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A note on generalized Fibonacci sequences

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ABSTRACT

Consider the generalized Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ having initial conditions $q_0 = 0$, $q_1 = 1$ and recurrence relation $q_n = aq_{n-1} + q_{n-2}$ (when n is even) or $q_n = bq_{n-1} + q_{n-2}$ (when n is odd), where a and b are nonzero real numbers. These sequences arise in a natural way in the study of continued fractions of quadratic irrationals and combinatorics on words or dynamical system theory. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. Pell's sequence is $\{q_n\}$ with $a = b = 2$ and the k-Fibonacci sequence is $\{q_n\}$ with $a = b = k$. In this article, we study numerous new properties of these sequences and investigate a sequence closely related to these sequences which can be regarded as a generalization of Lucas sequence of the first kind.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is a sequence of nonnegative integers starting with the integer pair 0 and 1, where $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$. The first few Fibonacci numbers are: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,... The Fibonacci sequence is perhaps one of the most well-known sequence and it has many interesting properties and important applications to diverse disciplines such as Mathematics, Statistics, Biology, Physics, Finance, Architecture, Computer Science, etc. For the history, properties, and rich applications of Fibonacci sequence and some of its variants, see [1–6,10,11,19].

Some authors [7–9,20] have generalized the Fibonacci sequence by preserving the recurrence relation and altering the first two terms of the sequence, while others [4,12–15,17,21] have generalized the Fibonacci sequence by preserving the first two terms of the sequence but altering the recurrence relation slightly. In [3], Edson and the author introduced and studied a new generalized Fibonacci sequence that depends on two real parameters used in a non-linear (piecewise linear) recurrence relation as defined below.

Definition 1. For any two nonzero real numbers a and b, the generalized Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ is defined recursively by

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}, \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \quad (n \geqslant 2).$$

It is not hard to see that when $a = b = 1$, we have the classical Fibonacci sequence and when $a = b = 2$, we get the Pell numbers. If we set $a = b = k$, for some positive integer k, we get the k-Fibonacci numbers. If $a = 1$ and $b = 2$, then members of the sequence $\{q_n\}$ are denominators of continued fraction convergents to $\sqrt{3}$ (see A002530 in [18]).

Edson and Yayenie has shown in [3] that the sequence $\{q_n\}$ is given by the extended Binet's Formula

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$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}}\right)\frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$, $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$, and $\xi(n) := n - 2\lfloor \frac{n}{2} \rfloor$. Note that α and β are roots of the quadratic equation $x^2 - abx - ab = 0$ and $\xi(m) = 0$ when m is even and $\xi(m) = 1$ when m is odd. The generalized Fibonacci sequences have word combinatorial interpretation and they are also closely related to continued fraction expansion of quadratic irrationals (see [3]).

In this article, we obtain numerous new identities of the generalized Fibonacci sequences and we investigate a much broader class of sequences which can be regarded as the generalization of many integer sequences, such as Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, etc. Furthermore, we will demonstrate that many of the properties of the Fibonacci sequence can be stated and proven for these new sequences.

2. Main results

2.1. New identities of generalized Fibonacci sequences

First we give numerous new identities of the generalized Fibonacci sequences.

Theorem 1. *For any nonnegative integer n, we have*

$$q_{n+6} = (ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + q_n.$$

Proof. One can use the extended Binet's formula to derive the above result. However, we prefer to use the recurrence relation as follows. For any positive integer $m \geqslant 2$, we have

$$q_{m+2} = (ab+2)q_m - q_{m-2}.$$

Hence,

$$\begin{aligned} q_{n+6} &= (ab+2)q_{n+4} - q_{n+2} = (ab+2)\left[a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + q_{n+2}\right] - q_{n+2} \\ &= (ab+2)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + (ab+1)q_{n+2} \\ &= (ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + (ab+1)q_{n+2} - a^{1-\xi(n)}b^{\xi(n)}q_{n+3} \\ &= (ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + (ab+1)q_{n+2} - a^{1-\xi(n)}b^{\xi(n)}\left[a^{\xi(n)}b^{1-\xi(n)}q_{n+2} + q_{n+1}\right] \\ &= (ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + (ab+1-ab)q_{n+2} - a^{\xi(n)}b^{1-\xi(n)}q_{n+1} \\ &= (ab+3)a^{1-\xi(n)}b^{\xi(n)}q_{n+3} + q_n. \qquad \square \end{aligned}$$

When $a = b = 1$ the above result reduces to a known identity of Fibonacci numbers

$$F_{n+6} = 4F_{n+3} + F_n.$$

Theorem 2. *For any positive integer m, we have*

$$q_m = a^{\xi(m-1)} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k}.$$

Proof. We will use the principle of mathematical induction to show the validity of the above formula. It is clear that the result is true when $m = 1$, since

$$q_1 = 1 = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-k-1}{k} (ab)^{\lfloor \frac{m-1}{2} \rfloor - k}.$$

