ITERATED FUNCTION SYSTEMS, FRACTALS AND KLEINIAN GROUPS

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This is a *personal* geodesic along the topics in the title. It is not a textbook, nor an attempt at it. It does *not* constitute a syllabus for my course. Definitions, lemmas, and examples are merged in the text, and it is up to the reader to discern which is which. Corrections, suggestions, observations, . . ., are most welcome. Version of February 2, 2026.

1. Basic definitions

All topological spaces are Hausdorff and 1st countable (every point has a contable open basis); for example, discrete more than countable spaces are fine. For 1st countable spaces being Hausdorff amounts to the fact that converging sequences have a unique limit. Good spaces are *Polish* (that is, 2nd countable and metrizable by a complete metric).

Theorem 1.1. *Let X be a metrizable space*; *the following conditions are equivalent.*

(1) *X is compact.*
(2) *every infinite subset has an accumulation point in X.*
(3) *X is sequentially compact.*
(4) *X is complete and totally bounded w.r.t. every complete metric.*
(5) *X is complete and totally bounded w.r.t. at least one complete metric.*

Definition 1.2. X is *locally compact at x* if there exists an open O and a compact K with $x \in O \subseteq K$. It is *locally compact* if it is locally compact at every point.

The space $\mathbb{Q}$ with the topology induced by $\mathbb{R}$ is not locally compact at any point. A theorem by Weil says that a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ is locally compact if and only if it has finite algebraic dimension. Thus, for example, ℓ_2, $\mathcal{P}([0,1])$, and $C([0,1], \|{-}\|_\infty)$ are not locally compact.

The *Cantor space* is the only nonempty, metrizable, compact, perfect, 0-dimensional (i.e., with a basis of clopen sets), 2nd countable space. The *Baire space* is the only nonempty, metrizable in a complete way, perfect, 0-dimensional, not locally compact at any point, 2nd countable space.

Theorem 1.3. *Let $A : X \to Y$ be continuous, with Y locally compact. T.f.a.e., and define a* proper *map.*

(a) *The counterimage of every compact set is compact.*
(b) *For every sequence (x_n) whose A-image converges to y, there exists a subsequence converging to a point in the fiber of y.*
(c) *A sends sequences* escaping to infinity (*that is, that intersect every compact set in finitely many points*) *to sequences escaping to infinity.*

Neither $x \to \exp(2\pi i x)$ from $[0,1)$ to $\mathbb{C}$ or $x \mapsto \sin(x)$ from $\mathbb{R}$ to $\mathbb{R}$ are proper.

Corollary 1.4. *Every proper map is closed.*

Corollary 1.5. *If A is injective, with X compact and Y locally compact, then $A : X \to A[X]$ is a homeomorphism.*

Definition 1.6. Let A be a continuous map between metric spaces. Then A is *c-Lipschitz* (or c is a *Lipschitz coefficient* for A) if $\omega(\delta) = c\,\delta$ is a module of continuity for A; the infimum of such c is the Lipschitz coefficient of A, written $\operatorname{Lip}(A)$. Also, A is *u-Hölder* (or u is a *Hölder exponent* of A) if there exists c such that $c\,\delta^u$ is a module of continuity; the supremum of such u is the *Hölder exponent* of A.

Definition 1.7. Let (X,d) be a metric space. A *weak contraction* (in the sense of Rakotch) is a map $A : X \to X$ that has a module of continuity of the form $\omega(\delta) = R(\delta)\,\delta$, for some continuous nonincreasing function $R : (0, \operatorname{diam} X) \to (0,1)$. That is, for every $x \neq x'$, we have $d(Ax, Ax') \leq R\big(d(x,x')\big)\,d(x,x')$. A weak contraction is Lipschitz with constant $r = \sup_{\delta>0} R(\delta) \leq 1$, hence continuous. If $r < 1$, then A is a *strong contraction*.

Theorem 1.8. *Every strong contraction A on a complete metric space has a unique fixed point, then can be obtained as $\lim_{n\to\infty} A^n x_0$ for an arbitrary x_0.*

The group of *similitudes* of $\mathbb{R}^d$ is the group in the middle of the short exact sequence

$$1 \to \operatorname{Transl} \to \operatorname{Sim} \to \mathbb{R}_{>0} \times \operatorname{O}_n \mathbb{R} \to 1,$$

while its affine group is the one in

$$1 \to \operatorname{Transl} \to \operatorname{Aff} \to \operatorname{GL}_n \mathbb{R} \to 1.$$

Sim and Aff act on Transl via $GT_xG^{-1} = T_{Gx}$.

Definition 1.9. Let (X,d) be a metric space. We write $x_{<\varepsilon}$ for the open ball of points at distance strictly less than ε from x, and $x_{\leq\varepsilon}$ for the closed one (in $\mathbb{R}^d$ the second is the closure of the first, but this may fail in other spaces). If Y is a nonepty subset of X, then

$$Y_{<\varepsilon} = \bigcup \{y_{<\varepsilon} : y \in Y\},$$

and analogously for $Y_{\leq\varepsilon}$.

The function $\rho(Y,Z) = \inf\{d(y,z) : y \in Y, z \in Z\}$ is useful, but is not even a pseudometric (the latter being defined as a metric, except that the distance between different points may be 0). The *asymmetric distance* between two nonempty subsets of X is

$$\overrightarrow{d}(Y,Z) = \sup\{\rho(y,Z) : y \in Y\}.$$

It is the supremum of distances an enemy can force me to travel when moving from a point of Y (chosen by her) to a point of Z (chosen by me).

Let $E = \{\varepsilon > 0 : \forall y\, \rho(y, Z) < \varepsilon\}$. Then E is upwards closed, which implies

$$\begin{aligned}\inf\{\varepsilon > 0 : Y \subseteq Z_{<\varepsilon}\} &= \inf E \\ &= \sup(\mathbb{R}_{\geq 0} \setminus E) \\ &= \sup\{t \geq 0 : \exists y\, \rho(y, Z) \geq t\} \\ &= \sup\{\rho(y, Z) : y \in Y\} \\ &= \overrightarrow{d}(Y, Z).\end{aligned}$$

The symmetrization

$$d(Y, Z) = \overrightarrow{d}(Y, Z) \vee \overleftarrow{d}(Y, Z)$$

is a pseudometric ($d\big((0, 1), [0, 1]\big) = 0$) on the set of nonempty subsets of X. It becomes the *Hausdorff metric*, still denoted d, when restricted to the space $\mathcal{X}$ of all nonempty compact subsets of X; the natural embedding of X in $\mathcal{X}$ is of course an isometry. It can be proved that the topology determined by d on $\mathcal{X}$ is the *Vietoris topology*. By definition, the latter is the topology generated by the sets $\{K : K \subseteq O\}$ and $\{K : K \cap O \neq \emptyset\}$, as O varies in the open sets of X; if X is Polish then $\mathcal{X}$ is Polish.

Theorem 1.10. *If* (X, d) *is complete* (*respectively, compact*)*, then* $(\mathcal{X}, d)$ *is complete* (*respectively, compact*)*. The limit of the Cauchy sequence* $K_0, K_1, \ldots$ *is*

$$\bigcap_{n \geq 0} \Big(\bigcup_{k \geq n} K_k \Big)^f.$$

Proof. See Appendix 2 of [BP17]. □

Definition 1.11. Let X be either $\mathbb{R}^d$ or a compact subset of $\mathbb{R}^d$, and let $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$ be a finite set of injective selfmaps of X. The metric d is always the one induced by the euclidean norm $\|{-}\|_2 = |{-}|$. Assume that:

- either every A_a is a strong contraction;
- or X is compact and all A_a are weak contractions for the same R.

Then $\mathcal{A}$ is an *Iterated Function Systems* (IFS); if all A_a are similitudes with the same contraction rate, necessarily < 1, then $\mathcal{A}$ is *homogeneous*.

In holomorphic dynamics one requires that $X = \mathcal{E}$ is a simply connected bounded domain, and that all A_a are holomorphic univalent maps such that the closure of the image of $\mathcal{E}$ is contained in $\mathcal{E}$. These are strict contractions by the Schwarz Lemma.

The map $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{A}(K) = \bigcup \{A_a[K] : a \in m\}$$

is the *Hutchinson operator*.

We fix notation: $u, v, \ldots \in m^{<\omega}$ are words, $\mathbf{a}, \mathbf{b}, \ldots \in m^\omega$ sequences. If $u = a_0 \ldots a_{t-1}$, then A_u is the composition $A_{a_0} \cdots A_{a_{t-1}}$ and x_u the fixed point of A_u. If A_a is r_a-Lipshitz and $r_u = r_{a_0} \cdots r_{a_{t-1}}$, then A_u is r_u-Lipschitz. The words u and v are *incomparable* if neither is a prefix of the other. For $Y \subseteq X$, we set $\operatorname{diam}(Y) = \sup\{d(y, y') : y, y' \in Y\}$, $Y_u = A_u[Y]$, and have $\operatorname{diam}(Y_u) \leq r_u \operatorname{diam}(Y)$.

Note that each $A_u : X \to X_u$ is a homeomorphism; if X is compact this follows from Corollary 1.5, and if $X = \mathbb{R}^d$ from Brouwer's Invariance of Domain Theorem.

Lemma 1.12. *Given finitely many pairs (H_a, K_a) of compact sets, we have*

$$d\big(\bigcup_a H_a, \bigcup_a K_a\big) \leq \max_a d(H_a, K_a).$$

Theorem 1.13. *Let $\mathcal{A} = \{A_0, \dots, A_{m-1}\}$ with all A_a r-Lipschitz. Then $\mathcal{A}$ is r-Lipschitz.*

Proof. If $\mathcal{A} = \{A\}$, we have

$$\begin{aligned} \overrightarrow{d}(AH, AK) &= \sup_h \big(\inf_k d(Ah, Ak)\big) \\ &\leq \sup_h \big(\inf_k r\, d(h,k)\big) \\ &= r\, \overrightarrow{d}(H, K). \end{aligned}$$

It holds for $\overleftarrow{d}$ as well, and thus for d.

For the general case,

$$d\Big(\bigcup_a A_a H, \bigcup_a A_a K\Big) \leq \max_a d(A_a H, A_a K) \leq r\, d(H, K).$$

□

Definition 1.14. The IFS $\mathcal{A}$ satisfies the *Strong Separation Condition* if there exists a compact H such that $H_0, \dots, H_{a_{m-1}}$ are pairwise disjoint subsets of H. It satisfies the *Open Set Condition* if there exists a nonempty bounded open set $O \subseteq X$ such that $O_0, \dots, O_{a_{m-1}}$ are pairwise disjoint and contained in O. If, moreover, $O \cap K \neq \emptyset$ (K being the attractor of $\mathcal{A}$, to be defined soon), then $\mathcal{A}$ satisfies the *Strong Open Set Condition*.

