be an element of that set. By inductive hypothesis, we know that it exists \( r_1 \in \mathcal{A}[\mathcal{A}[A_1]]_{\mathcal{F}[D]} \) such that \( d \cdot s_1' \in \text{prefix}(r_1 \parallel_d) \) with \( s_1' \in B^s[[D \cdot A_1']_{d_1'}]. \) Moreover, it exists \( r_2 \in \mathcal{A}[\mathcal{A}[A_2]]_{\mathcal{F}[D]} \) such that \( r_2 \parallel_d = d. \) We distinguish two cases (corresponding to the two agents that can make the agent \( A_2 \) not to proceed) in order to prove that \( s \in \text{prefix}((r_1 \parallel r_2) \parallel_d). \)

\( A_2 = \text{skip}, \) In this case, the behavior of the parallel composition is that of \( A_1 \) since \( A_2 \) makes no contribution to the computation. Then, \( (r_1 \parallel \Box) \parallel_d = d \cdot s' \) with \( s' \in B^s[[D \cdot A_1']_{d_1'} = B^s[[D \cdot A_1']_{d_1'} \parallel A_2']_d, \) thus \( s \in \text{prefix}((r_1 \parallel \Box) \parallel_d). \)

\( A_2 = \Sigma_{i=1}^n \text{ask}(c_i) \rightarrow B_i \) Consider \( r_2 = \text{stutt}((c_1, \ldots, c_n) \cdot r_2' \) with \( r_2' \in \mathcal{A}[\mathcal{A}[A_2]]_{\mathcal{F}[D]} \). We can assume that \( d \not\parallel c_i \) for all \( c_i, \) otherwise, the agent \( A_2 \) would proceed.

\[ \text{prefix}((r_1 \parallel \text{stutt}(c_1, \ldots, c_n) \cdot r_2') \parallel_d) = \]

\[ = \text{prefix}((d \cdot (r_1' \parallel r_2') \parallel_d)_{d_1'} \mid r_1' \in \mathcal{A}[\mathcal{A}[A_1]]_{\mathcal{F}[D]} \text{ and } r_2' \in \mathcal{A}[\mathcal{A}[A_2]]_{\mathcal{F}[D]}\}) \]

[by Lemma A.8]

\[ = \text{prefix}((d \cdot (r_1' \parallel r_2') \parallel_d)_{d_1'} \mid r_1' \in \mathcal{A}[\mathcal{A}[A_1]]_{\mathcal{F}[D]} \text{ and } r_2' \in \mathcal{A}[\mathcal{A}[A_2]]_{\mathcal{F}[D]}\}) \]

[by Equation (3.1)]

\[ = \{\epsilon, d\} \cup \{d \cdot (s_1' \parallel s_2') \mid s_1' \in \text{prefix}(\mathcal{A}[\mathcal{A}[A_1]]_{\mathcal{F}[D]} \parallel_d) \text{ and } s_2' \in \text{prefix}(\mathcal{A}[\mathcal{A}[A_2]]_{\mathcal{F}[D]} \parallel_d)\}

[by Inductive Hypothesis]

\[ = \{\epsilon, d\} \cup \{d \cdot (s_1' \parallel s_2') \mid s_1' \in B^s[[D \cdot A_1']_{d_1'} \text{ and } s_2' \in B^s[[D \cdot A_2']_{d_2'}\}

[by Definition A.7 and by Definition 3.1]

\[ = \{\epsilon, d\} \cup \{d \cdot (s_1' \parallel s_2') \mid s_1' \in B^s[[D \cdot A_1']_{d_1'} \parallel B^s[[D \cdot A_2']_{d_2'}\}

[by Rule R7, Rule R8 and Equation (A.1)]

\[ = \{\epsilon, d\} \cup \{d \cdot (s_1' \parallel s_2') \mid s_1' \in B^s[[D \cdot A_1' \parallel A_2']_{d_1'}\}

It follows directly that \( s \in \text{prefix}((r_1 \parallel r_2) \parallel_d). \)
(3) If \( \langle A_1, d \rangle \not \vdash \langle A_2, d \rangle \rightarrow \langle A'_1, d'_2 \rangle \), then the situation is symmetric to the previous case, so we can conclude that \( B^{ss}[D \, . \, A_1 \parallel A_2][d] \subseteq \text{prefix}(A[A_1 || A_2][x[D] \parallel d]) \).

(4) Finally, if \( \langle A_1, d \rangle \not \vdash \langle A_2, d \rangle \not \vdash \), then we can reason similarly to Point 2, considering, for both \( A_1 \) and \( A_2 \), the two cases in which they cannot proceed.

We can conclude that \( B^{ss}[D \, . \, A_1 || A_2][x[D] \parallel d] \subseteq \{ \epsilon, d \} \subseteq \text{prefix}(A[A_1 || A_2][x[D] \parallel d]) \).

\( \exists x \, A_1 \) We prove the two directions independently.

\( \subseteq \) We show that: \( \text{prefix}(A[\exists x \, A_1][x[D] \parallel d]) \subseteq B^{ss}[D \, . \, \exists x \, A_1][d] \). Let \( r = \exists x \, r_1 \) such that \( r_1 \in A[A_1][x[D]] \) and \( r_1 \) is \( x \)-self-sufficient. We show that the prefixes of \( (\exists x \, r_1) \parallel d \) are included in the behavior \( B^{ss}[D \, . \, \exists x \, A_1][d] \) by structural induction on \( r_1 \):

\[
\begin{align*}
&\text{Let } r_1 = \epsilon, \text{ the statement directly holds.} \\
&{r_1 = \emptyset}, \text{ then } \emptyset \parallel d = d, \text{ which belongs to } B^{ss}[D \, . \, \exists x \, A_1][d]. \\
&{r_1 = \eta \Rightarrow l \cdot r'_1}, \text{ by Definition 3.15, we have that } r'_1 \in A[A'_1][x[D]] \text{ and, by inductive hypothesis, there exists a transition } (A_1, d) \rightarrow (A'_1, d'). \\
&\text{Since } r_1 \text{ is } x \text{-self-sufficient, also } r'_1 \text{ is } x \text{-self-sufficient. Now, we have three cases.}
\end{align*}
\]