Assume that it is true for any n such that $1 \leqslant n \leqslant m$. Then by the induction assumption and a known fact about binomial coefficients, we get

$$
\begin{aligned}
q_{m+1} &= a^{\xi(m)}b^{1-\xi(m)}q_m + q_{m-1} = a^{\xi(m)}b^{1-\xi(m)}a^{\xi(m-1)}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m-1}{2}\rfloor-k} + a^{\xi(m-2)}\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-k-2}{k}(ab)^{\lfloor\frac{m-2}{2}\rfloor-k} \\
&= ab^{\xi(m+1)}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m-1}{2}\rfloor-k} + a^{\xi(m)}\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-k-2}{k}(ab)^{\lfloor\frac{m-2}{2}\rfloor-k} \\
&= a^{\xi(m)}(ab)^{\xi(m+1)}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m-1}{2}\rfloor-k} + a^{\xi(m)}\sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-k-2}{k}(ab)^{\lfloor\frac{m-2}{2}\rfloor-k} \\
&= a^{\xi(m)}\left[\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m-1}{2}\rfloor-k+\xi(m-1)} + \sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-k-2}{k}(ab)^{\lfloor\frac{m-2}{2}\rfloor-k}\right] \\
&= a^{1-\xi(m+1)}\left[\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m}{2}\rfloor-k} + \sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-k-2}{k}(ab)^{\lfloor\frac{m}{2}\rfloor-k-1}\right] \\
&= a^{\xi(m)}\left[\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}(ab)^{\lfloor\frac{m}{2}\rfloor-k} + \sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}\binom{m-k-1}{k-1}(ab)^{\lfloor\frac{m}{2}\rfloor-k}\right] \\
&= a^{\xi(m)}\left((ab)^{\lfloor\frac{m}{2}\rfloor} + (1-\xi(m)) + \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\left[\binom{m-k-1}{k} + \binom{m-k-1}{k-1}\right](ab)^{\lfloor\frac{m}{2}\rfloor-k}\right) \\
&= a^{\xi(m)}\left((ab)^{\lfloor\frac{m}{2}\rfloor} + (1-\xi(m)) + \sum_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k}{k}(ab)^{\lfloor\frac{m}{2}\rfloor-k}\right) = a^{\xi(m)}\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m-k}{k}(ab)^{\lfloor\frac{m}{2}\rfloor-k}.
\end{aligned}
$$

Thus, the given formula is true for any positive integer m. □

When $a = b = 1$ the above result reduces to a known identity of Fibonacci numbers

$$F_m = \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-k-1}{k}$$

which shows relationship between Fibonacci numbers and Pascal's triangle (due to Lucas).

The following properties of α and β are extremely useful to follow the proofs of some of the results presented in the current section:

$$(\alpha+1)(\beta+1) = 1, \quad \alpha+\beta = ab, \quad \alpha\cdot\beta = -ab, \quad ab(\alpha+1) = \alpha^2, \quad -\beta(\alpha+1) = \alpha.$$

Theorem 3. *For any two positive integers m and n, we have*

$$q_{m+n-1} = a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}q_m q_n + a^{-\xi(mn)}b^{\xi(mn)}q_{m-1}q_{n-1}. \tag{1}$$

Proof. We will prove the above result using the extended Binet's formula. First, note that $\xi(m+n) = \xi(m) + \xi(n) - 2\xi(m)\xi(n)$ and

$$
\begin{aligned}
a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}q_m q_n &= \frac{a^{1-\xi(m)-\xi(n)+\xi(mn+n-m)}b^{1-\xi(mn+n-m)}}{(ab)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}}\,\frac{\alpha^m-\beta^m}{\alpha-\beta}\,\frac{\alpha^n-\beta^n}{\alpha-\beta} \\
&= \frac{a^{1-\xi(m)\xi(n)}b^{\xi(m+1)\xi(n+1)}}{(ab)^{\lfloor\frac{m}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}}\,\frac{\alpha^{m+n}+\beta^{m+n}-\alpha^n\beta^m-\alpha^m\beta^n}{(\alpha-\beta)^2} \\
&= \frac{a^{1+\xi(m+n)/2}b^{1-\xi(m+n)/2}}{(ab)^{\frac{m+n}{2}}}\,\frac{\alpha^{m+n}+\beta^{m+n}-\alpha^n\beta^m-\alpha^m\beta^n}{(\alpha-\beta)^2}.
\end{aligned}
$$