Theorem 1.15. *Let $\mathcal{A}$ be an IFS as above*; *then the following statements hold.*

(1) *There exists $H \in \mathcal{X}$ such that $H \supseteq \mathcal{A}H$.*
(2) *There exists precisely one $K \in \mathcal{X}$ which is a fixed point for $\mathcal{A}$*; *it can be constructed as the intersection of the chain*

$$H \supseteq \mathcal{A}H \supseteq \mathcal{A}^2 H \supseteq \cdots,$$

for any H as in (1).
(3) *For every u and $t \geq 1$, we have $K_u = \bigcup\{K_{uw} : w \in m^t\}$.*
(4) *For every $\mathbf{a}$ we have a strictly descending chain*

$$K \supset K_{\mathbf{a}\restriction 1} \supset K_{\mathbf{a}\restriction 2} \supset \cdots$$

whose intersection is a singleton. Denoting by $\pi(\mathbf{a})$ the element of that singleton, the map $\pi : m^\omega \to K$ is surjective and continuous; *moreover, $A_u(\pi(\mathbf{a})) = \pi(u\mathbf{a})$. It is injective iff it is a homeomorphism iff the SSC holds*; *if so, then K is a Cantor set.*
(5) *We have $\pi(u^\omega) = x_u$ and $\pi(\mathbf{a}) = \lim_{t\to\infty} x_{\mathbf{a}\restriction t}$*; *therefore, $K = \{x_u : u \in m^{<\omega}\}^f$.*
(6) *Let $L \in \mathcal{X}$. Then*:
   - *$\mathcal{A}^t L$ converges to K in the Hausdorff metric, as $t \to \infty$.*
   - *For every $\mathbf{a}$, $d(L_{\mathbf{a}\restriction t}, \pi(\mathbf{a}))$ converges to 0, as $t \to \infty$, uniformly* (*i.e., with bounds on convergence not depending on* $\mathbf{a}$).
   - *In the definition of $\pi(\mathbf{a})$, K can be replaced by any H as in* (1).

There are two handy strategies to draw the *limit set*, or *attractor*, K of $\mathcal{A}$.

(1) If you know a reasonable approximation K_0 of K, draw $\mathcal{A}^t K_0$, for some t about $\log_m(5000)$.
(2) The *Chaos Game*: choose x_0 and generate $x_1, x_2, \dots$ with $x_{t+1} = A_{a_t}(x_t)$ for a random $\mathbf{a}$. Plot $\{x_t : 50 \leq t < 10000\}$ (if x_0 is known to be in K, replace 50 by 0).

Example 1.16. (1) The Sierpinski triangle and carpet.
(2) The Koch curve and snowflake..
(3) The Peano-like curves.
(4) The Lévy and Heighway dragons.

2. Topological dimension

Let X be any metrizable 2nd countable space.

- If $X = \emptyset$, then X has *topological dimension* -1.
- Otherwise, X has topological dimension $n \geq 0$, where n is the least integer such that for every $x \in X$ and every $\varepsilon > 0$ there exists a neighborhood (not necessarily open) $x \in U \subseteq x_{<\varepsilon}$ whose boundary ∂U has topological dimension $< n$.

3. The box dimension

Let $\emptyset \neq Y \subset \mathbb{R}^d$ be bounded and let $\delta > 0$. We classify balls according to diameter rather than radius.

- The *covering number* C_δ of Y is the minimum cardinality of a δ-*cover* $\mathcal{E}$ of Y (i.e., $\mathcal{E}$ is at most countable, $\bigcup \mathcal{E} \supseteq Y$, and $0 \leq \operatorname{diam}(E) \leq \delta$ for every $E \in \mathcal{E}$).
- The *net number* N_δ is the minimum cardinality of a $\delta/2$-net for Y, that is the covering number by closed balls of diameter δ.
- The *packing number* P_δ is the maximum number of nonoverlapping open balls of diameter δ with centers in Y, that is the maximum cardinality of a δ-separated subset of Y.
- The *box number* B_δ of Y is the number of closed mesh boxes with vertices in $\delta\mathbb{Z}^d$ that intersect Y.

Since Y is bounded, these numbers are finite. If Y is a "nice full dimensional" set, then $\lim_{\delta \to 0} B_\delta \, \delta^d$ converges to the volume of Y; thus B_δ grows like $\mathrm{const}\, \delta^{-d}$, with $\mathrm{const} \neq 0$. This suggests defining the dimension of Y as the exponent $0 \leq h \leq d$ such that B_δ grows like $\mathrm{const}\, \delta^{-h}$. Thus

$$\lim_{\delta \to 0} \frac{B_\delta}{\delta^{-h}} \to \mathrm{const} \in \mathbb{R}_{>0};$$

taking logarithms and dividing by $\log \delta^{-1}$ we see that we must have

$$h = \lim_{\delta \to 0} \frac{\log B_\delta}{\log \delta^{-1}}.$$

We then have

$$\left| \frac{\log B_\delta}{\log \delta^{-1}} - h \right| = q(\delta) = o(1).$$

Therefore

$$h - q(\delta) \leq \frac{\log B_\delta}{\log \delta^{-1}} \leq h + q(\delta),$$

and one writes $B_\delta = (\delta^{-1})^{h\pm o(1)}$.

All of this can be formalized as follows.

Theorem 3.1. *We have the indentities*

$$\liminf_{\delta\to 0} \frac{\log C_\delta}{\log \delta^{-1}} = \liminf_{\delta\to 0} \frac{\log N_\delta}{\log \delta^{-1}} = \liminf_{\delta\to 0} \frac{\log P_\delta}{\log \delta^{-1}} = \liminf_{\delta\to 0} \frac{\log B_\delta}{\log \delta^{-1}},$$

and that number defines the lower box dimension *of Y, denoted $\underline{\dim}_B(Y)$. An analogous statement holds for* limsup*, and we obtain the* upper box dimension $\overline{\dim}_B(Y)$ *of Y. If $\underline{\dim}_B(Y) = \overline{\dim}_B(Y)$, this defines the* box dimension *of Y, $\dim_B(Y)$ (sometimes box dimensions are called* Minkowski dimensions*).*

Proof. See [Fal14, §2.1]. We need to establish inequalities of the form $C_{r\delta} \le P_\delta$ and $P_{s\delta} \le C_\delta$ for appropriate $r, s \ge 1$. Indeed, such inequalities imply

$$\frac{\log C_{r\delta}}{\log(r\delta)^{-1}} \le \frac{\log P_\delta}{\log r^{-1} + \log \delta^{-1}},$$

and for $\delta \to 0$ the summand $\log r^{-1}$ disappears in $\liminf$ and $\limsup$.

For example, let $\{y_1, \dots, y_{P_\delta}\}$ be a maximal δ-separated subset of Y. Then every point of Y must be at distance $< \delta$ from at least one y_i, otherwise the set could be enlarged. This implies that the open balls $(y_i)_{<\delta}$ cover Y, and therefore $C_{2\delta} \le P_\delta$.

Conversely, let $\{E_1, \dots, E_{C_\delta}\}$ be a minimal δ-covering of Y. Every (1.1δ)-separated subset of Y is contained in the union of the E_i, and every E_i contains at most one element of the subset; therefore $P_{1.1\delta} \le C_\delta$. □

Example 3.2. Suppose that, for $k = 0, 1, 2, \dots$, Y can be covered by m^k "almost disjoint" sets of diameter const r^k. Then

$$\dim_B Y = \lim_{k\to\infty} \frac{\log m^k}{\log(\text{const}\, r^k)^{-1}} = \frac{\log m}{\log r^{-1}}.$$

Thus the classical Cantor set, the Sierpinski gasket, and the Koch curve have box dimension $\log 2/\log 3$, $\log 3/\log 2$, $\log 4/\log 3$, respectively.

Lemma 3.3. *Let $\emptyset \ne Y = Y_0 \cup \cdots \cup Y_{m-1}$, with every Y_i a copy of Y scaled by a factor $0 < r < 1$. Fot $t \ge 0$, let $\mathcal{Y}^t$ be the set of m^t copies of Y scaled by r^t. Assume that there exists $q \ge 1$ such that, for every t and every $q+1$ elements of $\mathcal{Y}^t$, at least 2 of them are at ρ-useful-function $\ge r^t$. Then $\dim_B Y = \log m/\log r^{-1}$.*

Proof. Of course $C_{r^t} \le m^t$; therefore

$$\limsup_{t\to\infty} \frac{\log C_{r^t}}{(-\log) r^t} \le \limsup \frac{m^t}{(-\log) r^t} = \frac{\log m}{\log r^{-1}}.$$

Conversely, every set of diameter $< r^t$ can intersect at most q elements of $\mathcal{Y}^t$. Therefore $C_\delta \ge m^t/q$, for t the exponent in the sandwich $r^{t+1} \le \delta < r^t$. Hence

$$\liminf_{\delta\to 0} \frac{\log C_\delta}{(-\log)\delta} \ge \liminf_{t\to\infty} \frac{\log m^t/q}{(-\log) r^{t+1}} = \frac{\log m}{\log r^{-1}}.$$

□

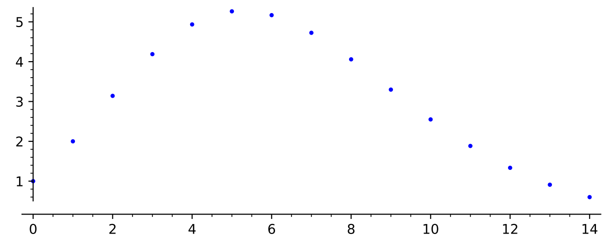

FIGURE 1. Volume of unit ball

Box dimension has a simple presentation, but has disadvantages. For example, using the N_δ-definition it is clear that $\underline{\dim}_B(Y) = \underline{\dim}_B(Y^f)$, and similarly for $\overline{\dim}_B$ and $\dim_B$. Thus $\mathbb{Q}^2 \cap [0,1]^2$ has box-dimension 2, which is questionable. The issue arises since we are considering *finite* converings only.

4. The Hausdorff measure

Let Ω be a nonempty set. An *outer measure* on Ω is a function $\mu^* : \mathscr{P}(\Omega) \to [0, \infty]$ which satisfies $\mu^*(\emptyset) = 0$ (it may be trivial), $\mu^*(A) \leq \mu^*(B)$ for $A \subseteq B$, and is σ-subadditive. A set A is μ^**-measurable* if for every B we have

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Theorem 4.1 (Carathéodory, about 1918)**.** *The family $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and contains every set of μ^*-measure 0. The restriction μ of μ^* to $\mathcal{M}$ is a (possibly trivial) measure, so that $(\Omega, \mathcal{M}, \mu)$ is a complete measure space. If Ω is a metric space and $\mu^*(Y \cup Z) = \mu^*(Y) + \mu^*(Z)$ if $\rho(Y, Z) > 0$, then $\mathcal{M}$ contains all Borel sets.*

Proof. [RF10, Chapter 20]. □

For $d = 0, 1, 2, \ldots$, the closed unit ball unit in $\mathbb{R}^d$ has d-dimensional volume

$$\gamma(d) = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}.$$

Fix a "dimension" $s \in \mathbb{R}_{\geq 0}$; the number $\gamma(s)$ still makes sense, so we define the *s-dimensional volume* of the ball of diameter δ to be $\gamma(s)(\delta/2)^s$.

Fix $\delta > 0$ and let Y be any subset of $\mathbb{R}^d$. We agree that $0^0 = 1$ and define

$$\begin{aligned}(\mathcal{H}^s_\delta)^*(Y) &= \inf\left\{\sum_{E \in \mathcal{E}} (\operatorname{diam} E)^s : \mathcal{E} \text{ is a } \delta\text{-cover of } Y\right\} \\ &= \left(\frac{\gamma(s)}{2^s}\right)^{-1} \inf\left\{\sum_{E \in \mathcal{E}} \gamma(s)\left(\frac{\operatorname{diam} E}{2}\right)^s : \mathcal{E} \text{ is a } \delta\text{-cover of } Y\right\}.\end{aligned}$$

Lemma 4.2. *$(\mathcal{H}^s_\delta)^*$ is an outer measure.*

Proof. Let $Y = \bigcup_{k<\omega} Y_k$, fix $\varepsilon > 0$ and, for every k, choose a δ-cover $\mathcal{E}_k$ of Y_k such that

$$\sum_{E \in \mathcal{E}_k} (\operatorname{diam} E)^s < (\mathcal{H}^s_\delta)^*(Y_k) + \varepsilon/2^{k+1}.$$

Since $\bigcup_{k<\omega} \mathcal{E}_k$ is a δ-cover of Y, we have

$$(\mathcal{H}^s_\delta)^*(Y) \leq \varepsilon + \sum_k (\mathcal{H}^s_\delta)^*(Y_k).$$

Since ε is arbitrary, we have σ-subadditivity. $\square$

If $0 < \beta < \delta$, then clearly $(\mathcal{H}^s_\beta)^*(Y) \geq (\mathcal{H}^s_\delta)^*(Y)$; this implies that

$$(\mathcal{H}^s)^*(Y) = \lim_{\delta\to 0}(\mathcal{H}^s_\delta)^*(Y) = \sup_{\delta>0}(\mathcal{H}^s_\delta)^*(Y)$$

exists and is still an outer measure (easy). Applying Caratheodory's theorem we obtain the *s-dimensional Hausdorff measure* $\mathcal{H}^s$ on the class of $(\mathcal{H}^s)^*$-measurable sets.