\( d \models \exists x \, \eta \) and \( d \models \mathbf{ff} \)

\[
\begin{align*}
&\text{prefix}(r_1 \parallel d) \\
&= \text{prefix}(\{ \exists x \, (\eta \Rightarrow l \cdot r'_1) \parallel d | r'_1 \in A[A'_1][x[D]] \text{ and } r'_1 \text{-self-sufficient} \}) \\
&\quad \text{by Definition 3.24} \\
&= \text{prefix}(\{ d \cdot (\exists x \, r'_1) \parallel d \& \exists x \, i | r'_1 \in A[A'_1][x[D]] \text{ and } r'_1 \text{-self-sufficient} \}) \\
&\quad \text{by Equation (3.1)} \\
&= \{ \epsilon, d \} \cup \{ d \cdot s | s \in \text{prefix}(A[\exists x \, A_1][x[D] \parallel d \& \exists x \, i]) \} \\
&\quad \text{by Inductive Hypothesis} \\
&\subseteq \{ \epsilon, d \} \cup \{ d \cdot s | s \in B^{ss}[D \, . \, \exists x \, A_1][d \& \exists x \, i] \} \\
&\quad \text{by Rule R9} \\
&\subseteq B^{ss}[D \, . \, \exists x \, A_1][d] \\
\end{align*}
\]

\( d \models \mathbf{ff} \). We have that \( \langle \exists x \, A_1, \mathbf{ff} \rangle \not \vdash \), thus \( B^{ss}[D \, . \, \exists x \, A_1][\mathbf{ff}] = \{ \epsilon, \mathbf{ff} \} \). On the other hand, since \( d \models \exists x \, \eta \), we have that \( \text{prefix}(r_1 \parallel d) = \{ \epsilon, \mathbf{ff} \} \) which corresponds to the small-step behavior \( B^{ss}[D \, . \, \exists x \, A_1][\mathbf{ff}] \).

\( d \models \exists x \, \eta \) Then, the operator \( \parallel d \) is undefined for the conditional trace, thus \( \text{prefix}(r_1 \parallel d) = \emptyset \subseteq B^{ss}[D \, . \, \exists x \, A_1][d] \).

\( r_1 = \text{stut}((c_1, \ldots, c_n)) \cdot r'_1 \) By Definition 3.15, \( r'_1 \in A[\sum_{i=1}^{n} \text{ask}(n_i) \rightarrow B_i][x[D]] \). If it exists no index \( 1 \leq j \leq n \) such that \( d \vdash c_j \), then this implies that \( d \vdash \exists x \, c_j \). In such case, we have

\[
\begin{align*}
&\text{prefix}(r_1 \parallel d) = \text{prefix}(\exists x \, (\text{stut}((c_1, \ldots, c_n)) \cdot r'_1) \parallel d) \\
&= \text{prefix}(\text{stut}((\exists x \, c_1, \ldots, \exists x \, c_n)) \cdot \exists x \, r'_1 \parallel d) \\
&= \text{prefix}(\exists x \, r'_1 \parallel d) \\
&\quad \text{by Definition 3.24} \\
&\subseteq B^{ss}[D \, . \, \exists x \, A_1][d] \\
\end{align*}
\]
Otherwise, if it exists an index $j$ such that $d \vdash c_j$, then $r_j^d$ is undefined, thus $\text{prefix}(r_j^d) = \emptyset \subseteq B^x[\emptyset]$.  

2 From Rule R9, we know that, if $d \neq ff$, then $B^x[D \cdot A_1]_{\emptyset \emptyset \emptyset}^d = l' \cdot B^x[D \cdot A_1]_{\emptyset \emptyset}^d$, where $l$ and $l'$ are local stores. Moreover, $l = tt$ because it is the initial (local) store for $A_1$. In the following, we show that $d \cdot B^x[D \cdot \exists x A_1]_{\emptyset \emptyset \emptyset}^d \in A[\exists x A_1]_{\emptyset \emptyset \emptyset}^d$, i.e., it exists a trace $r \in A[\exists x A_1]_{\emptyset \emptyset \emptyset}^d$ such that $r_j^d = d \cdot s$ with $s \in B^x[D \cdot \exists x A_1]_{\emptyset \emptyset \emptyset}^d$. By inductive hypothesis, $B^x[D \cdot A_1]_{\emptyset \emptyset}^d \subseteq \text{prefix}(A[A_1]_{\emptyset \emptyset \emptyset}^d)_{\emptyset \emptyset \emptyset}^d$, and by Rule R9, it holds that there exists $r_1 \in A[A_1]_{\emptyset \emptyset \emptyset}^d$ such that $r_1 \downarrow_{\emptyset \emptyset \emptyset}^d = 3_\emptyset^d \cdot B^x[D \cdot \exists x A_1]_{\emptyset \empty \emptyset}^d$.

Now, $r_1$ is $\emptyset$-self-sufficient since the only external information is provided by $3_\emptyset$, which in fact does not contain information about $x$. Moreover, $r_1$ is of the form $\eta \mapsto l' \cdot r_1'$ with $r_1' \in A[A_1]_{\emptyset \empty \empty \empty \empty \empty \empty \empty}^d$. Therefore, it exists $r \in A[\exists x A_1]_{\emptyset \empty \empty \empty \empty \empty \empty \empty}^d$ such that $r = 3_x^d r_1$. Then,

\[
\begin{align*}
  r_j^d &= (3_x \eta \mapsto 3_x l' \cdot 3_x r_1')_j^d \\
  &= (3_x \eta \mapsto 3_x l' \cdot 3_x r_1')_j^d \\
  &= d \cdot (3_x r_1')_j^d \downarrow_{\emptyset \empty \empty \empty \empty \empty \empty}^d \\
  &= d \cdot s \quad \text{with} \ s \in A[\exists x A_1]_{\emptyset \empty \empty \empty \empty \empty \empty \empty}^d \\
  &= d \cdot s \quad \text{with} \ s \in B^x[D \cdot \exists x A_1]_{\emptyset \empty \empty \empty \empty \empty \empty \empty}^d
\end{align*}
\]

If $d = ff$, then we have that $\text{prefix}(r_j^d) = \{1, ff\}$, which corresponds to the small-step behavior $B^x[D \cdot \exists x A_1]_{\emptyset \empty \empty \empty \empty \empty \empty \empty}^d$ since the transition relation $\rightarrow$ is not defined for $(\exists x A_1, ff)$.