Similarly, one can show:

$$a^{-\xi(mn)}b^{\xi(mn)}q_{m-1}q_{n-1} = \frac{a^{1+\xi(m+n)/2}b^{1-\xi(m+n)/2}}{(ab)^{\frac{m+n-2}{2}}}\,\frac{\alpha^{m+n-2}+\beta^{m+n-2}-\alpha^{n-1}\beta^{m-1}-\alpha^{m-1}\beta^{n-1}}{(\alpha-\beta)^2}.$$

Therefore the right hand side (RHS) of (1) is given by

$$
\begin{aligned}
RHS &= \frac{a^{1+\xi(m+n)/2}b^{1-\xi(m+n)/2}}{(ab)^{\frac{m+n}{2}}}\frac{\alpha^{m+n-1}(\alpha+ab\alpha^{-1})+\beta^{m+n-1}(\beta+ab\beta^{-1})-[\alpha^n\beta^m+\alpha^m\beta^n](1+ab/\alpha\beta)}{(\alpha-\beta)^2}\\
&= \frac{a^{1+\xi(m+n)/2}b^{1-\xi(m+n)/2}}{(ab)^{\frac{m+n}{2}}}\frac{\alpha^{m+n-1}(\alpha-\beta)+\beta^{m+n-1}(\beta-\alpha)}{(\alpha-\beta)^2} = \frac{a^{1+\xi(m+n)/2}b^{1-\xi(m+n)/2}}{(ab)^{\frac{m+n}{2}}}\frac{\alpha^{m+n-1}-\beta^{m+n-1}}{\alpha-\beta}\\
&= \frac{a^{1+(\xi(m+n)-\xi(m+n-1)-1)/2}b^{1-(\xi(m+n)+\xi(m+n-1)+1)/2}}{(ab)^{\lfloor\frac{m+n-1}{2}\rfloor}}\frac{\alpha^{m+n-1}-\beta^{m+n-1}}{\alpha-\beta} = \frac{a^{1-\xi(m+n-1)}}{(ab)^{\lfloor\frac{m+n-1}{2}\rfloor}}\frac{\alpha^{m+n-1}-\beta^{m+n-1}}{\alpha-\beta} = q_{m+n-1}. \qquad \square
\end{aligned}
$$

When $a=b=1$ the above result reduces to a known identity of Fibonacci numbers

$$F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1}.$$

Theorem 4 (*Generalized Lucas Identity*). *For any positive integer $m \geqslant 2$, we have*

$$q_{m+1}q_m - q_{m-1}q_{m-2} = aq_{2m-1}.$$

Proof. Using Theorem 3, we get

$$
\begin{aligned}
aq_{2m-1} &= a^{\xi(m+1)}b^{1-\xi(m+1)}q_{m+1}q_{m-1} + a^{1-\xi(m+1)}b^{\xi(m+1)}q_mq_{m-2} = a^{1-\xi(m)}b^{\xi(m)}q_{m+1}q_{m-1} + a^{1-\xi(m+1)}b^{\xi(m+1)}q_mq_{m-2}\\
&= q_{m+1}\left(a^{1-\xi(m)}b^{\xi(m)}q_{m-1}\right) + a^{1-\xi(m+1)}b^{\xi(m+1)}q_mq_{m-2} = q_{m+1}(q_m - q_{m-2}) + a^{1-\xi(m+1)}b^{\xi(m+1)}q_mq_{m-2}\\
&= q_{m+1}q_m - q_{m-2}\left(q_{m+1} - a^{1-\xi(m+1)}b^{\xi(m+1)}q_m\right) = q_{m+1}q_m - q_{m-2}q_{m-1}. \qquad \square
\end{aligned}
$$

When $a=b=1$ the above result reduces to a known identity of Fibonacci numbers (due to Lucas)

$$F_{2m-1} = F_{m+1}F_m - F_{m-1}F_{m-2}.$$

Remark. The above result can be obtained from the following identity, which is much more general and can be proved in a similar way:

$$q_{m+n-2} = a^{\xi(mn)-1}b^{-\xi(mn)}[q_mq_n - q_{m-2}q_{n-2}] \quad (m,\ n \geqslant 2).$$

Theorem 5 (*Generalized Catalan's Identity*). *For any positive integer m, we have*

$$q_m = \frac{a^{\xi(m+1)}}{2^{m-1}}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2k+1}(ab)^{\lfloor\frac{m-1}{2}\rfloor-k}(ab+4)^k.$$

The following lemma will be used to prove the above result.