From now on we will write $\mathcal{H}^s(Y)$ for $(\mathcal{H}^s)^*(Y)$, no matter what Y is.

Lemma 4.3. *Let $s < t$. Then $\mathcal{H}^s(Y) < \infty$ implies $\mathcal{H}^t(Y) = 0$ (equivalently, $\mathcal{H}^t(Y) > 0$ implies $\mathcal{H}^s(Y) = \infty$).*

Proof. For every E in every δ-cover of Y, we have $(\operatorname{diam} E)^t \leq \delta^{t-s}(\operatorname{diam} E)^s$. Therefore, $\mathcal{H}^t_\delta(Y) \leq \delta^{t-s}\, \mathcal{H}^s_\delta(Y)$, and the statement follows by letting δ tend to 0. $\square$

Theorem 4.4. *Let $A : Y \to \mathbb{R}^m$ be a u-Hölder map for some $u, c > 0$; then the following statements are true.*

(1) $\mathcal{H}^{s/u}(A[Y]) \leq c^{s/u}\mathcal{H}^s(Y)$.
(2) *In particular, since every projection* Proj (*not necessarily orthogonal*) *towards any subspace is* 1*-Lipschitz, we have* $\mathcal{H}^s\big(\operatorname{Proj}[Y]\big) \leq \mathcal{H}^s(Y)$.
(3) *If A is a similitude of ratio r, then $\mathcal{H}^s(A[Y]) = r^s\mathcal{H}^s(Y)$.*

Proof. Let $\mathcal{E}$ be a δ-cover of Y. Then $\mathcal{G} = \{A[E \cap Y] : E \in \mathcal{E}\}$ is a $c\delta^u$-cover of $A[Y]$. Noting that

$$\Big(\operatorname{diam}\big(A[E\cap Y]\big)\Big)^{s/u} \leq c^{s/u}\Big(\big(\operatorname{diam}(E\cap Y)\big)^u\Big)^{s/u} \leq c^{s/u}\operatorname{diam}(E)^s,$$

we obtain

$$\mathcal{H}^{s/u}_{c\delta^u}(A[Y]) \leq \sum_{G\in\mathcal{G}}(\operatorname{diam} G)^{s/u} \leq c^{s/u}\sum_{E\in\mathcal{E}}(\operatorname{diam} E)^s.$$

Since $\mathcal{E}$ is arbitrary, we get $\mathcal{H}^{s/u}_{c\delta^u}(A[Y]) \leq c^{s/u}\mathcal{H}^s_\delta(Y)$, and the statement follows by letting δ tend to 0. Let A be a similitude of ratio r. Taking $u = 1$ and applying (1) to both Y, A and $A[Y], A^{-1}$ we obtain (3). $\square$

Theorem 4.5. (i) *$\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^d$.*
(ii) *$\mathcal{H}^0 = \sharp$ and $\mathcal{H}^d = \gamma(n)^{-1}2^d\lambda^d$.*
(iii) *If $s > d$ then $\mathcal{H}^s$ is trivial.*
(iv) *If $s < d$ then $\mathcal{H}^s$ is not σ-finite, is not finite on compacta, and does not induce a continuous functional $C_c(\mathbb{R}^d) \to \mathbb{R}$; thus, it is not a Radon measure.*
(v) *If $0 < m < d$ and Y is a smooth m-dimensional manifold in $\mathbb{R}^d$, then the restriction of $\mathcal{H}^m$ to Y equals $\gamma(m)^{-1}2^m$ times the m-dimensional volume form on Y in the sense of differential geometry.*

Proof. (i) follows from Theorem 4.1. (ii) We have $\mathcal{H}^0_\delta(Y) = C_\delta(Y)$, whose limit for $\delta \to 0$ is $\sharp Y$.

The second statement crucially uses the *isodiametric inequality*: for every nonempty subset Y of $\mathbb{R}^d$ we have
$$\lambda^d(Y) \le \gamma(d)2^{-d}(\operatorname{diam} Y)^d.$$
□

5. The Hausdorff dimension

Definition 5.1. The number
$$\dim_H Y = \sup\{s : \mathcal{H}^s(Y) = \infty\} = \inf\{t : \mathcal{H}^t(Y) = 0\}.$$
is the *Hausdorff dimension* $\dim_H Y$ of $Y \subseteq \mathbb{R}^d$. An *s-set* is a Borel Y of Hausdorff dimension s such that $\mathcal{H}^s(Y) \neq 0, \infty$.

There is a way of defining the Hausdorff dimension bypassing the measure. The *s-dimensional Hausdorff content* of Y (which is *not* a measure) is defined by
$$\mathcal{H}^s_\infty(Y) = \inf\left\{\sum_{E\in\mathcal{E}} (\operatorname{diam} E)^s : \mathcal{E} \text{ is a countable cover of } Y\right\}.$$

Lemma 5.2. *We have $\mathcal{H}^s(Y) = 0$ iff $\mathcal{H}^s_\infty(Y) = 0$ and, hence,*
$$\dim_H Y = \inf\{s : \mathcal{H}^s_\infty(Y) = 0\}.$$

Proof. For every $\delta > 0$ we plainly have
$$\mathcal{H}^s(Y) \ge \mathcal{H}^s_\delta(Y) \ge \mathcal{H}^s_\infty(Y),$$
and the left-to-right direction is clear. Assume, conversely, that $\mathcal{H}^s_\infty(Y) = 0$. Then, for every $t > 0$ there exists a countable cover $\mathcal{E}$ of Y with
$$\sum\{(\operatorname{diam} E)^s : E \in \mathcal{E}\} < t.$$
Such an $\mathcal{E}$ is necessarily a $t^{1/s}$-cover and thus $\mathcal{H}^s_{t^{1/s}}(Y) < t$. Since $t^{1/s} \to 0$ for $t \to 0$, we have $\mathcal{H}^s(Y) = 0$. □

Lemma 5.3.
$$\dim_H \bigcup_{i<\omega} Y_i = \sup_{i<\omega} \dim_H Y_i. \tag{5.1}$$

Proof. If $Y \subseteq Z$ and $\mathcal{H}^s(Z) = 0$ for some s, then $\mathcal{H}^s(Y) = 0$. This implies $\dim_H Y \le \dim_H Z$ and the $\ge$ inequality. If the inequality were strict there would exist a sandwiched s. The $\mathcal{H}^s$-measure of each Y_i is then 0, and thus so is the $\mathcal{H}^s$-measure of the union, which is a contradiction. □

Example 5.4. Every at most countable Y has Hausdorff dimension 0, but the 0-sets are precisely the finite nonempty ones.

Let $0 < r_0 < r_1 < \cdots$ converge to $1/2$. Then we have Cantor sets $K_t \subset [0,1]$ with $\dim_H K_t = \log 2/\log r_t^{-1}$. Thus $\bigcup_t K_t$ has Hausdorff dimension 1, but it is not a 1-set. Analogously for $\mathbf{Bad} = \bigcup_{b\ge 2} \mathbf{Bad}(b)$, by the Jarník theorem..

Lemma 5.5. *Let $Y \subset \mathbb{R}^d$ be bounded. Then*

$$\dim_H Y \leq \underline{\dim}_B Y.$$

Taking $Y = [0,1]^2 \cap \mathbb{Q}^2$ we see that the inequality may be strict.

Proof. It suffices to show that if $\mathcal{H}^s(Y) = \infty$ then $s \leq \underline{\dim}_B Y$. For such an s, and for every small enough δ, we have $1 < \mathcal{H}^s_\delta(Y) \leq C_\delta \delta^s$. Taking logarithms we get $0 < \log C_\delta + s \log \delta$; therefore $s < \log C_\delta / \log \delta^{-1}$ and $s \leq \underline{\dim}_B Y$. □

Lemma 5.6. *Let A be as in Theorem* 4.4; *then $\dim_H A[Y] \leq (\dim_H Y)/u$.*

Proof. Let $s > 0$ be such that $\mathcal{H}^s(Y) = 0$; by Theorem 4.4 we have $\mathcal{H}^{s/u}(A[Y]) = 0$, and thus $\dim_H A[Y] \leq s/u$. □

Lemma 5.7. *Lipschitz maps do not increase $\underline{\dim}_B$, $\overline{\dim}_B$, $\dim_H$; bi-Lipschitz ones* (*e.g., affine maps*) *leave them invariant.*

Proof. By definition, A is bi-Lipschitz if it is invertible and both A and A^{-1} are Lipschitz; thus the second statement follows from the first. For Hausdorff dimension, apply Lemma 5.6. Let $A : Y \to \mathbb{R}^m$ be Lipschitz with constant $c > 0$, choose $\delta > 0$, and let $\mathcal{E}$ be a finite δ-covering of Y of cardinality $C_\delta(Y)$. Then $\mathcal{G}$ as in the proof of Theorem 4.4 witnesses that $C_{c\delta}(A[Y]) \leq C_\delta(Y)$. Taking logarithms and dividing by $\log(c\delta)^{-1} + \log c = \log \delta^{-1}$, we obtain

$$\frac{\log C_{c\delta}(A[Y])}{\log(c\delta)^{-1} + \log c} \leq \frac{\log C_\delta(Y)}{\log \delta^{-1}}.$$

The conclusion follows by applying $\liminf_{\delta\to 0}$ and $\limsup_{\delta\to 0}$. □

Lemma 5.8. *If $\dim_H Y < 1$ then Y is totally disconnected, hence a Cantor set.*

Proof. Let x_0, x_1 be distinct elements of Y. The function $Ax = |x - x_0|$ from $\mathbb{R}^d$ to $\mathbb{R}$ is Lipschitz, by the triangle inequality. Therefore $\dim_H A[Y] < 1$, and thus does not contain intervals. In particular there exists $q \notin A[Y]$ with $0 = Ax_0 < q < Ax_1$, and $(x_0)_{<q}$ is a clopen that separates x_0 from x_1. □

Lemma 5.9. *Let $\delta(0) > \delta(1) > \cdots$ converge to* 0 *not too fast* (*i.e., $\delta(n+1) \geq c\delta(n)$, for some $0 < c < 1$*). *Then*

$$\limsup_{\delta\to 0} \frac{\log(C_\delta)}{(-\log)(\delta)} = \limsup_{n\to\infty} \frac{\log(C_{\delta(n)})}{(-\log)(\delta(n))},$$

and analogously for liminf, *N_δ, P_δ, B_δ.*

Corollary 5.10. *Suppose that, for a sequence $\delta(n)$ as above, Y can be covered by $M(n)$ sets of diameter $\leq \delta(n)$. Then*:

(1)
$$\underline{\dim}_B Y \leq \liminf_n \frac{\log(M(n))}{(-\log)(\delta(n))}.$$

(2)
$$\overline{\dim}_B Y \leq \limsup_n \frac{\log(M(n))}{(-\log)(\delta(n))}.$$

(3) *If* $M(n) = O(\delta(n)^{-s})$ *for some* $s \geq 0$ (*i.e.,* $\{M(n)\delta(n)^s : n \geq 0\}$ *is bounded*)*, then* $\mathcal{H}^s(Y) < \infty$ *and* $\dim_H Y \leq s$.