$P(\bar{x})$. We have to distinguish two sub-cases.

\[
\begin{align*}
  d \neq ff \\
  \text{prefix}(A[p(\bar{x})]_{\emptyset \empty \empty \empty}^d) &= \text{prefix}((tt, \emptyset) \mapsto tt \cdot r' \mid r' \in D[p(\bar{x})])_j^d) \\
  &= \text{prefix}((tt, \emptyset) \mapsto tt \cdot r' \mid r' \in D[p(\bar{x})])_j^d) \\
  &= \text{prefix}((tt, \emptyset) \mapsto tt \cdot r' \mid r' \in D[p(\bar{x})])_j^d) \\
  &= \text{prefix}((tt, \emptyset) \mapsto tt \cdot r' \mid r' \in A[B]_{\emptyset \empty \empty \empty}^d, p(\bar{x}) \vdash B \in D)_j^d) \\
  &= \text{prefix}((d \cdot s' \mid s' \in A[B]_{\emptyset \empty \empty \empty}^d, p(\bar{x}) \vdash B \in D)_j^d) \\
  &= \text{prefix}((d \cdot s' \mid s' \in B^x[D \cdot B]_{\emptyset \empty \empty \empty}^d, p(\bar{x}) \vdash B \in D)_j^d) \\
  &= \text{prefix}((d \cdot s' \mid s' \in B^x[D \cdot B]_{\emptyset \empty \empty \empty}^d, p(\bar{x}) \vdash B \in D)_j^d) \\
  &= B^x[D \cdot p(\bar{x})]_d
\end{align*}
\]

Notice that, in the second last equality, the structural induction hypothesis cannot be applied because $B$ can be structurally greater than $p(\bar{x})$. For this reason, we have to introduce a second induction on the number of $p(\bar{x})$ present on $B$. If $B$ does not contain any process call $p(\bar{x})$, then we can directly apply structural induction. Otherwise, if the agent contains one process call $p(\bar{x})$, it is sufficient to replace the call with the body of the declaration. In this way, $B$ has less process calls $p(\bar{x})$ than $A$ and we can apply the inductive hypothesis.
\[ d = ff \]

In this case, the transition relation \( \rightarrow \) is not defined for the configuration \( (p(\bar{x}), ff) \), hence

\[
\text{prefix}(A[p(\bar{x})]_x[D]_y) = \text{prefix}(((tt, \varnothing) \Rightarrow tt \cdot r')_y | r' \in \mathcal{F}[D](p(\bar{x})))
\]

\[
= \{ \epsilon, ff \}
\]

\[
= \mathcal{B}^*\mathcal{D}. p(\bar{x})_y
\]

\[ \square \]

**PROOF.** [of Theorem 3.26] By Theorem 3.25 it follows that for each program \( P \) and each \( c \in C \), \( \text{prefix}(P[P]_c) = \mathcal{B}^*\mathcal{P}[P]_c \). Thus, we show that \( \mathcal{P}[P_1] = \mathcal{P}[P_2] \iff \forall c \in C. \text{prefix}(\mathcal{P}[P_1]_c) = \text{prefix}(\mathcal{P}[P_2]_c) \).  

\( \Rightarrow \) Follows directly from Definition 3.24 and by definition of \( \text{prefix} \).

\( \Leftarrow \) To prove this implication we first need to show that \( \mathcal{P}[P_1] \neq \mathcal{P}[P_2] \Rightarrow \exists \bar{\epsilon} \in C. \mathcal{P}[P_1]_c \neq \mathcal{P}[P_2]_c \). Without loss of generality, assume that \( \mathcal{P}[P_1] > \mathcal{P}[P_2] \), thus, it exists \( r_1 \in \mathcal{P}[P_1] \) such that \( r_1 \notin \mathcal{P}[P_2] \). We can distinguish two cases: \( \mathcal{P}[P_2] \) is empty or \( \mathcal{P}[P_2] \) contains at least one conditional trace.

If \( \mathcal{P}[P_2] = \varnothing \), then \( \mathcal{P}[P_2]_c \) is empty for any possible \( c \in C \). Now, if we choose \( \bar{\epsilon} \) to be the \( \text{lub}(\varnothing) \) of all the positive conditions occurring in \( r_1 \), then \( r_1 \varnothing \) is a valid trace. Therefore, \( \mathcal{P}[P_1]_c \geq \{ r_1 \varnothing \} \neq \varnothing \).

If \( \mathcal{P}[P_2] \neq \varnothing \), by the initial assumptions, it exists a conditional trace \( r_2 \in \mathcal{P}[P_2] \) such that \( r_1 \neq r_2 \). Without loss of generality, assume that \( \text{length}(r_1) \leq \text{length}(r_2) \) and that \( r_1 \) differs from \( r_2 \) at position \( k \), with \( k \in [1, \text{length}(r_1)] \). The index \( k \) is guaranteed to exist.\(^7\) We consider the six possible cases, corresponding to the possible forms of the conditional state at position \( k \), in order to prove that there exists a store \( \bar{\epsilon} \) such that \( \mathcal{P}[P_1]_c \neq \mathcal{P}[P_2]_c \). In the following, the stores \( \bar{\epsilon}_1 \) and \( \bar{\epsilon}_2 \) correspond to the \( \text{lub}(\varnothing) \) of all the positive conditions occurring in \( r_1 \) and \( r_2 \), respectively.