Lemma 1. *If $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$, then for any nonnegative integer m, we have*

$$\alpha^m - \beta^m = \frac{\alpha-\beta}{2^{m-1}}\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m}{2j+1}(ab)^{m-j-1}(ab+4)^j.$$

Proof. Note that α and β are roots of the quadratic equation $x^2 - abx - ab = 0$. Since $2\alpha = ab + \sqrt{ab(ab+4)}$ and $2\beta = ab - \sqrt{ab(ab+4)}$, it is clear to see that

$$
\begin{aligned}
(2\alpha)^m &= \left(ab + \sqrt{ab(ab+4)}\right)^m = \sum_{k=0}^{m}\binom{m}{k}(ab)^{m-k/2}(ab+4)^{k/2},\\
(2\beta)^m &= \left(ab - \sqrt{ab(ab+4)}\right)^m = \sum_{k=0}^{m}\binom{m}{k}(-1)^k(ab)^{m-k/2}(ab+4)^{k/2}.
\end{aligned}
$$

Therefore, we get

$$(2\alpha)^m - (2\beta)^m = \sum_{k=0}^{m} \binom{m}{k} \left(1-(-1)^k\right)(ab)^{m-k/2}(ab+4)^{k/2},$$

$$2^m(\alpha^m - \beta^m) = 2\sqrt{ab(ab+4)} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} (ab)^{m-j-1}(ab+4)^j,$$

$$\alpha^m - \beta^m = \frac{\alpha-\beta}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1} (ab)^{m-j-1}(ab+4)^j.$$

Proof of Theorem 5. Using the formula for q_m and Lemma 1, we get

$$q_m = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right)\frac{\alpha^m-\beta^m}{\alpha-\beta} = \left(\frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor}}\right)\frac{1}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}(ab)^{m-j-1}(ab+4)^j$$

$$= \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}(ab)^{m-\lfloor \frac{m}{2} \rfloor-j-1}(ab+4)^j = \frac{a^{\xi(m+1)}}{2^{m-1}} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}(ab)^{\lfloor \frac{m-1}{2} \rfloor-j}(ab+4)^j. \qquad \square$$

When $a = b = 1$ the above result reduces to a known identity established by Catalan

$$F_m = \frac{1}{2^{m-1}} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2k+1} 5^k.$$

The following results are generalizations of the identities for Fibonacci numbers established by Rao (see [16]) except part (d). We believe that part (d) provides a some what new identity for Fibonacci sequence, due mainly to lack of references.

Theorem 6. *For any positive integer n, we have*

(a) $\sum_{j=1}^{2n} q_j q_{j+1} = \frac{1}{b}[q_{2n+1}^2 - 1]$.
(b) $q_n q_{n+2} = (\frac{a}{b})^{\xi(n+1)}[(\frac{b}{a})^{\xi(n)} q_{n+1}^2 + (-1)^{n+1}]$.
(c) $\sum_{j=1}^{2n} (\frac{a}{b})^{\xi(j)} q_j q_{j+2} = \frac{1}{b}[q_{2n+1}q_{2n+2} - a]$
(d) $\sum_{j=1}^{2n} q_j q_{j+3} = \frac{1}{b}[q_{2n+1}q_{2n+3} - (ab+1)]$.

Proof. We will prove part (a) and (d). To prove (a), first note that

$$q_j q_{j+1} = \frac{a}{(\alpha-\beta)^2 (ab)^{\lfloor \frac{j}{2} \rfloor + \lfloor \frac{j+1}{2} \rfloor}} \left[\alpha^{2j+1} + \beta^{2j+1} - ab(\alpha\beta)^j\right] = \frac{a}{(\alpha-\beta)^2 (ab)^j}\left[\alpha^{2j+1} + \beta^{2j+1} - ab(\alpha\beta)^j\right]$$

$$= \frac{a}{(\alpha-\beta)^2}\left[\alpha\left(\frac{\alpha^2}{ab}\right)^j + \beta\left(\frac{\beta^2}{ab}\right)^j - ab(-1)^j\right].$$

Therefore,

$$\sum_{j=1}^{2n} q_j q_{j+1} = \frac{a}{(\alpha-\beta)^2}\left[\alpha\sum_{j=1}^{2n}\left(\frac{\alpha^2}{ab}\right)^j + \beta\sum_{j=1}^{2n}\left(\frac{\beta^2}{ab}\right)^j - ab\sum_{j=1}^{2n}(-1)^j\right] = \frac{a}{(\alpha-\beta)^2}\left(\frac{\alpha^2}{ab}\left[\left(\frac{\alpha^2}{ab}\right)^{2n} - 1\right] - \frac{\beta^2}{ab}\left[\left(\frac{\beta^2}{ab}\right)^{2n} - 1\right]\right)$$

$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2} + \beta^{4n+2} - (ab)^{2n}(\alpha^2+\beta^2)\right] = \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2} + \beta^{4n+2}\right] - \frac{ab+2}{b(ab+4)}.$$

The right hand side of part (a) is given by

$$\frac{1}{b}[q_{2n+1}^2 - 1] = \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left(\alpha^{4n+2} + \beta^{4n+2} - 2(\alpha\beta)^{2n+1}\right) - \frac{1}{b}$$