Definition 5.11. Let $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$ be an IFS of similitudes, with attractor K and ratios $0 < r_0, \ldots, r_{m-1} < 1$. The function $r_0^s + \cdots + r_{m-1}^s$ from $\mathbb{R}_{\geq 0}$ to $(0, m]$ is strictly decreasing, and therefore there exists a unique D such that $(r_0^D, \ldots, r_{m-1}^D)$ is a probability vector. We call D the *similarity dimension* of K; if the IFS is homogeneous with ratio r, then $D = \log m / \log r^{-1}$.

Very roughly, upper bounds on the dimension depend on the size of the chunks in $\mathcal{A}^n K_0$, while lower bounds on the spacings between these chunks. The following is called the Mass Distribution Principle.

Lemma 5.12. *Let* μ *be a probability on* $\mathbb{R}^d$*, with* $\mu(Y) = 1$*. Let* $c, \delta_0 > 0$ *be such that for every* E *of diameter* $< \delta_0$ *we have* $\mu(E) \leq c\,(\operatorname{diam} E)^s$*. Then* $c^{-1} \leq \mathcal{H}^s(Y)$*, and thus* $s \leq \dim_H Y$.

Proof. For every δ-covering $\mathcal{E}$ of Y, with $\delta < \delta_0$, we have

$$1 \leq \sum_{E \in \mathcal{E}} \mu(E) \leq c \sum_{E \in \mathcal{E}} (\operatorname{diam} E)^s.$$

Therefore $c^{-1} \leq \mathcal{H}^s_\delta(Y) \leq \mathcal{H}^s(Y)$. □

Example 5.13. Let $m \geq 2$ and let K be the attractor of an IFS consisting of m similitudes of H by the factor $r < 1$, with "almost disjoint" images. Then $\dim_H K = \dim_B K = D = \log m / \log r^{-1}$ = entropy over Lyapunov exponent.

- For the usual $(1/3)$-Cantor set, $D = \log 2 / \log 3 \sim 0.6309$.
- For the Koch curve, $D = \log 4 / \log 3 \sim 1.2618$.
- For the $(1/2)$-Sierpinski triangle, $D = \log 3 / \log 2 \sim 1.5849$.
- For the Heighway dragon, $D = \log 2 / \log \sqrt{2} = 2$.

6. Devil staircases

Let $[a, b]$ be a compact interval in $\mathbb{R}$, and let $\mu \in \mathcal{M}_{[0,\infty)}([a, b])$. We say that μ is *of pure type* if the decomposition of its repartition function

$$M = M_{\text{jump}} + M_{\text{ac}} + M_{\text{sing}}$$

(in which the first summand is the repartition functions of an at most countable combination of Diracs, the second is absolutely continuous, and the third is continuous with derivative 0 λ-everywhere) contains precisely one summand. A purely singular M is called a *devil staircase*, which is *ordinary* if $\lambda(\operatorname{supp} \mu) = 0$ and *splippery* if $\operatorname{supp} \mu = [a, b]$.

Theorem 6.1. *For every IFS of similitudes on* $\mathbb{R}$ *and every product measure* P *on* m^ω*, the pushforward measure* $\mu = \pi_* P$ *is of pure type* (*thus absolutely continuous or purely singular, because the jump component is always* 0)*. Moreover, if* μ *is purely singular, then its repartition function is either an ordinary devil staircase or a slippery one.*

Example 6.2. Let $0 < r_0, r_1, p < 1$ and let $A_0(x) = r_0 x$, $A_1(x) = r_1 x + (1 - r_1)$. Let P be the power measure on 2^ω with weights $(1 - p, p)$ and $\mu = \pi_* P$. If $r_0 + r_1 < 1$, M is an ordinary devil staircase, and if $r_0 + r_1 = 1$ it is a slippery one; see Figure 2.

Let K be attractor of $\mathcal{A}$; then $\dim_H K$ is an upper bound for the Hölder exponent of M. Indeed we have $M[K] = [0, 1]$ (which, in the $r_0 + r_1 < 1$ case, is a good specimen of the old

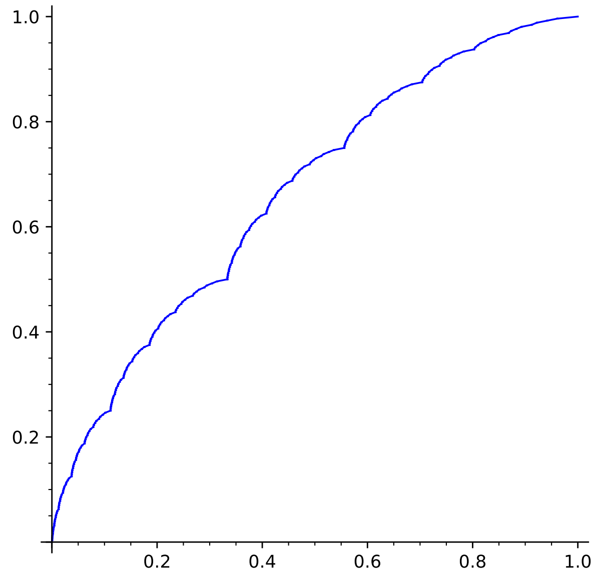

FIGURE 2. $r_0 = 1/3, r_1 = 2/3, p = 1/2$

theorem according to which every compact space is the continuous image of the Cantor set). Let u be a Hölder exponent of M. A fortiori, u is a Hölder exponent of $M \upharpoonright K$ and therefore, by Lemma 5.6, it is bounded above by $\dim_H K$.

7. Rectifiable curves

Let $\gamma : [0,1] \to \mathbb{R}^d$ be a parametrized simple curve, i.e., an injective continuous map from $[0,1]$ onto $\Gamma = \gamma[0,1]$. If γ is of bounded variation, then we say that Γ is *rectifiable* and has *length*

$$\ell(\Gamma) = \sup\Big\{\sum |\gamma t_{i+1} - \gamma t_i| : \cdots < t_i < t_{i+1} < \cdots \text{ are finitely many points in } [0,1]\Big\}.$$

Clearly these notions do not depend on the parametrization.

Theorem 7.1. *If Γ is rectifiable, then $\ell(\Gamma) = \mathcal{H}^1(\Gamma)$.*

Proof. Let $0 \le t < t' \le 1$, and let Proj be the orthogonal projection from Γ to the line through γt and $\gamma t'$. Then Proj is Lipschitz and

$$|\gamma t' - \gamma t| \le \mathcal{H}^1(\mathrm{Proj}[\gamma[t,t']]) \le \mathcal{H}^1(\gamma[t,t']).$$

Since $\mathcal{H}^1$ is a measure, this easily implies $\ell(\Gamma) \le \mathcal{H}^1(\Gamma)$.

Conversely, one shows that for every $0 \le q \le \ell(\Gamma)$ there exists precisely one t such that $\ell(\gamma[0,t]) = q$. This gives another parametrization, by arclenght, $\eta : [0, \ell(\Gamma)] \to \Gamma$, and again η is Lipschitz. Therefore $\ell(\Gamma) = \mathcal{H}^1([0,\ell(\Gamma)]) \le \mathcal{H}^1(\Gamma)$. ☐

All *repartition curves* (that is, repartition functions made continuous curves by addding "vertical segments") are rectifiable (since they are BV), and therefore their graphs are always 1-sets.

8. Stationary measures

Let (X,d) be a compact metric space, and let $\mathcal{P}(X) \subset \mathcal{M}_{[0,\infty)}(X) \subset \mathcal{M}_{\mathbb{R}}(X)$ be the spaces of all Borel probabilities, finite measures, and signed measures on X, endowed with the weak topology, namely the one induced by the immersion $\mathcal{M}_{\mathbb{R}}(X) \to \mathbb{R}^{C(X)}$. The restriction to finite measures is essential: $\mathcal{H}^{0.5}$ and $\mathcal{H}^{0.6}$ integrate in the same way —and very badly— continuous functions on $[0,1]$. Clearly $\mathcal{M}_{[0,\infty)}(X)$ is a positive cone. By Banach-Alaoglu $\mathcal{P}(X)$ is compact, and is clearly closed under affine combinations.

A *pairing* of $\mu_1 \in \mathcal{P}(X_1)$ and $\mu_2 \in \mathcal{P}(X_2)$ is a probability $\sigma \in \mathcal{P}(X_1 \times X_2)$ whose marginals are μ_1 and μ_2; the pairing is *deterministic* if there exist a measurable map $T : X_1 \to X_2$ such that $\sigma = (\mathrm{id} \times T)_* \mu_1$. In the language of the Monge-Kantorovich theory, pairings and deterministic pairings are called *transport plans* and *transport maps*.

Theorem 8.1 (Kantorovich)**.** *Given $\mu, \nu \in \mathcal{P}(X)$, we have*

$$\inf\left\{\int_{X^2} d(x,y) :\, \mathrm{d}\sigma(x,y) : \sigma \text{ is a pairing of } \mu \text{ and } \nu\right\}$$
$$= \sup\left\{\int_X u\, \mathrm{d}(\mu-\nu) : u \text{ is a 1-Lipshitz function from } X \text{ to } \mathbb{R}\right\}.$$

The above number defines the (1-)Wasserstein distance $d(\mu,\nu)$. *It is a metric on $\mathcal{P}(X)$ inducing the weak topology*; *we have* $\operatorname{diam}(\mathcal{P}(X)) < \infty$ *and* $d(\delta_x, \delta_y) = d(x,y)$.

Remark 8.2. More generally one tries to minimize not distances, but costs, a *cost* being a lower semicontinuous function $c : X \times Y \to \mathbb{R}_{\geq 0}$; for example, $\mathbb{1}_{\{x \neq y\}}$, or the *quadratic cost* $|x-y|^2$ in $\mathbb{R}^d$. Lower semicontinuity guarantees that *optimal* transport plans, that is, plans realizing the minimum, always exist; the problem for maps is more difficult.

Remark 8.3. For $p = 1, 2, \ldots$, the function

$$d_p(\mu,\nu) = \inf\left\{\left(\int_{X^2} d^p \,\mathrm{d}\sigma :\ \sigma \text{ is a pairing of } \mu \text{ and } \nu\right)^{1/p}\right\}$$

is a again a distance that metrizes the weak topology.

Suppose that $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$ is a set of continuous maps on X, and let P^1 be any probability on the free monoid $m^{<\omega}$. Then P^1 acts on $\mathcal{P}(X)$ by convolution:

$$P^1 * \mu = \int_{m^{<\omega}} (A_w)_* \mu \,\mathrm{d}P^1(w) = \sum_{w \in m^{<\omega}} P^1(w) \cdot (A_w)_* \mu.$$

Note that convolution is associative: $P^1 * (Q^1 * \mu) = (P^1 * Q^1) * \mu$. A fixed point for $P^1 * -$ is a P^1-*stationary* probability. The simplest P^1 are the ones supported on the free generators; in this case we write $P^1 = (p_0, \ldots, p_{m-1})$ and set $P^n = P^1 * \cdots * P^1$ (n times).