(1) Let be \( (\eta_1^1, \eta_1^2) \Rightarrow d_1 \) and \( (\eta_2^1, \eta_2^2) \Rightarrow d_2 \) the \( k \)-th conditional tuple in \( r_1 \) and \( r_2 \), respectively. There are three possible ways in which these two tuples can differ:

\[ \eta_1^1 \neq \eta_2^1, \eta_1^2 \neq \eta_2^2, \eta_1^1 \neq \eta_2^1 \neq \eta_2^2. \]  

Let us assume that \( \eta_1^1 \rightarrow \eta_2^1 \) and \( \eta_2^2 \neq \eta_1^2 \). Notice that \( r_1 \) has to come from the semantics of an \texttt{ask} or a \texttt{now} construct since they are the only \texttt{tccp} agents that can add information to the positive condition (see Definition 3.15). Hence, there exists also a conditional trace \( \bar{r}_1 \in \mathcal{P}[P_1] \) in which \( \eta_1^1 \) occurs in a negative condition (corresponding to the else branch of a \texttt{now} agent) or in a \texttt{stutt} construct (corresponding to the suspension of an \texttt{ask} agent) of the sequence. There are two cases in which \( \bar{r}_1 \) does not exists, but both are in contradiction with the hypothesis: (1) when \( \eta_1^2 \neq tt \), but this contradicts \( \eta_1^2 \rightarrow \eta_1^1 \) or (2) when a constraint \( d \) stronger than \( \eta_1^1 \) (\( d \neq \eta_1^1 \)) is propagated. In this last case, the trace \( \bar{r}_1 \) does not exists since the condition is in contradiction with the propagated store. However, since \( \eta_1^1 \rightarrow \eta_2^2 \), it follows that \( d \) entails also \( \eta_2^2 \) (\( d \otimes \eta_1^2 = d \otimes \eta_2^2 = d \)). Therefore, the propagation of \( d \) makes \( r_1 \) and \( r_2 \) equal. Since they were supposed to be different only at this point, this is a contradiction with the hypothesis \( r_1 \neq r_2 \). Therefore, \( \bar{r}_1 \) exists and belongs to \( \mathcal{P}[P_1] \). Furthermore, \( \bar{r}_1 \) differs from any trace in \( \mathcal{P}[P_2] \) for at least the negative part of a condition or the body of a \texttt{stutt}, otherwise, reasoning in a similar way as above, \( r_1 \) would also belong to \( \mathcal{P}[P_2] \), and this is not possible.

\(^7\)There are two cases in which \( k \) does not exist, but both are in contradiction with the initial hypothesis: (1) \( r_1 = r_2 \) or (2) one of the traces is a prefix of the other.
If \( \eta_1^j \neq \eta_2^j \) and \( \eta_1^j \vdash \eta_2^j \), we can reason in a symmetric way, thus concluding that it exists \( r_2 \in \mathcal{P}[P_2] \) that differs from any trace in \( \mathcal{P}[P_1] \) for at least the negative part of a condition or the body of a \textit{stutt}.

Finally, if \( \eta_1^j \neq \eta_2^j \) and \( \eta_2^j \neq \eta_1^j \), we can reason as before and deduce that there exist two traces \( \tilde{r}_1 \in \mathcal{P}[P_1] \) and \( r_2 \in \mathcal{P}[P_2] \), which contains respectively \( \eta_1^j \) and \( \eta_2^j \) in the negative part of the condition, and such that \( \tilde{r}_1 \notin \mathcal{P}[P_2] \) and \( r_2 \notin \mathcal{P}[P_1] \).

In case \( \tilde{r}_1 \) (respectively \( \tilde{r}_2 \)) comes from an \textit{ask} agent we remand to the following Points 2, 3 and 4 of the proof, where we deal with the conditional traces containing \textit{stutt} constructs. Otherwise, if \( r_1 \) comes from a \textit{now} agent we can reduce to the following case where we deal with the negative part of the conditions (\( \eta_1^j \neq \eta_2^j \)).

\( \eta_1^j \neq \eta_2^j \) Let us first assume that \( \eta_1^j \in \eta_2^j \). This means that the store at position \( k \) in \( r_2 \) has to satisfy a stronger condition than the one in \( r_1 \). Let \( \tilde{c} := c_1 \otimes h_2 \), with \( h_2 \in \eta_2^j \setminus \eta_1^j \). Under these conditions, \( r_1 \| \tilde{c} \) computes a behavioral timed trace whereas \( r_2 \| \tilde{c} \) computes no trace since, at position \( k \), \( \tilde{c} \) entails one of the stores in the negative condition.

For the case in which \( \eta_1^j \in \eta_2^j \) we choose \( c = c_2 \otimes h_1 \), with \( h_1 \in \eta_1^j \setminus \eta_2^j \) and reason in an symmetric way.

Finally, if \( \eta_1^j \notin \eta_2^j \) and \( \eta_2^j \notin \eta_1^j \), we can choose indifferently \( \tilde{c} = c_1 \otimes h_2 \) or \( \tilde{c} = c_2 \otimes h_1 \) and conclude that \( r_1 \| \tilde{c} \) computes a behavioral timed trace but \( r_2 \| \tilde{c} \) is not defined, or vice-versa.

Thus, we can conclude that \( \mathcal{P}[P_1] \| \tilde{c} \neq \mathcal{P}[P_2] \| \tilde{c} \).

\( d_1 \neq d_2 \) Consider \( \tilde{c} = c_1 = c_2 \). There are two possible cases. Assume first that \( \tilde{c} \vdash d_1 \) and \( \tilde{c} \vdash d_2 \). Both \( r_1 \) and \( r_2 \) must be compatible with their own conditions, thus, being the store monotonic, it happens that \( r_1 \| \tilde{c} \) and \( r_2 \| \tilde{c} \) are both defined. Moreover, we know that \( \eta_1^j = \eta_2^j \) and from Property A.1 \( d_1 \vdash \eta_1^j \) and \( d_2 \vdash \eta_1^j \). Since \( \tilde{c} \vdash d_1 \) and \( \tilde{c} \vdash d_2 \), we can conclude that in \( r_1 \| \tilde{c} \) at position \( k \) we have the store \( d_1 \), whereas in \( r_2 \| \tilde{c} \) at the same position we find the store \( d_2 \) that is different from \( d_1 \) by the initial assumptions. Thus \( r_1 \| \tilde{c} \neq r_2 \| \tilde{c} \).

