Lyapunov methods in robustness—Part E

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A preliminary result. Consider the system

$$\dot{x}(t) = f(x(t), w(t)) + B(I + \Delta_B)u$$

where $\Delta_B(x(t), w(t))$ represents a matched uncertainty.

Proposition

Assume that $\Psi(x)$ is a control Lyapunov function with associated with

$$u = -\gamma B^T \nabla \Psi(x)^T$$

for the system with $\Delta_B = 0$. Then it is also a CLF for the uncertain system provided that

$$\Delta_B^T \Delta_B < \lambda^2 I, \quad \textit{with} \quad \lambda < 1.$$

$$\begin{aligned}
\dot{\Psi}(x) &= \nabla\Psi(x)f(x(t),w(t)) - \gamma\nabla\Psi(x)BB^T\nabla\Psi(x)^T \\
&- \gamma\nabla\Psi(x)B\Delta_B B^T\nabla\Psi(x)^T = \\
&= \nabla\Psi(x)f(x(t),w(t)) - (1-\lambda)\gamma\nabla\Psi(x)BB^T\nabla\Psi(x)^T \\
&\underbrace{-\gamma\nabla\Psi(x)B(\lambda I - \Delta_B)B^T\nabla\Psi(x)^T}_{\leq 0} \\
&\leq \nabla\Psi(x)f(x(t),w(t)) - \hat{\gamma}\nabla\Psi(x)BB^T\nabla\Psi(x)^T \leq -\phi(\|x\|)
\end{aligned}$$

where $\hat{\gamma} = (1-\lambda)\gamma$.
The prize to contrast Δ_B is to increase γ as $\gamma = \hat{\gamma}/(1-\lambda)$.

Robotic manipulators 3

A typical robotic manipulator has equation

$$M(q(t))\ddot{q}(t) + H(q(t), \dot{q}(t))\dot{q}(t) + G(q(t)) = \tau(t)$$

$M(q)$ nonsingular for all q.

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Procedure

- *Consider the control*

  $$\tau(t) = M(q)u(t) + H(q, \dot{q})\dot{q} + G(q)$$

  the resulting system is $M(q(t))[\ddot{q}(t) - u(t)] = 0$

- *Since* $M(q)$ *is invertible we get* m *decoupled equations*

  $$\ddot{q}_i(t) = u_i(t).$$

Problem: the model is uncertain, so we must consider an estimated model.

$$\tau = \tilde{M}(q)u(t) + \tilde{H}(q,\dot{q})\dot{q} + \tilde{G}(q)$$

$$\begin{aligned}\ddot{q}(t) &= u + \underbrace{M(q)^{-1}[\tilde{M}(q) - M(q)]}_{-I+\Delta_B} u(t) + \\ &+ \underbrace{M(q)^{-1}[\tilde{H}(q,\dot{q}) - H(q,\dot{q})]\dot{q} + M(q)^{-1}[\tilde{G}(q) - G(q)]}_{\Delta_f}\end{aligned}$$

$$\ddot{q}(t) = [I + \Delta_B]u(t) + \Delta_f$$

If $\tilde{M}(q)$ is a sufficiently accurate estimate of $M(q)$ we can assume that

$$\Delta_B = M^{-1}(\tilde{M} - M)$$

satisfies

$$\Delta_B^T \Delta_B < \lambda^2 I, \quad \text{with} \quad \lambda < 1.$$

along with

$$\|\Delta_f(q, \dot{q})\| \leq \alpha + \beta \|x\|.$$

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Robotic manipulators 6

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3. *Find a control* $u(t) = -\gamma B^T P x(t)$ *and the CLF* $x^T P x$,

$$QA^T + AQ + R^T B^T + BR = -S < 0, \quad P = Q^{-1}$$

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4. *Practically stabilize the actual system by taking* γ *large.*