Lemma 8.4. *Every A_* distributes on positive linear combinations of finite measures*; *therefore $P^1 * -$ distributes on affine combinations of probabilities.*

Theorem 8.5. *Assume that $\mathcal{A}$ is an IFS of strict contractions with attractor K. Let $r = \max r_a$, and let $P^1 = (p_0, \ldots, p_{m-1})$ be supported on the free generators. Then $P^1 * -$ is r-Lipschitz on* *the compact space $(\mathcal{P}(K), d)$, and therefore has a unique fixed point μ, that can be obtained as*

$$\mu = \lim_{n\to\infty} P^n * \mu_0 = \lim_{n\to\infty} \sum_{|w|=n} p_w \cdot (A_w)_* \mu_0,$$

for any $\mu_0 \in \mathcal{P}(K)$. Let P be the product measure on m^ω with weights $(p_0, \dots, p_{m-1})$; then $\mu = \pi_* P$, and its support is K. The map $P^1 \mapsto \mu$ is Lipschitz, even for graph-directed IFS (see Rauzy fractals of random substitutions *by Gohlke at al. on ArXiv, and references therein).*

9. The dimension of a measure

Let $\mu \in \mathcal{M}_{[0,\infty)}(\mathbb{R}^d)$. The *Hausdorff dimension* of μ is

$$\dim_H \mu = \inf\{\dim_H Y : Y \text{ is a Borel set of full measure}\}.$$

Given $x \in \mathbb{R}^d$ and $s \in \mathbb{R}_{\geq 0}$, the *upper s-dimensional density of μ at x* is

$$\overline{\theta}(\mu, x, s) = \operatorname*{limsup}_{\delta\to 0} \frac{\mu(x_{\leq\delta})}{\gamma(s)\,\delta^s}.$$

The *lower* one $\underline{\theta}(\mu, x, s)$ is defined by replacing limsup with liminf, and the usual jarcon apply. Replacing μ with a positive multiple, and dropping $\gamma(s)$, do not change the three basic cases of $\overline{\theta}(\mu, x, s)$ being 0, in $\mathbb{R}_{>0}$, or ∞.

The main example is the *Lebesgue density theorem.*

Theorem 9.1. *Let $A \subseteq \mathbb{R}^d$ be Lebesgue measurable with $\lambda^d(A) > 0$. Then for λ^d-all $x \in \mathbb{R}^d$ we have*

$$\overline{\theta}(\lambda^d \llcorner A, x, d) = \underline{\theta}(\lambda^d \llcorner A, x, d) = \mathbb{1}_A(x).$$

In other words, for λ^d-all x,

$$\lim_{\delta\to 0} \frac{\lambda^d(A \cap x_{\leq\delta})}{\gamma(d)\delta^d} = \lim_{\delta\to 0} \lambda^d(A|x_{\leq\delta}) = \mathbb{1}_A(x).$$

A partial inversion appears in [Fed69, p. 181].

Theorem 9.2. *Let $\mu \in \mathcal{M}_{[0,\infty)}(\mathbb{R}^d)$ and let A be Borel.*

(1) *If $c_1 \leq \overline{\theta}(\mu, -, s)$ on A, then $c_1\mathcal{H}^s(A) \leq \mu(A)$.*
(2) *If $\overline{\theta}(\mu, -, s) \leq c_2$ on A, then $\mu(A) \leq c_2 2^s \mathcal{H}^s(A)$.*

In particular, if $\overline{\theta}(\mu, -, s)$ is bounded from below and above on A, then $\mu(A)$ equals 0, a finite number, or ∞ iff $\mathcal{H}^s(A)$ equals 0, a finite number, or ∞.

Lemma 9.3. (1) *If $s < t$ and $\overline{\theta}(\mu, x, t) < \infty$, then $\overline{\theta}(\mu, x, s) = 0$, and analogously for $\underline{\theta}$.*
(2) *We have the identities*

$$\sup\{s : \overline{\theta}(\mu, x, s) = 0\} = \inf\{s : \overline{\theta}(\mu, x, s) = \infty\} = \operatorname*{limsup}_{\delta\to 0} \frac{\log \mu(x_{\leq\delta})}{\log \delta},$$

$$\sup\{s : \underline{\theta}(\mu, x, s) = 0\} = \inf\{s : \underline{\theta}(\mu, x, s) = \infty\} = \operatorname*{liminf}_{\delta\to 0} \frac{\log \mu(x_{\leq\delta})}{\log \delta}.$$

They define the upper local dimension of μ at x, $\overline{\dim}(\mu, x)$*, and the lower one* $\underline{\dim}(\mu, x)$*. If they agree, the common value is written* $\dim(\mu, x)$.

A finite measure μ is *exact dimensional* if $\dim(\mu, x)$ exists and is constant μ-everywhere. For measures which are not exact dimensional, making global the local notions is intricate; for example, taking the μ-essential supremum/infimum of $\overline{\dim}(\mu, x)$ and $\underline{\dim}(\mu, x)$ results, a priori, in four distinct notions. By [You82, Theorem 4.4], if $K \subset \mathbb{R}^d$ is compact and $\mu \in \mathcal{P}(K)$ is exact dimensional then $\dim \mu = \dim_H \mu$ (and many more altenative definitions of the dimension of μ agree).

Example 9.4. Smaller $\dim(\mu, x)$ corresponds to more mass around x.

(1) If $x \notin \operatorname{supp}(\mu)$, then $\dim(\mu, x) = \infty$.
(2) $\dim(\delta_x, x) = 0$.
(3) Let $\mu = \delta_0 + (\lambda^d \llcorner 0_{<1}) \in \mathcal{M}_{[0,\infty)}(\mathbb{R}^d)$. Then $\dim(\mu, x)$ equals 0 at $x = 0$, equals ∞ for $x \notin 0_{\leq 1}$, and equals d otherwise.
(4) Lebesgue's theorem says that every $\lambda^d \llcorner A$ is exact dimensional.

Definition 9.5. Let (X, d) be a metric space, $\mu \in \mathcal{M}_{[0,\infty)}$. If there exist $0 < c_1 \leq c_2 < \infty$ and $s \in \mathbb{R}_{\geq 0}$ such that for every $x \in \operatorname{supp} \mu$ and every $0 < \delta \leq \delta(x)$ we have

$$c_1 \leq \frac{\mu(x_{\leq \delta})}{\delta^s} \leq c_2, \tag{9.1}$$

then we say that μ is *Ahlfors-regular*. This is stronger than exact dimensionality (multiply (9.1) by δ^s and take logarithms). Indeed, $\dim(\mu, x) = s$ means that for every $0 < \varepsilon$ and every $0 < \delta \leq \delta(x, \varepsilon)$, we have

$$\delta^{\varepsilon} \leq \frac{\mu(x_{\leq \delta})}{\delta^s} \leq \delta^{-\varepsilon}.$$

Theorem 9.6 ([FH09]). *For every conformal IFS on $\mathbb{R}^d$ and every power measure P on m^{ω}, the pushforward measure $\pi_* P$ is exact dimensional.*

Theorem 9.7 (Rapaport). *Let P be a probability on $\mathrm{GL}_{d+1} \mathbb{R}$ supported on m matrices. Assume that the group generated by these matrices contains a proximal element and does not preserve any finite union of proper subspaces. Then the* Furstenberg measure (*that is, the only probability on $\mathrm{P}^d \mathbb{R}$ which is fixed by P*) *is exact dimensional*

The relationship between the following local characteristics (when defined) is quite intricate, most intricate for non-conformal IFS.

- The local dimension of μ at $x = \pi(\mathbf{a})$.
- The Hölder exponent at x of a "repartition function" for μ.
- The ratio of the local entropy and the first Lyapunov exponent of $\mathcal{A}$ at x:

$$\lim_{t \to \infty} \frac{\log \mu\big(\pi[\mathbf{a} \restriction t]\big)}{\log \operatorname{diam}\big(\pi[\mathbf{a} \restriction t]\big)} = \lim_{t \to \infty} \frac{-\log \mu\big(\pi[\mathbf{a} \restriction t]\big)/t}{-\log \operatorname{diam}\big(\pi[\mathbf{a} \restriction t]\big)/t} = \frac{h_\mu(x)}{\chi_1^{\mathcal{A}}(x)}.$$

- $$\lim_{t \to \infty} \frac{d \, \log \mu\big(\pi[\mathbf{a} \restriction t]\big)}{\log \lambda\big(\pi[\mathbf{a} \restriction t]\big)}.$$

Example 9.8. Consider the devil's staircases of Example 6.2 with parameters r_0, r_1, p. By the Strong Law of Large Numbers the set N_p of all $\mathbf{a} \in 2^{\omega}$ such that 1 appears in $\mathbf{a}$ with frequency p

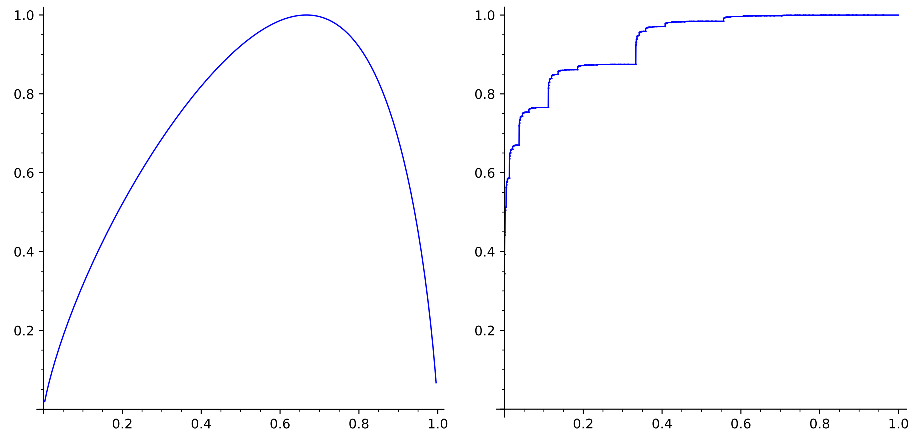

FIGURE 3. $\dim \mu$ as a function of p

has full P-measure. The set $\pi[N_p]$ is not closed, has full μ-measure, and its Hausdorff dimension equals $\dim \mu$. Given $x = \pi(\mathbf{a}) \in \pi[N_p]$, the above characteristics should agree, obtaining that

$$
\begin{aligned}
\lim_{t\to\infty} \frac{\log \mu\big([\mathbf{a} \upharpoonright t]\big)}{\log \lambda\big([\mathbf{a} \upharpoonright t]\big)} &= \lim_{t\to\infty} \frac{\log\big((1-p)^{(1-p)t} p^{pt}\big)}{\log\big(r_0^{(1-p)t} r_1^{pt}\big)} \\
&= \frac{(1-p)(-\log)(1-p) + p(-\log)(p)}{(1-p)(-\log)(r_0) + p(-\log)(r_1)} \\
&= \frac{h_P(S)}{\chi_1^{\mathcal{A}}(P)},
\end{aligned}
$$

equals $\dim \mu$. It should also equal the Hölder exponent of M at x, but this has to be checked.

For $r_0 = 2/5$, $r_1 = 3/5$, the graph of $\dim \mu$ as a function of p is in Figure 3 left. If $p = 1/8$, then $\dim \mu$ is small; the points of $\pi[N_{1/8}]$ correspond to the "vertical" segments in the repartition function in Figure 3 right.

10. THE SIMILARITY DIMENSION

In this section $\mathcal{A}$ and D are as in Definition 5.11.

Lemma 10.1. (i) $\mathcal{H}^D(K) < \infty$, *and hence* $\dim_H K \le D$.
(ii) *If K is an s-set, then $s = D$ iff for every $a \neq b$ we have $\mathcal{H}^s(K_a \cap K_b) = 0$.*

Definition 10.2. If the monoid homomorphism $w \mapsto A_w$ from $m^{<\omega}$ to $\operatorname{Aff} \mathbb{R}^d$ is not injective, then we say that $\mathcal{A}$ has *exact overlaps*. This means that there are different words u, v such that $A_u = A_v$. Replacing u, v with uv, vu, and observing that $A_u = A_v$ implies that u and v are incomparable, which implies $uv \neq vu$, we may assume $|u| = |v|$.

Theorem 10.3. *If $\mathcal{A}$ has exact overlaps, then* $\dim_H K < D$.