Assume now that \( \tilde{c} \) contains more information than the store \( d_1 \) (respectively \( d_2 \)). Then, we know that, at certain point in \( r_1 \) (respectively \( r_2 \)), the positive condition is stronger than \( d_1 \) (respectively \( d_2 \)). Therefore, we can reason as in the previous case when \( \eta_1^j \neq \eta_2^j \) and \( r_1 \) (respectively \( r_2 \)) are produced by the semantics of an \textit{ask} or a \textit{now} agent.

(2) Let \textit{stutt}(\( \eta_1^j \)) (respectively \textit{stutt}(\( \eta_2^j \))) be the \( k \)-th conditional state in \( r_1 \) (respectively \( r_2 \)). It is sufficient to proceed as in Point 1 of this proof (case \( \eta_1^j \neq \eta_2^j \)) to show that there exists a store \( \tilde{c} \) such that \( r_1 \| \tilde{c} \) is well defined while \( r_2 \| \tilde{c} \) is not. For instance, if \( \eta_1^j \subset \eta_2^j \) we set \( \tilde{c} = c_1 \otimes h_2 \), with \( h_2 \in \eta_2^j \setminus \eta_1^j \). It is easy to notice that \( r_1 \| \tilde{c} \) computes a behavioral timed trace but \( r_2 \| \tilde{c} \) recovers no trace since at position \( k \) the constraint \( h_2 \) belongs to the negative part of the condition. Therefore, \( \mathcal{P}[P_1] \| \tilde{c} \neq \mathcal{P}[P_2] \| \tilde{c} \).

(3) Let \( \eta_1 \rightarrow d_1 \) be the \( k \)-th conditional tuple in \( r_1 \) and \textit{stutt}(\( \eta_2^j \)) the \( k \)-th element in \( r_2 \). Consider \( \tilde{c} = c_1 \). Up to instant \( k \), \( r_1 \| \tilde{c} \) and \( r_2 \| \tilde{c} \) coincide and, as \( r_1 \) and \( r_2 \) differ only at position \( k \), \( \tilde{c} \) satisfies all the conditions in \( r_1 \) and in \( r_2 \) till up that position. The behavioral timed trace \( r_2 \| \tilde{c} \) ends at position \( k \) since a \textit{stutt} has been encountered (see Definition 3.24). However, since \( r_1 \) is maximal, \( r_1 \| \tilde{c} \) does not end at position \( k \) but continues with at least another state, otherwise we would have found an ending symbol \( \Box \). In conclusion, \( r_2 \| \tilde{c} \) is at least one store longer than \( r_1 \| \tilde{c} \), thus, \( \mathcal{P}[P_1] \| \tilde{c} \neq \mathcal{P}[P_2] \| \tilde{c} \).
A Condensed Goal-Independent Bottom-Up Fixpoint Semantics Modeling the Behavior of \textit{tccpApp}–21

(4) If \( \eta_2 \mapsto d \) is the \( k \)-th element in \( r_2 \) and \( \text{stutt}(\eta_1^-) \) that in \( r_1 \), then the proof is symmetric to previous Point 3.

(5) Let \( \varepsilon \) and \( \eta_2 \mapsto d \) be the \( k \)-th states of \( r_1 \) and \( r_2 \), respectively. We can reason similarly to Point 3 in this proof, by choosing \( \hat{c} = \hat{c}_2 \). By hypothesis, \( r_1 \) and \( r_2 \) differ only at position \( k \), thus, \( r_1 \downarrow_{\varepsilon} \) and \( r_2 \downarrow_{\varepsilon} \) compute the same behavioral timed trace up to position \( k \)-th. However, while \( r_1 \downarrow_{\varepsilon} \) stops at instant \( k \) (an ending symbol \( \varepsilon \) is found), \( r_2 \downarrow_{\varepsilon} \) is at least one store longer. Thus, \( \mathcal{P}[P_1] \downarrow_{\varepsilon} \neq \mathcal{P}[P_2] \downarrow_{\varepsilon} \).

(6) Let \( \varepsilon \) be the \( k \)-th element of \( r_1 \) and \( \text{stutt}(\eta_2) \) the conditional state occurring in \( r_2 \) at the same position. We set \( \hat{c} = c_1 \otimes h_2^2 \), with \( h_2^2 \in \eta_2^2 \setminus \eta_1^2 \). In this way, \( r_1 \downarrow_{\varepsilon} \) is defined but \( r_2 \downarrow_{\varepsilon} \) computes no trace since, at position \( k \), the constraint \( h_2^2 \) is required not to be entailed by the current store. Thus, \( \mathcal{P}[P_1] \downarrow_{\varepsilon} \neq \mathcal{P}[P_2] \downarrow_{\varepsilon} \).

