On convolved generalized Fibonacci and Lucas polynomials

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A R T I C L E   I N F O

Keywords:
Convolved $h(x)$-Fibonacci polynomials
$h(x)$-Fibonacci polynomials
$h(x)$-Lucas polynomials
Hessenberg matrices

A B S T R A C T

We define the convolved $h(x)$-Fibonacci polynomials as an extension of the classical convolved Fibonacci numbers. Then we give some combinatorial formulas involving the $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Moreover we obtain the convolved $h(x)$-Fibonacci polynomials from a family of Hessenberg matrices.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art (e.g., see [1]). The Fibonacci numbers $F_n$ are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$, wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature. In particular, a generalization is the $k$-Fibonacci Numbers.

For any positive real number $k$, the $k$-Fibonacci sequence, say $\{F_k(n)\}_{n \in \mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1$$

In [2], $k$-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [2–7].

The convolved Fibonacci numbers $F^{(r)}_j$ are defined by

$$(1 - x - x^2)^{-r} = \sum_{j=0}^{\infty} F^{(r)}_j x^j, \quad r \in \mathbb{Z}^+.$$

If $r = 1$ we have classical Fibonacci numbers. These numbers have been studied in several papers; see [8–10]. Convolved $k$-Fibonacci numbers have been studied in [11].

Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers [1]. Such polynomials, called Fibonacci polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geq 1.$$ (2)

The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = xJ_n(x) + J_{n-1}(x), \quad n \geq 1.$$ (3)

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \geq 1.$$ (4)

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0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.

http://dx.doi.org/10.1016/j.amc.2013.12.049
In [12], the authors introduced the \( h(x) \)-Fibonacci polynomials. That generalize Catalan’s Fibonacci polynomials \( F_{\Delta}(x) \) and the \( k \)-Fibonacci numbers \( F_{k,x} \). In this paper, we introduce the convolved \( h(x) \)-Fibonacci polynomials and we obtain new identities.

2. Some properties of \( h(x) \)-Fibonacci polynomials and \( h(x) \)-Lucas polynomials

**Definition 1.** Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Fibonacci polynomials \( \{F_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation

\[
F_{h,0}(x) = 0, \quad F_{h,1}(x) = 1, \quad F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geq 1. \tag{5}
\]

For \( h(x) = x \) we obtain Catalan’s Fibonacci polynomials, and for \( h(x) = k \) we obtain \( k \)-Fibonacci numbers. For \( k = 1 \) and \( k = 2 \) we obtain the usual Fibonacci numbers and the Pell numbers.

The characteristic equation associated with the recurrence relation (5) is \( \nu^2 = h(x) \nu + 1 \). The roots of this equation are

\[
r_1(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \quad r_2(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.
\]

Then we have the following basic identities:

\[
r_1(x) + r_2(x) = h(x), \quad r_1(x) - r_2(x) = \sqrt{h(x)^2 + 4}, \quad r_1(x)r_2(x) = -1. \tag{6}
\]

Some of the properties that the \( h(x) \)-Fibonacci polynomials verify are summarized below (see [12] for the proofs).

- Binet formula: \( F_{h,n}(x) = \frac{r_1(x)^n - r_2(x)^n}{2^{n-1}} \).
- Combinatorial formula: \( F_{h,n}(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1}{i} h^{n-1-2i}(x) \).
- Generating function: \( g_i(t) = \frac{1}{1 - r_1(x)t - r_2(x)t^2} \).

**Definition 2.** Let \( h(x) \) be a polynomial with real coefficients. The \( h(x) \)-Lucas polynomials \( \{L_{h,n}(x)\}_{n \in \mathbb{N}} \) are defined by the recurrence relation

\[
L_{h,0}(x) = 2, \quad L_{h,1}(x) = h(x), \quad L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geq 1. \tag{7}
\]

For \( h(x) = x \) we obtain the Lucas polynomials, and for \( h(x) = k \) we have the \( k \)-Lucas numbers. For \( k = 1 \) we obtain the usual Lucas numbers.