Proof. Let $u \neq v$, $|u| = |v| = n \geq 1$, $A_u = A_v$, and let $\mathcal{A}' = \mathcal{A}^n$. Then $\mathcal{A}$ and $\mathcal{A}'$ have the same attractor, and $\mathcal{A}'$ has cardinality strictly less than m^n. Therefore

$$\sum_{|w|=n} r_w^D = 1 \quad \text{and} \quad \sum_{B \in \mathcal{A}'} \big(r(B)\big)^D < 1.$$

Thus $\mathcal{A}'$ has similarity dimension $D' < D$, and $\dim_H K \leq D' < D$. □

Lemma 10.4. *Assume that $\mathcal{A}$ satisfies the OSC w.r.t. O.*

(i) *For every u, $K_u \subseteq O_u^f$.*
(ii) *If u and v are incomparable, then $O_u \cap O_v^f = \emptyset$.*

Proof. Every A_u is a homeomorphism of $\mathbb{R}^d$, and hence commutes with the topological operators. Thus (i) reduces to $K \subseteq O^f$. Let $x = \pi(\mathbf{a}) \in K$ and $y \in O$; then $x = \lim_n A_{\mathbf{a}\restriction n} y$. Since every $A_{\mathbf{a}\restriction n} y$ belongs to O, we have $x \in O^f$. Analogously, (ii) reduces to $O_{au'} \cap O_{bv'}^f = \emptyset$, for any $a \neq b$. Since $O_a \cap O_b = \emptyset$ and O_a^f is closed, we have $O_a \cap O_b^f = \emptyset$; as $O_{au'} \subseteq O_a$ and $O_{bv'}^f \subseteq O_b^f$, the conclusion follows. □

Theorem 10.5. *Assume that $\mathcal{A}$ satisfies the OSC w.r.t. O, let $p_a = r_a^D$, let P be the product probability on m^ω with weights $(p_0, \ldots, p_{m-1})$, and let $\mu = \pi_* P$.*

(1) *There exist constants $0 < c_1 \leq c_2 < \infty$ such that, for every $x \in K$ and every $0 < \varepsilon \leq \varepsilon(x)$ we have*

$$c_1 \leq \frac{\mu(K \cap x_{\leq \varepsilon})}{\gamma(D) \varepsilon^D} \leq c_2.$$

(2) $\dim_H K = D \leq d$*, and actually K is a D-set.*
(3) $\pi_* P = \mathcal{H}^D(-|K)$*; this measure, as well as $\mathcal{H}^D \llcorner K$, has local dimension D in every point of K; in particular, it is exact-dimensional.*
(4) *If $D = d$, then $K = O^f$ and K has nonempty interior. Thus K is a regular closed set and replacing O with K^o the OSC is still satisfied.*

11. Bernoulli convolutions

Let $0 < r < 1$, $A_{-1}(x) = -1 + rx$, $A_1(x) = 1 + rx$, P the product probability on $\{-1, 1\}^\omega$ with weights $(1/2, 1/2)$. The interval $[-1/(1-r), 1/(1-r)]$ is mapped into itself by $\mathcal{A}$, so $K \subseteq [-1/(1-r), 1/(1-r)]$.

Given w of length n, one proves by induction that

$$A_w(x) = \left(\sum_{k=0}^{n-1} w_k r^k\right) + r^n x.$$

Therefore K is the limit in the Hausdorff metric of the sets $\mathcal{A}^n(0) = \{\sum w_k r^k : |w| = n\}$, while μ is the weak limit of

$$\mu_n = \frac{1}{2^n} \sum \{\delta_z : z \in \mathcal{A}^n(0)\} = \frac{\delta_{-1} + \delta_1}{2} * \frac{\delta_{-r} + \delta_r}{2} * \cdots * \frac{\delta_{-r^{n-1}} + \delta_{r^{n-1}}}{2}.$$

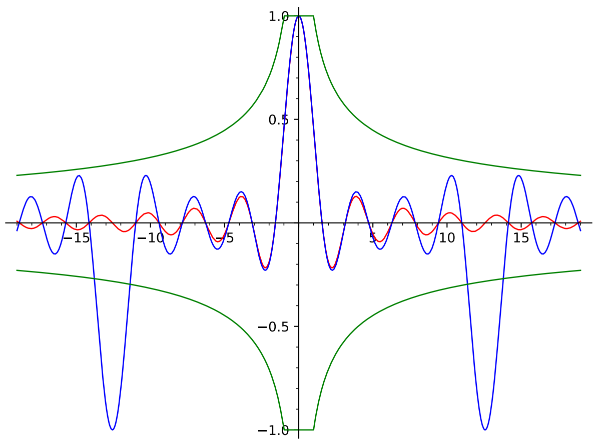

FIGURE 4. Convergence to $\lambda(-|[-2,2])$

Classically all of this is described by considering a sequence $X_0, X_1, \dots$ of independent random variable such that X_k takes values $\pm r^k$ with equal probability. Thus μ_n is the distribution of $X_0 + \cdots + X_{n-1}$ and, since $\varphi_{X_k}(\xi) = \cos(r^k\xi)$, we have

$$\hat{\mu}_n(\xi) = \cos(\xi) \cdot \cos(r\xi) \cdots \cos(r^{n-1}\xi).$$

- If $r \in (0, 1/2)$ then the OSC is satisfied and K and μ have both dimension $D = \log(2)/\log(r^{-1}) < 1$.
- If $r = 1/2$, the OCS is still satisfied, $K = [-2, 2]$ and $\mu = \lambda(-|[-2, 2])$; see Figure 4 for the graphs of $\hat{\mu}_3(\xi)$ and $\hat{\mu}(\xi) = \sin(2\xi)/(2\xi)$.
- If $r \in (1/2, 1)$ then the OSC is not satisfied (for otherwise K and μ would have dimension $D > 1$).

Definition 11.1. A *Pisot-Vijayaraghavan number* is an algebraic integer $\alpha \in \mathbb{R}_{>1}$ all of whose conjugates have modulus < 1.

The set of PV numbers is closed; its smallest element is the real root $1.32471\dots$ of $x^3 - x - 1$, and the second one $1.38027\dots$ one of the two real roots of $x^4 - x^3 - 1$. The smallest accumulation point is the golden ratio.

Theorem 11.2 (Erdös, 1939)**.** *Let $r \in (1/2, 1)$ be such that r^{-1} is PV. Then $\hat{\mu}(\xi)$ does not converge to 0 as $|\xi| \to \infty$; therefore, by the Riemann-Lebesgue lemma, μ is not absolutely continuous. This has been strenghtened by*

12. FRACTAL TRANSFORMATIONS

Lemma 12.1. *Let $\mathcal{A} = \{A_1, \dots, A_{m-1}\}$ be and IFS, not necessarily strictly contractive, but such that Theorem 1.15 holds. Then the topology of K is the final one, i.e., $Y \subseteq K$ is open iff $\pi^{-1}Y$ is open.*

Proof. One direction is clear. Since π is surjective we have $Y^c = \pi[\pi^{-1}Y^c]$. Therefore $\pi^{-1}Y$ open implies $\pi^{-1}Y^c$ closed compact, which implies $\pi[\pi^{-1}Y^c]$ compact closed, which implies Y open. □

Theorem 12.2. *Let $\mathcal{B} = \{B_1, \ldots, B_{m-1}\}$ be a second IFS as in Lemma 12.1. Assume that the fibers of $\pi_\mathcal{A}$ refine the fibers of $\pi_\mathcal{B}$ (that is, $\pi_\mathcal{A}(\mathbf{a}) = \pi_\mathcal{A}(\mathbf{b})$ implies $\pi_\mathcal{B}(\mathbf{a}) = \pi_\mathcal{B}(\mathbf{b})$).*

1. *The map $\Phi = \pi_\mathcal{B} \circ \pi_\mathcal{A}^{-1} : K_\mathcal{A} \to K_\mathcal{B}$ is a well defined continuous surjection.*
2. *If $\pi_\mathcal{A}$ and $\pi_\mathcal{B}$ have the same fibers, then Φ is a homeomorphism.*

Fix now $P \in \mathcal{P}(m^\omega)$, and let $\mu_\mathcal{A}$, $\mu_\mathcal{B}$ be the pushforward measures.

3. *$\Phi_* \mu_\mathcal{A} = \mu_\mathcal{B}$.*
4. *If $K_\mathcal{B} = [0, 1]$ and $\mu_\mathcal{B} = \lambda$, then Φ determines the repartition function M of $\mu_\mathcal{A}$ (distinguish Φ order-preserving of reversing).*

Proof. Clearly Φ is well defined and surjective. Let $Y \subseteq K_\mathcal{B}$ be open. Then $\pi_\mathcal{A}^{-1}[\Phi^{-1}Y] = (\Phi \circ \pi_\mathcal{A})^{-1}Y = \pi_\mathcal{B}^{-1}Y$ is open, and Lemma 12.1 yields that $\Phi^{-1}Y$ is open. This shows (1), and (2) follows.
(3) We have $\Phi_*\mu_\mathcal{A} = \Phi_*(\pi_\mathcal{A})_*P = (\pi_\mathcal{B})_*P = \mu_\mathcal{B}$. □

13. Inversive geometry and the extended Möbius group

Let $d = 1, 2, 3, \ldots$; the *d-dimensional inversive space* is the one-point compactification $\Pi^d = \mathbb{R}^d \cup \{\infty\}$ [Wil81]. A $(d-1)$*-dimensional generalized sphere* is:

- either an ordinary sphere $\Gamma = \{\|{-}-\mathbf{v}\|^2 - r^2 = 0\}$ of center $\mathbf{v} \in \mathbb{R}^d$ and radius $r > 0$;
- or a hyperplane $\Gamma = \{\langle -, \mathbf{w}\rangle + d = 0\} \cup \{\infty\}$ for some $\mathbf{w} \in \mathbb{R} \setminus \{0\}$.

Theorem 13.1. *The generalized spheres are the sets of the form*

$$\Gamma = \{a\|\mathbf{x}\|^2 + \langle \mathbf{w}, \mathbf{x}\rangle + d = 0\},$$

with $\|\mathbf{w}\|^2 - 4ad > 0$.

1. *$0 \in \Gamma$ iff $d = 0$.*
2. *$a \neq 0$ iff Γ is the sphere of center $\mathbf{v} = -\mathbf{w}/(2a)$ and square of radius*
$$r^2 = \frac{\|\mathbf{w}\|^2 - 4ad}{4a^2};$$
in this case Möbius inversion in Γ *is given by*
$$M(\mathbf{x}) = \mathbf{v} + \frac{r^2}{\|\mathbf{x} - \mathbf{v}\|^2}(\mathbf{x} - \mathbf{v}).$$
3. *$a = 0$ iff Γ is a plane; in this case $\mathbf{w} \neq 0$ and M is given by*
$$M(\mathbf{x}) = \mathbf{x} - 2\frac{\langle \mathbf{x}, \mathbf{w}\rangle + d}{\|\mathbf{w}\|^2}\mathbf{w}.$$

Theorem 13.2. *The inversion M in the ordinary sphere $\Gamma = \{\|{-}-\mathbf{v}\|^2 - r^2 = 0\}$ has the following properties.*

1. *Fixes (globally) spheres perpendicular to Γ, in particular planes by $\mathbf{v}$.*
2. *Exchanges spheres Δ by $\mathbf{v}$ with planes Δ' not by $\mathbf{v}$, fixing the unique ray from $\mathbf{v}$ perpendicular to both Δ and Δ'. Conversely, given a sphere Δ by $\mathbf{v}$ and a plane Δ' not by $\mathbf{v}$ for which there exists a ray as above, stereographic projection through $\mathbf{v}$ from Δ to Δ' is the restriction of inversion in an appropriate sphere of center $\mathbf{v}$.*
3. *Exchanges spheres not by $\mathbf{v}$ with other spheres not by $\mathbf{v}$.*

(4) *Preserves angles and cross-ratio*

$$(x_0, x_1; x_2, x_3) = \frac{\|x_0 - x_2\| \|x_1 - x_3\|}{\|x_0 - x_3\| \|x_1 - x_2\|}.$$

The *extended Möbius group* is the subgroup $\text{Möb}_d^{\pm}$ of all homeomorphisms of Π^d generated by inversions in generalized spheres, the *Möbius group* Möb_d being its index-2 subgroup of orientation preserving homeomorphisms. By definition, these are the automorphism groups of d-dimensional inversive geometry.