$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2} + \beta^{4n+2}\right] - \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}2(-ab)^{2n+1} - \frac{1}{b}$$

$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2} + \beta^{4n+2}\right] + \frac{2}{b(ab+4)} - \frac{1}{b} = \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2} + \beta^{4n+2}\right] - \frac{ab+2}{b(ab+4)}.$$

This completes the proof of (a). Parts (b) and (c) can be shown in the same way. To show part (d), note that

$$q_j q_{j+3} = \frac{a}{(\alpha-\beta)^2}\left[\alpha\left(\frac{\alpha^2}{ab}\right)^{j+1} + \beta\left(\frac{\beta^2}{ab}\right)^{j+1} - ab(ab+3)(-1)^j\right].$$

Hence

$$\sum_{j=1}^{2n} q_j q_{j+3} = \frac{a}{(\alpha-\beta)^2}\left[\alpha\sum_{j=1}^{2n}\left(\frac{\alpha^2}{ab}\right)^{j+1} + \beta\sum_{j=1}^{2n}\left(\frac{\beta^2}{ab}\right)^{j+1} - ab(ab+3)\sum_{j=1}^{2n}(-1)^j\right]$$
$$= \frac{a}{(\alpha-\beta)^2}\left(\frac{\alpha^4}{(ab)^2}\left[\left(\frac{\alpha^2}{ab}\right)^{2n}-1\right] - \frac{\beta^4}{(ab)^2}\left[\left(\frac{\beta^2}{ab}\right)^{2n}-1\right]\right)$$
$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+2}}\left(\alpha^{4n+4}+\beta^{4n+4}-(ab)^{2n}(\alpha^4+\beta^4)\right) = \frac{a}{(\alpha-\beta)^2(ab)^{2n+2}}\left[\alpha^{4n+4}+\beta^{4n+4}\right] - \frac{ab(ab+4)+2}{b(ab+4)}.$$

The right hand side of part (d) is given by

$$\frac{1}{b}\left[q_{2n+1}q_{2n+3}-(ab+1)\right] = \frac{a}{(\alpha-\beta)^2(ab)^{2n+2}}\left[\alpha^{4n+4}+\beta^{4n+4}-(\alpha\beta)^{2n+1}(\alpha^2+\beta^2)\right] - \frac{ab+1}{b}$$
$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+2}}\left[\alpha^{4n+4}+\beta^{4n+4}\right] + \frac{(ab+2)}{b(ab+4)} - \frac{ab+1}{b}$$
$$= \frac{a}{(\alpha-\beta)^2(ab)^{2n+1}}\left[\alpha^{4n+2}+\beta^{4n+2}\right] - \frac{ab(ab+4)+2}{b(ab+4)}.$$

This completes the proof of (d). □

When $a = b = 1$ part (d) of the above result reduces to an identity for Fibonacci numbers given by

$$\sum_{j=1}^{2n} F_j F_{j+3} = F_{2n+1}F_{2n+3} - 2.$$

We do not think that this identity is new, however we could not find a reference for it.

2.2. Modified generalized Fibonacci sequences

In this section we introduce a new sequence that are obtained by modifying the recurrence relation of the generalized Fibonacci sequence while preserving the initial conditions. Unlike the variation discussed in many articles in the past (see [3,4,7–9,12–15,17,20,21]), this new generalization depends on four real parameters used in a non-linear recurrence relation as shown below. These sequences can be viewed as the generalization of many integer sequences, such as Fibonacci, Lucas, Pell, Pell–Lucas, Jacobsthal, etc. One of the main objective of this paper is to derive Binet's like formula for the terms of these sequences and in addition to demonstrate that many of the properties of the Fibonacci sequence can be stated and proven for these new sequences.

Definition 2. For any four real numbers a, b, c, and d, the generalized Fibonacci sequence $\{Q_n\}_{n=0}^{\infty}$ is defined recursively by

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_n = \begin{cases} aQ_{n-1} + cQ_{n-2}, & \text{if } n \text{ is even}, \\ bQ_{n-1} + dQ_{n-2}, & \text{if } n \text{ is odd}, \end{cases} \quad (n \geqslant 2).$$

2.2.1. Special cases

1. $c = d = 1 : \{Q_n\}$ is the generalized Fibonacci sequence $\{q_n\}$.
2. $a = b = k,\ c = d = 1 : \{Q_n\}$ is the k-Fibonacci sequence.
3. $a = d = 1,\ b = c = 2 : \{Q_n\}$ is A005824 (see [18]).
4. $a = 2,\ b = c = d = 1 : \{Q_n\}$ is A048788. The terms of the sequence $\{Q_n\}$ are numerators of continued fraction convergents to $\sqrt{3} - 1$ (see [18]).
5. $a = b = 1,\ c = d = 2 : \{Q_n\}$ is A001045 (Jacobsthal sequence). The terms of the sequence $\{Q_n\}$ are the number of ways to tile a $3 \times (n-1)$ rectangle with 1×1 and 2×2 square tiles (see [18]).
6. $a = 2,\ b = 2,\ c = 0,\ d = -2 : Q_{2m+j} = 2^m (j = 0, 1)$ for all $m \geqslant 1$.