Some properties that the \( h(x) \)-Lucas numbers verify are summarized below (see [12] for the proofs).

- Binet formula: \( L_{h,n}(x) = r_1(x)^n + r_2(x)^n \).
- Relation with \( h(x) \)-Fibonacci polynomials: \( L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x), \quad n \geq 1 \).

3. Convolved \( h(x) \)-Fibonacci polynomials

**Definition 3.** The convolved \( h(x) \)-Fibonacci polynomials \( F_{h,j}^{(r)}(x) \) are defined by

\[
g_h^{(r)}(t) = (1 - h(x)t - t^2)^{-r} = \sum_{j=0}^{\infty} F_{h,j}^{(r)}(x)t^j, \quad r \in \mathbb{Z}^+.
\]

Note that

\[
F_{h,j+1}^{(r)}(x) = \sum_{j_1, j_2, \ldots, j_m} F_{h,j_1+1}(x)F_{h,j_2+1}(x) \cdots F_{h,j_m+1}(x). \tag{8}
\]

Moreover, using a result of Gould [13, p. 699] on Humbert polynomials (with \( n = j \), \( m = 2 \), \( x = h(x)/2 \), \( y = -1 \), \( p = -r \) and \( C = 1 \)), we have

\[
F_{h,j+1}^{(r)}(x) = \sum_{l=0}^{[r/2]} \binom{j + r - l - 1}{j - l} \binom{j - l}{l} h(x)^{l+2l}. \tag{9}
\]

If \( r = 1 \) we obtain the combinatorial formula of \( h(x) \)-Fibonacci polynomials. In Table 1 some polynomials of convolved \( h(x) \)-Fibonacci polynomials are provided. The purpose of this paper is to investigate the properties of these polynomials.
The following identities hold:

1. \( F_{n}^{(r)}(x) = rh(x) \).

2. \( F_{h,n}^{(r)}(x) = F_{h,n-1}^{(r-1)}(x) + h(x)F_{h,n-1}^{(r)}(x) + F_{h,n-2}^{(r)}(x), \quad n \geq 2. \)

3. \( nF_{h,n-1}^{(r)}(x) = r(h(x)F_{h,n-1}^{(r-1)}(x) + 2F_{h,n-1}^{(r)}(x)), \quad n \geq 1. \)

Proof.

1. Taking \( j = 1 \) in (9), we obtain

\[
F_{h,1}^{(r)}(x) = \begin{pmatrix} r \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} h(x) = rh(x).
\]

2. This identity is obtained from observing that

\[
\sum_{j=0}^{\infty} F_{h,j+1}^{(r)}(x)t^j = (h(x)t + t^2)\sum_{j=0}^{\infty} F_{h,j+1}^{(r)}(x)t^j + \sum_{j=0}^{\infty} F_{h,j}^{(r-1)}(x)t^j.
\]

3. Taking the first derivative of \( g_{h}^{(r)}(t) = (1 - h(x)t - t^2)^{-r} \), we obtain

\[
\left( g_{h}^{(r)}(t) \right)' = \sum_{j=1}^{\infty} F_{h,j+1}^{(r)}(x)t^{j-1} = r \left( 1 - h(x)t - t^2 \right)^{-r-1} \left( \frac{h(x) + 2t}{1 - h(x)t - t^2} \right) = r(h(x) + 2t)g_{h}^{(r+1)}(t).
\]

Therefore the identity is clear. \( \square \)

In the next theorem we show that the convolved \( h(x) \)-Fibonacci polynomials can be expressed in terms of \( h(x) \)-Fibonacci and \( h(x) \)-Lucas polynomials. This theorem generalizes Theorem 4 of [10] and Theorem 4 of [11].