A more symmetric model is the unit sphere $\Sigma^d = S^d$ in $\mathbb{R}^{d+1}$, which is bijective to Π^d via stereographic projection through $(0^d, 1) \in \mathbb{R}^{d+1}$ or, better, inversion Q in the sphere of center $(0^d, 1)$ and radius $\sqrt{2}$. The two realizations of $\text{Möb}_d^{\pm}$ in Π^d and in Σ^d are conjugate by Q as follows: let Γ be an extended $(d-1)$-sphere in Π^d. Then there exist a unique extended d-sphere Δ in Π^{d+1} which intersects Π^d perpendicularly in Γ. Now, Q maps Δ to a d-sphere Δ' that intersects perpendicularly Σ^d in $\Gamma' = Q[\Gamma]$, and inversion of Σ^d in Γ' is just the restriction to Σ^d of inversion in Δ'.

14. The hyperbolic space and its isometries

The *Lorentz form* on $\mathbb{R}^{d+1}$ is $[\mathbf{x}, \mathbf{y}] = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1}$, and the *$d$-dimensional De Sitter space* is $\mathcal{S}^d = \{\mathbf{x} : [\mathbf{x}, \mathbf{x}] = 1\}$. Given $\mathbf{c} \in \Sigma^{d-1}$ and $0 < \theta < \pi$, the *cap* of center $\mathbf{c}$ and angular radius θ is $\{\mathbf{x} \in \Sigma^{d-1} : \langle \mathbf{x}, \mathbf{c} \rangle > \cos\theta\}$.

Lemma 14.1. *The space of all caps in Σ^{d-1} is homeomorphic to $\mathcal{S}^d$ via*

$$(\mathbf{c}, \theta) \mapsto (\mathbf{c}/\sin\theta, \cot\theta)$$

$$\big((w_1, \ldots, w_d)/\sqrt{1 + w_{d+1}}, \operatorname{arccot} w_{d+1}\big) \leftarrow\!\shortmid \mathbf{w}.$$

The cap corresponding to $\mathbf{w} \in \mathcal{S}^d$ is $\{\mathbf{x} \in \Sigma^{d-1} : [(\mathbf{x}, 1), \mathbf{w}] > 0\}$.

Theorem 14.2. *Let $\mathcal{H}^d$ be a cap in Π^d or in Σ^d, and let the* absolute *be its boundary. Given $x_1 \neq x_2 \in \mathcal{H}^d$, let x_-, x_+ be the points of perpendicular intersection of the absolute with the unique 1-sphere through x_1, x_2. Then the function*

$$d(x_1, x_2) = \log(x_-, x_+; x_2, x_1)$$

is a metric on $\mathcal{H}^d$. There is equality in the triangular inequality for x_1, x', x_2 iff x_1, x', x_2 lie in that order on a 1-sphere perpendicular to the absolute.

Every cap in Π^d or in Σ^d is a model for *d-dimensional hyperbolic space*. For every $1 \leq q < d$ the *q-planes* (*lines*, ...) in $\mathcal{H}^d$ are the intersections of $\mathcal{H}^d$ with q-spheres perpendicular to the absolute. Each such plane is uniquely determined either by $q + 1$ points in general position, of by a point and q linearly independent vectors in the tangent space at the point.

Theorem 14.3. *Given two sets $x_0, \ldots, x_d$ and $x'_0, \ldots, x'_d$ of $d + 1$ points in general position in $\mathcal{H}^d$, with distances ordinately equal, there is precisely one isometry that ordinately sends the first set to the second. This isometry is the product of at most $d + 1$ inversions in spheres perpendicular to the absolute. The group of isometries of $(\mathcal{H}^d, d)$ is the subgroup of $\text{Möb}_d^{\pm}$ generated by all such inversions, and equals the stabilizer of $\mathcal{H}^d$ inside $\text{Möb}_d^{\pm}$. Since the absolute is a $(d-1)$-dimensional sphere, $\text{Isom}(\mathcal{H}^d, d) \simeq \text{Möb}_{d-1}^{\pm}$.*

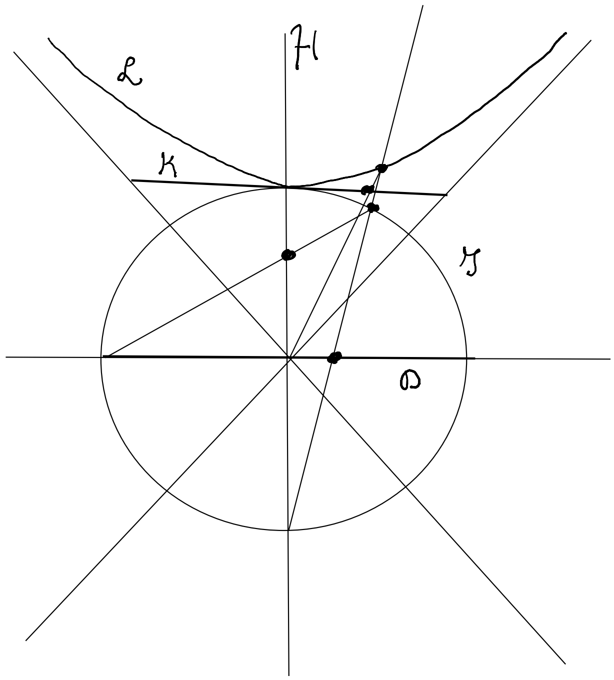

FIGURE 5. The five standard models

15. THE STANDARD MODELS

There are five standard models for $\mathcal{H}^d$, all sitting inside $\mathbb{R}^{d+1}$.

- the *hyperboloid model* $\mathcal{L}^d = \{\mathbf{x} : [\mathbf{x}, \mathbf{x}] = -1 \text{ and } x_{d+1} > 0\}$;
- the *Klein model* $\mathcal{K}^d = \{\mathbf{x} : [\mathbf{x}, \mathbf{x}] < 0 \text{ and } x_{d+1} = 1\}$;
- the *Poincaré model* $\mathcal{D}^d = \{\mathbf{x} : [\mathbf{x}, \mathbf{x}] < 1 \text{ and } x_{d+1} = 0\}$;
- the *upper hemisphere model* $\mathcal{J}^d = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{x} \rangle = 1 \text{ and } x_{d+1} > 0\}$;
- the *upper halfspace model* $\mathcal{H}^d = \{\mathbf{x} : x_1 = 0 \text{ and } x_{d+1} > 0\}$.

Lemma 15.1. (1) *$\mathcal{D}^d$, $\mathcal{J}^d$ and $\mathcal{L}^d$ are related by projection through $(0^d, -1)$, which is stereographic w.r.t. the first two. Analogously $\mathcal{J}^d$ and $\mathcal{H}^d$ are related by stereographic projection through $(-1, 0^d)$. These four models are conformal.*
(2) *$\mathcal{L}$ and $\mathcal{K}$ are related by projection through the origin.*

Given $A \in \mathrm{PSL}_2\,\mathbb{C}$, the *homography* and the *antihomography* determined by A are

$$(A, 0)(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad (A, 1)(z) = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}.$$

They form the group $\mathrm{PSL}_2\,\mathbb{C} \rtimes Z_2$, with composition

$$(A,0)(B,0) = (AB,0), \qquad (A,0)(B,1) = (AB,1),$$
$$(A,1)(B,0) = (AB,1), \qquad (A,1)(B,1) = (AB,0).$$

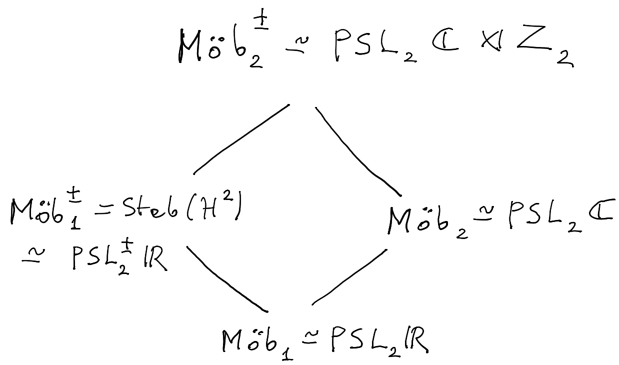

FIGURE 6. Hasse diagram

For example, inveersion in the circle of center α and radius r is

$$\left(\begin{bmatrix}1 & \alpha\\ & 1\end{bmatrix},0\right)\left(\begin{bmatrix}r & \\ & r^{-1}\end{bmatrix},0\right)\left(\begin{bmatrix} & i\\ i & \end{bmatrix},1\right)\left(\begin{bmatrix}1 & -\alpha\\ & 1\end{bmatrix},0\right)=\left(\frac{i}{r}\begin{bmatrix}\alpha & r^2-|\alpha|^2\\ 1 & -\bar{\alpha}\end{bmatrix},1\right).$$

Theorem 15.2. (1) *Identify* $\mathcal{H}^3$ *with* $\mathbb{C}\times\mathbb{R}_{>0}$, *with absolute* $\mathrm{P}^1\,\mathbb{C}$; *we then have* $\mathrm{Möb}_2^\pm \simeq \mathrm{PSL}_2\,\mathbb{C}\rtimes Z_2 = \{$*homographies and antihomographies*$\}$.

(2) *Every cap in the absolute is given by an inequality*

$$\begin{pmatrix}z\\1\end{pmatrix}^*\begin{pmatrix}a & \beta/2\\ \bar{\beta}/2 & d\end{pmatrix}\begin{pmatrix}z\\1\end{pmatrix} = a|z|^2+\mathrm{re}(\bar{\beta}z)+d<0$$

induced by a 2×2 *hermitian-symmetric matrix* H *of determinant* <0, *unique up to multiplication by a positive real number. If* $a=0$, *then the cap determined by* H *is an open halfplane. If* $a>0$, *it is the interior of the circle of center* $-\beta/(2a)$ *and square of radius* $r^2=\big(|\beta|^2-4ad\big)/(4a^2)$, *while if* $a<0$ *it is the exterior. In any case, the* A*-image of the cap determined by* H *is the cap determined by* $(A^{-1})^*HA^{-1}$.

(3) *Stereographic projection conjugates the action of* $\mathrm{SU}_2\,\mathbb{C}$ *on* $\mathrm{P}^1\,\mathbb{C}$ *with that of* $\mathrm{SO}_3\,\mathbb{R}$ *on* S^2. *This yields the adjoint representation of* $\mathrm{SU}_2\,\mathbb{C}$, *as well as the fact that action of* $\mathrm{PSL}_2\,\mathbb{C}$ *on* $\mathrm{P}^1\,\mathbb{C}$ *is transitive on caps.*

(4) *There are two natural choices*; *the first one is the upper-half plane* $\mathcal{H}^2$, *induced by* $H=\left(\begin{smallmatrix} & -i\\ i & \end{smallmatrix}\right)$, *and the second one the unit disk* $\mathcal{D}^2$, *induced by* $H=\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$. *The* Cayley matrix

$$C=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ -i & 1\end{bmatrix}\in\mathrm{PSU}_2\,\mathbb{C}$$

maps $\mathcal{H}^2$ *to* $\mathcal{D}^2$.