Theorem 7 (*Generating Function of* $\{Q_n\}$). *The generating function of the sequence* $\{Q_n\}_{n=0}^{\infty}$ *is given by*

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x(1 + ax - cx^2)}{1 - (ab + c + d)x^2 + cdx^4}.$$

Proof. It is not hard to see that the sequence under consideration satisfies the identity:

$$Q_n = (ab + c + d)Q_{n-2} - cdQ_{n-4} \quad (n \geqslant 4).$$

Note that

$$\begin{aligned}\left(1-(ab+c+d)x^2+cdx^4\right)H(x) &= \sum_{n=0}^{\infty} Q_n x^n - (ab+c+d)\sum_{n=0}^{\infty} Q_n x^{n+2} + cd\sum_{n=0}^{\infty} Q_n x^{n+4} \\ &= \sum_{n=0}^{\infty} Q_n x^n - (ab+c+d)\sum_{n=2}^{\infty} Q_{n-2} x^{n} + cd\sum_{n=4}^{\infty} Q_{n-4} x^{n} \\ &= Q_0 + Q_1 x + (Q_2-(ab+c+d)Q_0)x^2 + (Q_3-(ab+c+d)Q_1)x^3 \\ &\quad + \sum_{n=4}^{\infty}(Q_n-(ab+c+d)Q_{n-2}+cdQ_{n-4})x^n.\end{aligned}$$

Since $Q_n = (ab+c+d)Q_{n-2} - cdQ_{n-4} (n \geqslant 4),\ Q_0 = 0,\ Q_1 = 1,\ Q_2 = a$, and $Q_3 = ab+d$, we have

$$\left(1-(ab+c+d)x^2+cdx^4\right)H(x) = x + ax^2 - cx^3.$$

Therefore,

$$H(x) = \sum_{n=0}^{\infty} Q_n x^n = \frac{x(1+ax-cx^2)}{1-(ab+c+d)x^2+cdx^4}. \qquad \square$$

When $c = d = 1$, the above result reduces to the Theorem 4 of [3].

Theorem 8. (Generalized Binet's Formula). *The mth term of the generalized Fibonacci sequence* $\{Q_n\}$ *is given by*

$$Q_m = \frac{a^{1-\xi(m)}}{(ab)^{\lfloor \frac{m}{2}\rfloor}}\left(\frac{\alpha^{\lfloor \frac{m}{2}\rfloor}(\alpha+d-c)^{m-\lfloor \frac{m}{2}\rfloor} - \beta^{\lfloor \frac{m}{2}\rfloor}(\beta+d-c)^{m-\lfloor \frac{m}{2}\rfloor}}{\alpha-\beta}\right),$$

where $\alpha = \frac{ab+c-d+\sqrt{(ab+c-d)^2+4abd}}{2}$ *and* $\beta = \frac{ab+c-d-\sqrt{(ab+c-d)^2+4abd}}{2}$.

When $c = d = 1$, the above formula reduces to the formula for q_m given in [3].

Proof. First note that α and β are roots of the quadratic equation

$$x^2 - (ab+c-d)x - abd = 0.$$

The following properties of α and β will be used throughout the proof.

(i) $\alpha + \beta = ab + c - d$.
(ii) $\alpha \cdot \beta = -abd$.
(iii) $(\alpha + d)(\beta + d) = cd$.
(iv) $ab(\beta + d) = \beta(\beta + d - c)$.
(v) $ab(\alpha + d) = \alpha(\alpha + d - c)$.

Since $\frac{\alpha+d}{cd}$ and $\frac{\beta+d}{cd}$ are roots of $cdx^2 - (ab+c+d)x + 1 = 0$, we can rewrite the generating function $H(x)$ by using the partial fractions decomposition as

$$H(x) = \frac{1}{cd(\alpha-\beta)}\left[\frac{a(\alpha+d)-c\alpha x}{x^2-\frac{\alpha+d}{cd}} - \frac{a(\beta+d)-c\beta x}{x^2-\frac{\beta+d}{cd}}\right].$$