Theorem 5. Let \( j \geq 0 \) and \( r \geq 1 \). We have

\[
F_{h,j+1}^{(r)}(x) = \sum_{l=\max(0, j-r+1)}^{\min(r, j)} \binom{r + l - 1}{l} \binom{r - l + j - 1}{j} \frac{1}{(h(x)^2 + 4)^{(r-j)/2}} L_{h,j-l+1}^{(r)}(x)
\]

\[
+ \sum_{l=\max(0, j-r+1)}^{\min(r, j)} \binom{r + l - 1}{l} \binom{r - l + j - 1}{j} \frac{1}{(h(x)^2 + 4)^{(r-j)/2}} F_{h,j-l+1}^{(r)}(x)
\]

Proof. Given \( \alpha, \beta \in \mathbb{C} \), such that \( \alpha \beta \neq 0 \) and \( \alpha \neq \beta \). Then we have the following partial fraction decomposition:

\[
(1 - \alpha z)^{-r}(1 - \beta z)^{-r} = \sum_{l=\max(0, j-r+1)}^{\min(r, j)} \binom{-r}{l} \frac{\alpha^l \beta^{j-l+1}}{(\alpha - \beta)^{j-l+1}} (1 - \alpha z)^{-l} + \sum_{l=\max(0, j-r+1)}^{\min(r, j)} \binom{-r}{l} \frac{\beta^l \alpha^{j-l+1}}{(\beta - \alpha)^{j-l+1}} (1 - \beta z)^{-l},
\]

where \( \binom{t}{0} = 1 \) and \( \binom{t}{l} = \binom{t-1}{l-1} \pm \binom{t-1}{l-1} \) with \( t \in \mathbb{R} \). Using the Taylor expansion

\[
(1 - z)^t = \sum_{j=0}^{\infty} (-1)^j \binom{t}{j} z^j
\]
Then \((1 - xz)^{-1}(1 - \beta z)^{-1} = \sum_{j=0}^{\infty} (j) z^j\), where
\[
\gamma(j) = \sum_{l=0}^{r-1} \left( \frac{r}{l} \right) \frac{\alpha^l}{(x - \alpha)^{r-l}} (-1)^l \binom{l}{j} \alpha^l + \sum_{l=0}^{r-1} \left( \frac{r}{l} \right) \frac{\beta^l}{(\beta - \alpha)^{r-l}} (-1)^l \binom{l}{j} \beta^l
\]
Note that \(1 - h(x)z - z^2 = (1 - r_1(x)z)(1 - r_2(x)z)\). On substituting these values of \(r = r_1(x)\) and \(\beta = r_2(x)\) and using the identities (6), we obtain
\[
F_{n+1}^{(h)}(x) = \sum_{l=0}^{r-1} \left( \frac{r}{l} \right) \frac{1}{(h(x)^2 + 4)^{r-l}z^2} (-1)^l \binom{l}{j} (1 - r)^l \binom{l}{j} \beta^l
\]
Since that \((-1)^l \binom{l}{j} = \binom{r + l - 1}{l - 1} \) and \((-1)^l \binom{l}{j} = \binom{r - l + j - 1}{j} \), then
\[
F_{n+1}^{(h)}(x) = \frac{(r + l - 1)}{(r - l + j - 1)} \frac{1}{(h(x)^2 + 4)^{r-l}z^2} \binom{r + l - 1}{l - 1} \binom{r - l + j - 1}{j} \beta^l
\]
From the above equality and Binet formula, we obtain the Eq. (10). □

4. Hessenberg matrices and convolved \(h(x)\)-Fibonacci polynomials

An upper Hessenberg matrix, \(A_n\), is an \(n \times n\) matrix, where \(a_{ij} = 0\) whenever \(i > j + 1\) and \(a_{i,j+1} \neq 0\) for some \(j\). That is, all entries bellow the superdiagonal are 0 but the matrix is not upper triangular:
\[
A_n = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\
0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\
0 & 0 & a_{4,3} & \cdots & a_{4,n-1} & a_{4,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & a_{n,n}
\end{pmatrix}
\]
(11)
We consider a type of upper Hessenberg matrix whose determinants are \(h(x)\)-Fibonacci numbers. Some results about Fibonacci numbers and Hessenberg can be found in [14]. The following known result about upper Hessenberg matrices will be used.