In conclusion, we can always choose an adequate \( \hat{c} \) which differentiates \( \mathcal{P}[P_1] \downarrow_{\varepsilon} \) from \( \mathcal{P}[P_2] \downarrow_{\varepsilon} \). From Definition 3.15 and Definition 3.19, it can be noticed that the traces contained in \( \mathcal{P}[P_1] \) and \( \mathcal{P}[P_2] \) either end in \( \varepsilon \) or are infinite. From this observation, it follows directly that, if \( \mathcal{P}[P_1] \downarrow_{\varepsilon} \neq \mathcal{P}[P_2] \downarrow_{\varepsilon} \), then \( \text{prefix}(\mathcal{P}[P_1] \downarrow_{\varepsilon}) \neq \text{prefix}(\mathcal{P}[P_2] \downarrow_{\varepsilon}) \). Otherwise, there would exist a trace in \( \mathcal{P}[P_1] \) that is prefix of a trace in \( \mathcal{P}[P_2] \) (or viceversa), which is not possible since \( \varepsilon \) is a termination symbol and an infinite trace cannot prefix another infinite trace. Thus, we can conclude that if \( \mathcal{P}[P_1] \downarrow_{\varepsilon} \neq \mathcal{P}[P_2] \downarrow_{\varepsilon} \), then there exists \( \hat{c} \in \mathcal{C} \) such that \( \text{prefix}(\mathcal{P}[P_1] \downarrow_{\varepsilon}) \neq \text{prefix}(\mathcal{P}[P_2] \downarrow_{\varepsilon}) \), and this concludes the proof.

\[\square\]

**Proof.** [of Proposition 3.27]

\[\Rightarrow\] Straightforward.

\[\Leftarrow\] By Definition 3.19, \( \mathcal{P}[D_1 \cdot A] = A[D_1] \) and \( \mathcal{P}[D_2 \cdot A] = A[D_2] \). We have to check that \( \mathcal{F}[D_1] = \mathcal{F}[D_2] \). The only case depending on the interpretation is when \( A = p(\hat{x}) \). By hypothesis,

\[A[p(\hat{x})]D_1 = ((\text{true}, \varepsilon)) \mapsto \text{true} \cdot r | r \in \mathcal{F}[D_1](p(\hat{x})) \]

\[= \{(\text{true}, \varepsilon)) \mapsto \text{true} \cdot r | r \in \mathcal{F}[D_2](p(\hat{x})) \} = A[p(\hat{x})]D_2\]

We have to check that \( \mathcal{F}[D_1](p(\hat{x})) \) and \( \mathcal{F}[D_2](p(\hat{x})) \) coincide for each \( p(\hat{x}) \in \mathcal{P} \). Since \( \mathcal{F}[D_1] \) (respectively \( \mathcal{F}[D_2] \)) is the least fixpoint of \( D_1 \) (respectively \( D_2 \)), we know that it contains only information regarding the procedure calls in \( D_1 \) (respectively \( D_2 \)). So we can conclude that \( \mathcal{F}[D_1] = \mathcal{F}[D_2] \).

\[\square\]

**Proof.** [of Corollary 3.28] Consider \( D_1, D_2 \in \mathbb{P}_C \):

\( D_1 \approx_{\mathcal{F}} D_2 \iff \mathcal{F}[D_1] = \mathcal{F}[D_2] \)

[by Proposition 3.27]

\( \iff \forall A \in \mathbb{A}_C, \mathcal{P}[D_1 \cdot A] = \mathcal{P}[D_2 \cdot A] \)

[by Theorem 3.26]

\( \iff \forall A \in \mathbb{A}_C, \forall c \in \mathcal{C}, \text{prefix}(\mathcal{P}[D_1 \cdot A] \downarrow_{\varepsilon}) = \text{prefix}(\mathcal{P}[D_2 \cdot A] \downarrow_{\varepsilon}) \)

[by Theorem 3.25]

\( \iff \forall A \in \mathbb{A}_C, \forall c \in \mathcal{C}, B^{ss}[D_1 \cdot A]_c = B^{ss}[D_2 \cdot A]_c \)

\( \iff D_1 \approx_{ss} D_2 \)

\[\square\]
Proofs of Section 4

**Lemma A.9.** \( (\mathbb{M}, \subseteq, \sqcup, \sqcap, \emptyset, \{\epsilon\}) \xrightarrow{\gamma_{io}} (\mathbb{I}, \subseteq, \sqcup, \sqcap, \emptyset, \emptyset) \)

**Proof.**

\( \alpha_{io} \text{ is monotonic} \)

Let \( R_1, R_2 \in \mathbb{M} \) such that \( R_1 \subseteq R_2 \), thus \( \alpha_{io}(R_1) \subseteq \alpha_{io}(R_2) \). Otherwise, if there exists an input-output pair belonging to \( \alpha_{io}(R_1) \) but not to \( \alpha_{io}(R_2) \), this means that the associated trace belongs to \( R_1 \) but not to \( R_2 \), and this contradicts the hypothesis.

\( \gamma_{io} \text{ is monotonic} \)

Let \( P_1, P_2 \in \mathbb{I} \) such that \( P_1 \subseteq P_2 \). Suppose that \( \gamma_{io}(P_1) \not\subseteq \gamma_{io}(P_2) \), in this case, there exists \( r_1 \in \gamma_{io}(P_1) \) but not \( r_2 \in \gamma_{io}(P_2) \) that extends \( r_1 \) (\( r_1 \) is a prefix of \( r_2 \)). It is easy to see that this situation is impossible since, by the definition of \( \gamma_{io} \), \( r_1 \) has to belong also to \( \gamma_{io}(P_2) \) (since \( P_1 \subseteq P_2 \)) and \( r_1 \) trivially extends itself.

\( (\gamma_{io} \circ \alpha_{io}) \text{ is extensive} \)

This means that for all \( R \in \mathbb{M}, R \subseteq \gamma_{io}(\alpha_{io}(R)) \). We show that \( r \in R \Rightarrow r \in \gamma_{io}(\alpha_{io}(R)) \); we distinguish three cases:

- \( r = \eta_1 \rightarrow c_1 \cdots \eta_n \rightarrow c_n \). We have that \( \alpha_{io}(R) \supseteq \{(c_0, \text{fin}(c)) \mid c_0 \in C \land \text{last}(r_{\|c_0}) = c\} \). Thus, by (4.2), it follows that \( r \in \gamma_{io}(\alpha_{io}(r)) \).
- \( r = \eta_1 \rightarrow c_1 \cdots \text{stutt}((\eta_n) \cdots) \). We have that \( \alpha_{io}(R) \supseteq \{(c_0, \text{fin}(c)) \mid c_0 \in C \land \text{last}(r_{\|c_0}) = c\} \). From (4.2), it follows that \( r \in \gamma_{io}(\alpha_{io}(r)) \).
- \( r = \eta_1 \rightarrow c_1 \cdots \eta_n \rightarrow c_n \cdots \) (an infinite sequence that does not contain any \( \text{stutt} \)). We have that \( \alpha_{io}(R) \supseteq \{(c_0, \text{fin}(c)) \mid c_0 \in C, r_{\|c_0} = c_0 \cdots c_i \cdots, \text{and } \otimes_{i \geq 0} c_i = c\} \). By (4.2), we have that \( r \in \gamma_{io}(\alpha_{io}(r)) \).