If we let

$$H_0(x) = \sum_{n=0}^{\infty} Q_{2n}x^{2n} \quad \text{and} \quad H_1(x) = \sum_{n=0}^{\infty} Q_{2n+1}x^{2n+1},$$

then $H(x) = H_0(x) + H_1(x)$. Using the Maclaurin's series expansion

$$\frac{A+Bz}{z^2-C} = -\sum_{n=0}^{\infty} AC^{-n-1}z^{2n} - \sum_{n=0}^{\infty} BC^{-n-1}z^{2n+1}$$

and the identities mentioned above, we simplify both $H_0(x)$ and $H_1(x)$ as follows:

$$H_0(x) = \frac{-a}{\alpha-\beta}\sum_{n=0}^{\infty}\left[\left(\frac{\alpha+d}{cd}\right)^{-n} - \left(\frac{\beta+d}{cd}\right)^{-n}\right]x^{2n} = \frac{a}{\alpha-\beta}\sum_{n=0}^{\infty}[(\alpha+d)^n - (\beta+d)^n]x^{2n}$$
$$= \sum_{n=0}^{\infty}\left[\frac{a}{(ab)^n}\right]\frac{\alpha^n(\alpha+d-c)^n - \beta^n(\beta+d-c)^n}{\alpha-\beta}x^{2n},$$

$$H_1(x) = \frac{cx}{cd(\alpha-\beta)}\sum_{n=0}^{\infty}\left[\alpha\left(\frac{\alpha+d}{cd}\right)^{-n-1} - \beta\left(\frac{\beta+d}{cd}\right)^{-n-1}\right]x^{2n} = \frac{-c}{cd(\alpha-\beta)}\sum_{n=0}^{\infty}\left[\beta(\alpha+d)^{n+1} - \alpha(\beta+d)^{n+1}\right]x^{2n+1}$$
$$= \frac{-1}{(\alpha-\beta)}\sum_{n=0}^{\infty}[(\beta-ab)(\alpha+d)^n - (\alpha-ab)(\beta+d)^n]x^{2n+1}$$
$$= -\sum_{n=0}^{\infty}\left[\frac{1}{(ab)^n}\right]\frac{(\beta-ab)\alpha^n(\alpha+d-c)^n - (\alpha-ab)\beta^n(\beta+d-c)^n}{\alpha-\beta}x^{2n+1}$$
$$= \sum_{n=0}^{\infty}\left[\frac{1}{(ab)^n}\right]\frac{\alpha^n(\alpha+d-c)^{n+1} - \beta^n(\beta+d-c)^{n+1}}{\alpha-\beta}x^{2n+1}.$$

Therefore,

$$H(x) = \sum_{m=0}^{\infty}\left[\frac{a^{1-\xi(m)}}{(ab)^{\lfloor\frac{m}{2}\rfloor}}\right]\frac{\alpha^{\lfloor\frac{m}{2}\rfloor}(\alpha+d-c)^{m-\lfloor\frac{m}{2}\rfloor} - \beta^{\lfloor\frac{m}{2}\rfloor}(\beta+d-c)^{m-\lfloor\frac{m}{2}\rfloor}}{\alpha-\beta}x^m.$$

Thus,

$$Q_m = \frac{a^{1-\xi(m)}}{(ab)^{\lfloor\frac{m}{2}\rfloor}}\left(\frac{\alpha^{\lfloor\frac{m}{2}\rfloor}(\alpha+d-c)^{m-\lfloor\frac{m}{2}\rfloor} - \beta^{\lfloor\frac{m}{2}\rfloor}(\beta+d-c)^{m-\lfloor\frac{m}{2}\rfloor}}{\alpha-\beta}\right). \qquad \square$$

In the following theorem, we list a number of mathematical properties including generalizations of Cassini's, Catalan's and d'Ocagne's identities for the classical Fibonacci numbers. The proof of some of them are provided and the others can be proved in the same way.

Theorem 9. *Suppose that $d = c$. Then the sequence $\{Q_m\}$ satisfies the following identities:*

(a) Cassini's Identity:

$$\left(\frac{a}{b}\right)^{\xi(m+1)}Q_{m-1}Q_{m+1} = \left(\frac{a}{b}\right)^{\xi(m)}Q_m^2 - \left(\frac{a}{b}\right)(-d)^{m-1}.$$

(b) Catalan's Identity:

$$\left(\frac{a}{b}\right)^{\xi(m+r)-\xi(r)}Q_{m-r}Q_{m+r} - \left(\frac{a}{b}\right)^{\xi(m)-\xi(r)}Q_m^2 = (-1)^{m+1-r}d^{m-r}Q_r^2.$$

(c) Binomial Sum:

$$\sum_{k=0}^{m}\binom{m}{k}a^{\xi(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}d^{n-k}Q_k = Q_{2m}.$$

(d) More General Binomial Sum:

$$\sum_{k=0}^{m}\binom{m}{k}a^{\xi(k+r)}(ab)^{\lfloor\frac{k}{2}\rfloor+\xi(r)\xi(k)}d^{n-k}Q_{k+r} = a^{\xi(r)}Q_{2m+r}.$$