(5) *The stabilizer of* $\mathcal{D}^2$ *in* $\mathrm{PSL}_2\,\mathbb{C}$ *is* (*by definition*)

$$\mathrm{PSU}_{1,1}\,\mathbb{C}=\left\{\begin{bmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{bmatrix} : |\alpha|^2-|\beta|^2=1\right\}.$$

(6) *The stabilizer of* $\mathcal{H}^2$ *in* $\mathrm{PSL}_2\,\mathbb{C}$ *is* $C^{-1}(\mathrm{PSU}_{1,1}\,\mathbb{C})C=\mathrm{PSL}_2\,\mathbb{R}$, *and in* $\mathrm{PSL}_2\rtimes Z_2$ *is* $\mathrm{PSL}_2^\pm\,\mathbb{R}$.

16. Classification of elements of $\mathrm{PSL}_2\,\mathbb{C}$

Let $G \in \mathrm{PSL}_2\,\mathbb{C}$ be different from the identity; its trace t is determined up to sign, and changing that sign the roots ρ, ρ^{-1} of its characteristic polynomial $x^2 - tx + 1$ become $-\rho^{-1}, -\rho$. Without loss of generality we have $|\rho^{-1}| \le 1 \le |\rho|$.

- If $\rho = \pm 1$, then G is not diagonalizable (otherwise it would be the identity). Therefore it is conjugate to a translation; this case is characterized by $t = \pm 2$, and G is *parabolic*.
- If $\rho \in S^1 \setminus \{\pm 1\}$, then G is conjugate to a rotation, and it is *elliptic*. This case is characterized by $t \in (-2, 2)$.
- Otherwise, G is conjugate to $z \mapsto \rho^2 z$, for some ρ of modulus > 1. This is characterized by $t \notin [-2, 2]$ and G is *loxodromic*. In the subcase $\rho \in \mathbb{R} \setminus [-1, 1]$, characterized by $t \in \mathbb{R} \setminus [-2, 2]$, we say that G is *hyperbolic*.

Parabolic elements have exactly one fixed point in $\mathrm{P}^1\,\mathbb{C}$, while elliptic and loxodromic elements have two fixed points.

17. Group actions

Let $\alpha : G \times M \to M$ be a continuous group action, with M locally compact. If $(G, m) \mapsto (gm, m)$ is a proper map from $G \times M$ to $M \times M$, then α is a *proper* action; this is weaker than requiring that α is proper. Since the restriction of a proper map to a closed subset of the domain remains proper, if a group acts properly so does any of its closed subgroups. Looking at $\alpha^{-1}(m, m)$, we see that every $\mathrm{Stab}_G(m)$ has to be compact; looking at $\alpha[G \times \{m\}]$ we see that every G-orbit is closed. Under our standing topological assumptions (being Hausdorff and 1st countable), properness is equivalent to any of the following:

(b) for every compact $K \subset M$, the set $\{g : gK \cap K \neq \emptyset\}$ has compact closure;
(c) if (m_k) converges and (g_k) is such that $(g_k m_k)$ converges, then a subsequence of (g_k) converges.

Properness means that every K is moved away from itself by most of G (all of G, except for a compact). In particular, every action of a compact group is proper, Conversely, no action of a discrete infinite group on a compact space can be proper. We recall two topological facts:

(1) X is Hausdorff iff the diagonal is closed in X^2;
(2) if $F : X \to Y$ is continuous, surjective, and either open or closed, then for every $Z \subseteq Y$ we have that Z is open (respectively, closed) iff $F^{-1}Z$ is open (respectively, closed).

Lemma 17.1. *Let G act on M and endow $G\backslash M$ with the quotient topology induced by $\pi : M \to G\backslash M$. Then π is [continuous and] open. The three conditions*

(1) *$G\backslash M$ is Hausdorff*;
(2) *the orbit-equivalence relation $\sim$ is closed in M^2*;
(3) *the map $(g, m) \mapsto (gm, m)$ has closed image*;

are equivalent, and hold if the action is proper.

Proof. Let $O \subseteq M$ be open. Then $\pi\pi^{-1}O$ is the union of all G-translates of O, which is open; therefore π is open. The equivalence of the second and third condition is clear, because the image of the map is $\sim$. Now, the map $\pi^2 : M^2 \to (G\backslash M)^2$ is continuous, surjective and open, and moreover the π^2-counterimage of the diagonal in $(G\backslash M)^2$ is $\sim$ in M^2. Thus, the two topological fact above yield the equivalence of the first and second condition. Finally, if the action is proper then the third condition holds. ☐

Suppose $\mathfrak{G}$ is a Lie group acting on a smooth manifold M. If the action is proper and free, then $\mathfrak{G}\backslash M$ carries a natural manifold structure, and the projection $\pi : M \to \mathfrak{G}\backslash M$ is smooth and a submersion (that is, is surjective on tangent spaces). Moreover, $f : \mathfrak{G}\backslash M \to N$ is smooth iff $f \circ \pi$ is smooth.

For Lie groups acting on smooth manifolds properness is equivalent to compactness of $\{G : G[K] \cap K \neq \emptyset\}$, for every compact K (Lee, *Introduction to Smooth Manifolds*). A *properly dicontinuous* action is a proper action of a discrete countable group, but we try to avoid this terminology. See Kapovich, *A note on properly discontinuous actions*, for an attempt at clarifying a very confusing dictionary; see also Schwartz *A criterion for Hausdorff quotients*.

Note that (a_b, b_n) escapes to infinity iff at least one of a_n, b_n does.

Lemma 17.2. *Let K be a compact subgroup of G. Then G acts properly on G/K.*

Proof. First the case $K = 1$, that is, every group (locally compact by hypothesis) acts properly on itself. Suppose that $(g_n h_n, h_n)$ converges to (t, h). Then $g_n h_n \to t$ and $h_n^{-1} \to h^{-1}$, so that $g_n \to th^{-1}$. Therefore (g_n, h_n) converges to (th^{-1}, h), which is in the fiber of (t, h).

In the general case, asssume that $(g_n, h_n K)$ escapes to infinity; we must show that $(g_n h_n K, h_n K)$ escapes. If $h_n K$ escapes, we are through. Otherwise, g_n escapes and $h_n K$ does not. Since $G \to G/K$ is proper, we know that h_n does not escape. As G acts properly on itself, $g_n h_n$ escapes, and therefore so does $g_n h_n K$. ☐

18. Kleinian and fuchsian groups

Lemma 18.1. *Let Γ be a discrete group acting* **by isometries** *on the metric locally compact space M; the following conditions are equivalent.*

1. *The action is proper.*
2. *For every x and every compact K, the set $\{G \in \Gamma : Gx \in K\}$ is finite.*
3. *Every point has discrete orbit and finite stabilizer.*

Example 18.2. Consider the action of $\mathbb{Z}$ on $M = \mathbb{R}^2 \setminus \{0\}$ given by $g_n = \mathrm{diag}(2, 2^{-1})$. By looking at the images of the unit circle we see that it is not proper. Alternatively, let $m = (0, 1)$, $m_n = (2^{-n}, 1)$, $m' = (1, 0)$; then (m_n) and $(g_n m_n)$ are both convergent, buth (g_n) has no convergent subsequence.

However condition (3) holds (in a stronger form, as it is a free action). Note that the quotient space $\mathbb{Z}\backslash M$ is not Hausdorff. Indeed $Gm \neq Gm'$, but for every two neighborhoods $O \ni m$, $O' \ni m'$ we have $GO \cap O' \neq \emptyset$; therefore $\sim$ is not closed in M^2. The sequence Gm_n has both Gm and Gm' as limit points.

Definition 18.3. An *extended klenian group* is a discrete subgroup $\Gamma^{\pm}$ of $\mathrm{M\ddot{o}b}_2^{\pm}$ that acts in a proper way on $\mathcal{H}^3$. Since the action of the full group is proper, the condition is equivalent to $\Gamma^{\pm}$ being a discrete subgroup. $\Gamma^{\pm}$ is said to be *extended fuchsian* if it fixes a cap: it is then a discrete subgroup of $\mathrm{M\ddot{o}b}_1^{\pm}$. Removing $^{\pm}$ we have kleinian/fuchsian groups. The quotient space $\Gamma/\mathcal{H}^3$ is a 3-dimensional hyperbolic orbifold, which is a manifold iff Γ is torsion-free. In the fuchsian case $\Gamma/\mathcal{H}^2$ is a 2-dimensional hyperbolic orbifold, which is a hyperbolic surface iff Γ is torsion-free.

Note that the action of $\Gamma^{\pm}$ on $\partial\mathcal{H}^3$ (respectively, $\partial\mathcal{H}^2$) is *not* proper.

Example 18.4. (1) The *modular group* $\mathrm{PSL}_2\,\mathbb{Z}$ and its *principal congruence subgroups* $\Gamma(l) = \{G \in \mathrm{PSL}_2\,\mathbb{Z} : G \equiv I \pmod{l}\}$, which are normal subgroups of finite index. If $\Gamma(1) \geq \Gamma \geq \Gamma(l)$, with l minimum with this property, then Γ is a *congruence subgroup of level l*.

(2) The *Bianchi modular groups* $\mathrm{PSL}_2\,\mathbb{O}$, where $\mathbb{O}$ is the integer ring of an imaginary quadratic field K. Relevant cases are $K = \mathbb{Q}(\sqrt{-n})$ for $n \in \{1, 2, 3, 7, 11\}$ (the only ones for which $\mathbb{O}$ has an euclidean algorithm w.r.t. the norm); adding $n \in \{19, 43, 67, 163\}$ we get the only cases of a p.i.d. The case $\mathbb{O} = \mathbb{Z}[i]$ gives the *Picard modular group*.

(3) Fix $z_0 \in \mathrm{P}^1\,\mathbb{C}$, $g \geq 2$, and $2g$ caps $C_0, C_0', \ldots, C_{g-1}, C_{g-1}'$, pairwise nonintersecting and none of them cointaining z_0. For each $0 \leq k < g$, choose a loxodromic G_k (which is not unique) that maps the cap complementary to C_k onto C_k'. The group generated by $G_0, \ldots, G_{g-1}$ is kleinian and is called a *Schottky group* (around 1877).

A virtually abelian kleinian group is said to be *elementary*; **we will only deal with nonelementary —in particular, infinite— ones**.

The hyperbolic metric in $\mathcal{D}^3$ is different from the euclidean one, but they induce the same topology. Let Γ be kleinian; given $z \in \mathcal{D}^3$, the infinite orbit Γz must have accomulation points in $\mathcal{D}^3 \cup \partial\mathcal{D}^3$. Since Γz is discrete in $\mathcal{D}^3$, the set Λ of accumulation points is a nonempty subset of $\partial\mathcal{D}^3$; it is called the *limit set* of Γ, while $\Omega = \partial\mathcal{D}^3 \setminus \Lambda$ is the *regular set*, or *discontinuity domain* of Γ.

Lemma 18.5. (1) *Λ and Ω do not depend on the initial point z.*

(2) *Λ is closed and Γ-invariant. The action of Γ on Λ is by homeomorphisms, and is* minimal (*every point has a dense Γ-orbit*).

(3) *Ω is an open subset of $\partial\mathcal{D}^3$. If it is empty, then Γ is a kleinian group* of the first kind*. Otherwise, Γ is* of the second kind*, acts on Ω in a proper way, Γ/Ω is a 2-dimensional surface* (*or orbifold*?)*, and Λ is a perfect set with empty interior.*

(4) *If Γ fixes a Jordan closed curve $J \subset \partial\mathcal{D}^3$, then $\Lambda \subseteq J$ and Γ is* quasifuchsian.

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