**Theorem 6.** Let \(a_1, p_{ij} (i \leq j)\) be arbitrary elements of a commutative ring \(R\), and let the sequence \(a_1, a_2, \ldots\) be defined by:
\[
a_{n+1} = \sum_{i=1}^{n} p_{i,n} a_i, \quad (n = 1, 2, \ldots).
\]
If
\[
A_n = \begin{pmatrix}
p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\
-1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\
0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\
0 & 0 & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & a_{n,n}
\end{pmatrix}
\]
then
\[
a_{n+1} = a_1 \det A_n.
\]
(12)
In particular, if
\[
F_n^{(h)} = \begin{pmatrix}
h(x) & 1 & 0 & \cdots & 0 & 0 \\
-1 & h(x) & 1 & \cdots & 0 & 0 \\
0 & -1 & h(x) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & h(x) & 1 \\
0 & 0 & 0 & \cdots & 0 & h(x)
\end{pmatrix}
\]
(13)
then from Theorem 6 we have that
\[
\det F_n^{(b)} = F_{h, n+1}(x), \quad (n = 1, 2, \ldots).
\]
(14)

It is clear that the principal minor \(M^{(b)}(i)\) of \(F_n^{(b)}\) is equal to \(F_{h,i}(x)F_{h,n-i+1}(x)\). It follows that the principal minor \(M^{(b)}(i_1, i_2, \ldots, i_l)\) of the matrix \(F_n^{(b)}\) is obtained by deleting rows and columns with indices \(1 \leq i_1 < i_2 < \cdots < i_l \leq n\):
\[
M^{(b)}(i_1, i_2, \ldots, i_l) = F_{h,i_1}(x)F_{h,i_2-i_1}(x) \cdots F_{h,n-i_l+1}(x)F_{h,n-i_l+1}(x).
\]
(15)

Then we have the following theorem.

**Theorem 7.** Let \(S_{n-l}^{(b)}(l = 0, 1, 2, \ldots, n-1)\) be the sum of all principal minors of \(F_n^{(b)}\) or order \(n - l\). Then
\[
S_{n-l}^{(b)} = \sum_{j_1+j_2+\cdots+j_l=n-l} F_{h,j_1+1}(x)F_{h,j_2+1}(x) \cdots F_{h,n-l+1}(x) = F_{h,n-l+1}(x).
\]
(16)

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

**Corollary 8.** The convolved \(h(x)\)-Fibonacci polynomials \(F_{h,n-1-l}^{(1-l)}(x)\) is equal, up to the sign, to the coefficient of \(t^l\) in the characteristic polynomial \(p_n(t)\) of \(F_n^{(b)}\).

**Corollary 9.** The following identity holds:
\[
F_{h,n-1-l}^{(1-l)}(x) = \sum_{l=0}^{\lfloor(n-l)/2\rfloor} \binom{n-i}{i} \binom{n-2i}{l} h(x)^{n-2i-l}.
\]

**Proof.** The characteristic matrix of \(F_n^{(b)}\) has the form
\[
\begin{pmatrix}
t - h(x) & 1 & 0 & \cdots & 0 & 0 \\
-1 & t - h(x) & 1 & \cdots & 0 & 0 \\
0 & -1 & t - h(x) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & t - h(x) & 1 \\
0 & 0 & 0 & \cdots & 1 & t - h(x)
\end{pmatrix}
\]
(17)

Then \(p_n(t) = F_{n+1}(t - h(x))\), where \(F_{n+1}(t)\) is a Fibonacci polynomial. Then from Corollary 8 and the following identity for Fibonacci polynomial [5]:
\[
F_{n-1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i},
\]
we obtain that
\[
F_{n-1}(t - h(x)) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \sum_{l=0}^{\lfloor n-2i/2 \rfloor} \binom{n-2i}{l} (-1)^{n-1} h(x)^{n-2i-l} t^l.
\]

Therefore the corollary is obtained. \(\square\)

**Acknowledgments**

The author would like to thank the anonymous referees for their helpful comments. The author was partially supported by Universidad Sergio Arboleda under Grant No. USA-II-2012-14.

**References**