(e) d'Ocagne's Identity:

$$a^{\xi(mn+m)}b^{\xi(mn+n)}Q_mQ_{n+1} - a^{\xi(mn+n)}b^{\xi(mn+m)}Q_{m+1}Q_n = a^{\xi(m-n)}(-d)^nQ_{m-n}.$$

Proof (*a*). Use the extended Binet's formula (Theorem 8) to see that

$$Q_{m-1}Q_{m+1} = \frac{a^{2-\xi(m+1)}b^{\xi(m+1)}}{(\alpha-\beta)^2(ab)^m}\left[\alpha^{2m} + \beta^{2m} - (\alpha\beta)^{m-1}(\alpha^2+\beta^2)\right] = \frac{a^{2-\xi(m+1)}b^{\xi(m+1)}}{(\alpha-\beta)^2}\left[\frac{\alpha^{2m}+\beta^{2m}}{(ab)^m} - (-1)^{m-1}(ab+2d)d^{m-1}\right]$$

and

$$Q_m^2 = \frac{a^{2-\xi(m)}b^{\xi(m)}}{(\alpha-\beta)^2}\left[\frac{\alpha^{2m}+\beta^{2m}}{(ab)^m} - 2(-1)^m d^m\right].$$

Therefore,

$$\left(\frac{a}{b}\right)^{\xi(m+1)} Q_{m-1}Q_{m+1} = \frac{a^2}{(\alpha-\beta)^2}\left[\frac{\alpha^{2m}+\beta^{2m}}{(ab)^m} - (-1)^{m-1}(ab+2d)d^{m-1}\right]$$

and

$$\left(\frac{a}{b}\right)^{\xi(m)} Q_m^2 = \frac{a^2}{(\alpha-\beta)^2}\left[\frac{\alpha^{2m}+\beta^{2m}}{(ab)^m} - 2(-1)^m d^m\right].$$

Hence,

$$\left(\frac{a}{b}\right)^{\xi(m+1)} Q_{m-1}Q_{m+1} - \left(\frac{a}{b}\right)^{\xi(m)} Q_m^2 = \frac{a^2}{(\alpha-\beta)^2}(-1)^m d^{m-1}(ab+4d) = \frac{a^2}{ab(ab+4d)}(-1)^m d^{m-1}(ab+4d) = \frac{a}{b}(-1)^m d^{m-1}.$$

This finalizes the proof of part (a). To prove (d), note that

$$a^{\xi(k+r)}(ab)^{\lfloor\frac{k}{2}\rfloor+\xi(r)\xi(k)} Q_{k+r} = \frac{a}{(ab)^{\lfloor\frac{r}{2}\rfloor}(\alpha-\beta)}\left[\alpha^{k+r}-\beta^{k+r}\right].$$

Hence

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k} a^{\xi(k+r)}(ab)^{\lfloor\frac{k}{2}\rfloor+\xi(r)\xi(k)} d^{n-k} Q_{k+r} &= \frac{a}{(ab)^{\lfloor\frac{r}{2}\rfloor}(\alpha-\beta)}\sum_{k=0}^{m}\binom{m}{k} d^{m-k}\left[\alpha^{k+r}-\beta^{k+r}\right]\\ &= \frac{a}{(ab)^{\lfloor\frac{r}{2}\rfloor}(\alpha-\beta)}\sum_{k=0}^{m}\binom{m}{k}\left[\alpha^{k+r}d^{m-k}-\beta^{k+r}d^{m-k}\right] = \frac{a}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\frac{\alpha^r(\alpha+d)^m-\beta^r(\beta+d)^m}{\alpha-\beta}\\ &= \frac{a}{(ab)^{\lfloor\frac{r}{2}\rfloor+m}}\frac{\alpha^{2m+r}-\beta^{2m+r}}{\alpha-\beta} = \frac{a}{(ab)^{\lfloor\frac{2m+r}{2}\rfloor}}\frac{\alpha^{2m+r}-\beta^{2m+r}}{\alpha-\beta} = a^{\xi(r)}Q_{2m+r}.\end{aligned}$$

This proves (d). The remaining parts can be proved in the same way. □

2.2.2. Open problem

The sequence $\{Q_n\}$ is eventually constant when $a = 1,\ b = 2,\ c = 0,\ d = -1$. For these choices, $Q_n = 1$ for all $n \geqslant 1$. In addition, if $a = 1,\ c = 0$, and $b + d = 1$, then $Q_n = 1$ for all $n \geqslant 1$ (this result was pointed to us by one of the referees). A necessary conditions on the parameters for the sequence to be eventually constant is that $a + c = 1$ and $b + d = 1$. Are there more general sufficiency conditions?

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