\( (\alpha_{io} \circ \gamma_{io}) \text{ is the identity for } \mathbb{I} \)

This means that for all \( P \in \mathbb{I}, P = \alpha_{io}(\gamma_{io}(P)) \). We show the two inclusions separately.

\( \subseteq \) We first show that \( p \in P \Rightarrow p \in \alpha_{io}(\gamma_{io}(P)) \) by distinguishing two sub-cases.

- \( p = (c_0, \text{fin}(c_n)) \). In this case, \( \gamma_{io}(P) \) contains all the conditional traces \( r \) such that \( \text{last}(r_{\|c_0}) = c_n \). By (4.1), \( p \in \alpha_{io}(\gamma_{io}(P)) \).
- \( p = (c_0, \text{inf}(c)) \). We have that \( \gamma_{io}(P) \) contains all the conditional state sequences \( r \) such that \( r_{\|c_0} = c_0 \cdots c_i \cdots \) and \( \otimes_{i \geq 0} c_i = c \). By (4.1), \( p \in \alpha_{io}(\gamma_{io}(P)) \).

\( \supseteq \) Now we show the other inclusion i.e., \( p \in \alpha_{io}(\gamma_{io}(P)) \Rightarrow p \in P \). We have to consider two sub-cases.

- \( p = (c_0, \text{fin}(c_n)) \). In this case, it exists \( r \in \gamma_{io}(P) \) such that \( \text{last}(r_{\|c_0}) = c_n \). Obviously, \( p \in P \), otherwise \( r \) would not belong to \( \gamma_{io}(P) \).
- \( p = (c_0, \text{inf}(c)) \). In this case, it exists \( r \in \gamma_{io}(P) \) such that \( r_{\|c_0} = c_0 \cdots c_i \cdots \) and \( \otimes_{i \geq 0} c_i = c \). It is easy to notice that \( p \in P \), otherwise, by using \( \gamma_{io} \), we would not obtain \( r \).

\( \square \)

**Proof of Proposition 4.4.** Consider \( D \in \mathbb{D}_I \) and \( A \in \mathbb{A}_C \), then \( \alpha_{io}(P[D.A]) = B^w[D \cdot A] \). We show the two inclusions independently.

\( \subseteq \)

Let \( r \in P[D.A] \) and \( c_0 \in C \) such that \( r_{\|c_0} \) is defined. In order to show that \( \alpha_{io}(\{r\}) \subseteq B^w[D \cdot A] \), we distinguish two cases.

1. In case \( r_{\|c_0} \) is finite, by (4.1), \( \alpha_{io}(\{r\}) = (c_0, \text{fin}(c_n)) \in \alpha_{io}(P[D.A]) \), where \( c_n := \text{last}(r_{\|c_0}) \). Moreover, by Definitions 3.19, 3.15 and 3.24, it is easy to notice that \( r \) must be of one of the following forms:
   - \( r \) ends with \( \mathbb{N} \),
   - \( r \) contains a \( \text{stutt} \) or...

ACM Transactions on Computational Logic, Vol. V, No. N, Article A, Publication date: January YYYY.
(c) $r$ contains a conditional store $\eta \mapsto d$ such that there is no $\text{stutt}$ before it and $c_0 \otimes d = \text{ff}$.

Now, let us show that on the behavioral part, when $A$, with initial store $c_0$, behaves as $\langle A, c_0 \rangle \rightarrow \ast \langle A_n, c_n \rangle \uparrow$ (the sequence is finite), $r$ takes also one of those forms. Looking at the agent semantics $A$ (Definition 3.15) we observe that:

(a) we obtain a sequence that ends with $n$ if a subagent of $A$ is equal to skip or tell, this means that, starting from an initial store $c_0$ such that $\text{last}(r\|c_0)$ is well defined, the operational semantics cannot perform any step from the reached configuration $(\text{skip}, c_n) \uparrow$;

(b) when $A$ contains an agent $\sum_{i=1}^n \text{ask}(g_i) \rightarrow A_i$ and $\forall i \in [1, n], g_i \neq \text{ff}$, then a $\text{stutt}(\cup_{i=1}^n)$ is introduced. Since we assume that $r\|c_0$ is well defined, it holds that the guards are not entailed by $c_0$ (merged with the store produced by the sequence up to that position), thus the operational semantics cannot perform any step from the reached configuration $\langle \sum_{i=1}^n \text{ask}(g_i) \rightarrow A_i, c_n \rangle \uparrow$;

(c) when $r$ contains a conditional state $\eta \mapsto d$ (that occurs before any $\text{stutt}$) such that $c_0 \otimes d = \text{ff}$, we can deduce that, starting from $\langle A, c_0 \rangle$, we reach in a finite number of operational steps the state $\langle A_n, c_n \rangle \uparrow$, from which no further derivation is possible since an inconsistent store has been produced.

Thus, by Definition 4.2, $\langle c_0, \text{fin}(c_n) \rangle \in B^\omega[D, A]$. 

(2) In case $r\|c_0 = c_0 \cdots c_n$ is infinite, let us define $c := \otimes_{i \geq 0} c_i$. By (4.1), $\alpha_{io}(\{r\}) = \langle c_0, \text{inf}(c) \rangle \in \alpha_{io}(P[D, A])$. By Theorem 3.25, it is easy to notice that $r\|c_0 \in B^\omega[D, A]$, in fact, agent $A$ with initial store $c_0$ behaves in the following way: $\langle A, c_0 \rangle \rightarrow \ldots \rightarrow \langle A_i, c_i \rangle \rightarrow \ldots$. By Definition 4.2, it follows that $\langle c_0, \text{inf}(c) \rangle \in B^\omega[D, A]$. Let $p \in IO$, we show that $p \in B^\omega[D, A] \Rightarrow p \in \alpha_{io}(P[D, A])$. Let us distinguish two cases:

- $p = \langle c_0, \text{fin}(c_n) \rangle$ By Definition 4.2, it follows that $\langle A, c_0 \rangle \rightarrow \ldots \rightarrow \langle A_n, c_0 \rangle \uparrow$, and by Definition 3.1, $c_0 \cdots c_n \in B^\omega[D, A]$. By Theorem 3.25, it exists $r \in P[D, A]$ such that $r\|c_0 = c_0 \cdots c_n$, and by (4.1) it follows that $\langle c_0, \text{fin}(c_n) \rangle \in \alpha_{io}(P[D, A])$.

- $p = \langle c_0, \text{inf}(c) \rangle$ By Definition 4.2, it follows that $\langle A, c_0 \rangle \rightarrow \ldots \rightarrow \langle A_i, c_i \rangle \rightarrow \ldots$, by Definition 3.1, $c_0 \cdots c_n \in B^\omega[D, A]$. By Theorem 3.25, it exists $r \in P[D, A]$ such that $r\|c_0 = c_0 \cdots c_n$, and by (4.1) it follows that $\langle c_0, \text{inf}(c) \rangle \in \alpha_{io}(P[D, A])$.

\end{proof}

\textbf{Proof of Theorem 4.6.} From Proposition 4.4 and by definition of $\pi_F$ (Definition 4.2), for each tcep program $P$, $\pi_F(\alpha_{io}(P[P])) = B^\omega[F[P]]$. Thus, it is sufficient to show that $O^{\omega}[P_1] = O^{\omega}[P_2] \iff \pi_F(\alpha_{io}(P_1[P])) = \pi_F(\alpha_{io}(P_2[P]))$ for $P_1$ and $P_2$ tcep programs such that no trace in $P_1[P] \cup P_2[P]$ is a failed conditional trace. We prove the two directions separately.

\begin{itemize}
  \item \Rightarrow \ We prove the equivalent implication: $\pi_F(\alpha_{io}(P_1[P_1])) \neq \pi_F(\alpha_{io}(P_2[P_2])) \Rightarrow O^{\omega}[P_1] \neq O^{\omega}[P_2]$. Let us assume, without loss of generality, that $\pi_F(\alpha_{io}(P_1[P_1])) \subset \pi_F(\alpha_{io}(P_2[P_2]))$, which means that there exist $r_2 \in P_2[P_2]$ and $c_0 \in C$ such that $r_2\|c_0 = c_0 \cdots c_n$, but it does not exist $r_1 \in P_1[P_1]$ such that $r_1\|c_0 = c_0 \cdots c_n$. Furthermore, $c_n \neq \text{ff}$, since, by hypothesis, $r_2$ is not a failed conditional trace. By Theorem 3.25, $c_0 \cdots c_n \in B^\omega[P_2]$ and, by Definition 4.2, $\langle c_0, c_n \rangle \in \text{fin}(P_2[P_2])$. Since $B^\omega[P_1] \neq O^{\omega}[P_1]$, on sequences terminating in $f$ and $c_n \neq \text{ff}$, it follows that $\langle c_0, c_n \rangle \in O^{\omega}[P_2]$. On the other hand, we have that $c_0 \cdots c_n \in B^\omega[P_1]$, thus $\langle c_0, c_n \rangle \in P_2[P_1]$. It is easy to see that, given a tcep program $P$, $O^{\omega}[P] \neq B^\omega[P]$, thus it holds that $\langle c_0, c_n \rangle \notin O^{\omega}[P_1]$. This means that $\langle c_0, c_n \rangle \notin O^{\omega}[P_2] \neq O^{\omega}[P_1]$ and we can conclude that $O^{\omega}[P_1] \neq O^{\omega}[P_2]$.

  \item \Leftarrow \ We prove the equivalent implication: $O^{\omega}[P_1] \neq O^{\omega}[P_2] \Rightarrow \pi_F(\alpha_{io}(P_1[P])) \neq \pi_F(\alpha_{io}(P_2[P]))$. Without loss of generality, assume that $O^{\omega}[P_1] \subset O^{\omega}[P_2]$, thus,
there exists \( \langle c_0, c_n \rangle \in \mathcal{O}^{io}[P_2] \) such that \( \langle c_0, c_n \rangle \notin \mathcal{O}^{io}[P_1] \). Since no trace in \( \mathcal{P}[P_1] \cup \mathcal{P}[P_2] \) is failed, we can assume that \( c_n \neq ff \). This means that, by using the transition relation defined in [de Boer et al. 2000], we have a derivation of the form \( \langle A_2, c_0 \rangle \rightarrow \ldots \langle A'_2, c_n \rangle \neq \top \), with \( A_2, A'_2 \in \mathbb{A}^{II}_C \), \( D_2 \in \mathbb{D}^{II}_C \) and \( P_2 = D_2 \cdot A_2 \); On the other hand, it can be noticed that, by using the transition relation of Figure 1, for \( P_1 \) there is no derivation starting with \( c_0 \) and ending in \( c_n \). Thus, we have that \( \langle c_0, c_n \rangle \in \mathcal{B}^{io}_F[P_2] \) and \( \langle c_0, c_n \rangle \notin \mathcal{B}^{io}_F[P_1] \). From Proposition 4.4, it follows that \( \langle c_0, c_n \rangle \in \pi_F(\alpha_{io}(\mathcal{P}[P_2])) \setminus \pi_F(\alpha_{io}(\mathcal{P}[P_1])) \) and we can conclude that \( \pi_F(\alpha_{io}(\mathcal{P}[P_1])) \neq \pi_F(\alpha_{io}(\mathcal{P}[P_2])) \).