First assume $\Delta_B = 0$. It can be shown that

$$\alpha \|B^T Sx\| - \gamma \|B^T Sx\|^2 \leq \frac{\alpha}{2\gamma}.$$

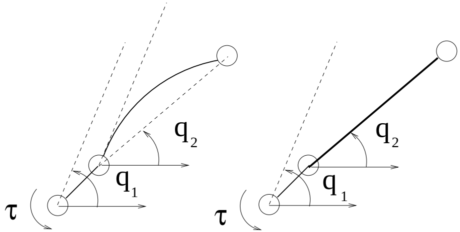
Fix $\varepsilon > 0$ γ_ε such that, for $x^T Px > \varepsilon$, $x^T Sx/2/ \geq \alpha/2\gamma_\varepsilon$. Let $\sigma_{min}(S) > 0$ the smallest eigenvalue of S

$$\begin{aligned}
\dot{\Psi}(x) &= -x^T Sx + x^T SB\Delta_f - \hat{\gamma} x^T SBB^T Sx \\
&\leq -x^T Sx + \alpha \|B^T Sx\| + \beta \|B^T Sx\| \|x\| - \hat{\gamma} \|B^T Sx\|^2 = \\
&= -x^T Sx/2/2 + \beta \|B^T Sx\| \|x\| - (\gamma - \gamma_\varepsilon) \|B^T Sx\|^2 \\
&+ \underbrace{-x^T Sx/2 + \alpha \|B^T Sx\| - \hat{\gamma}_\varepsilon \|B^T Sx\|^2}_{\leq 0} \\
&\leq -\sigma_{min}(S) \|x\|^2/2 + \beta \|B^T Sx\| \|x\| - (\hat{\gamma} - \gamma_\varepsilon) \|B^T Sx\|^2 < 0
\end{aligned}$$

For $\hat{\gamma}$ large enough. Finally accommodate Δ_B by taking $\gamma = \hat{\gamma}/(1 - \lambda)$.

The case with elastic joint Spong (1998) can be modeled as follows.

$$\begin{aligned} M(q_2)\ddot{q}_2 + C(\dot{q}_2, q_2)\dot{q}_2 + g(q_2) + K(q_1 - q_2) &= 0 \\ J\ddot{q}_1 + K(q_2 - q_1) &= \tau \end{aligned}$$



Strict feedback form: backstepping applies.

Switching and switched systems 1

A switching system is a system of the form

$$\dot{x}(t) = f(x(t), u(t), q(t))$$

with $q(t) \in \mathscr{Q}$, a finite set. Important cases:

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Uncontrolled switch – stability.

$$\dot{x}(t) = f(x(t), p(t))$$

We have to cope with the condition

$$\dot{\Psi}(x,p) = \nabla\Psi(x) f(x,p) \le -\phi(\|x\|). \quad \forall p \in \mathscr{P}$$

$\mathscr{P} = \{1, 2, \ldots, N_P\}$. The previous condition is equivalent to

$$\dot{\Psi}(x,\alpha) = \sum_{p=1}^{N_p} \alpha_p \;\; \nabla\Psi(x) f(x,p) \le -\phi(\|x\|)$$

In the linear case the stability of

$$\dot{x}(t) = A_p x(t), \quad \text{and of} \quad \dot{x}(t) = [\sum_{p=1}^{N_p} \alpha_p A_p] \; x(t)$$

are equivalent and they can be analyzed as previously described.

Switched systems – stabilizability.

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Proposition

Assume that there exist a system $\dot{x}(t) = \bar{f}(x)$ which is stable with Lyapunov function

$$\nabla\Psi(x)\bar{f}(x) \leq -\phi(\|x\|)$$

and such that $\bar{f}(x) \in conv\{f(x,p),\ p = 1, 2, \dots, N_p\}$, then there exists a stabilizing strategy of the form

$$p = \Phi(x) = \arg\min_p \ \nabla\Psi(x) f(x,p)$$

Switching and switched systems 3

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It is immediately seen that

$$\min_{p} \ \nabla\Psi(x)f(x,p) \leq \nabla\Psi(x)\bar{f}(x) \leq -\phi(\|x\|)$$

In the case of a switched linear plant

$$\dot{x}(t) = A_i x(t), \quad 1 = 1, 2, \ldots, r$$

the problem is easily solved if there exists

$$\tilde{A} \in conv\{A_i\}$$

which is stable. Take positive definite matrix P such that

$$\tilde{A}^T P + P\tilde{A} \leq -x^T Qx, \qquad Q > 0$$

Then the control law

$$\Phi(x) = \arg\min_i \; \dot{\Psi}(x) = \arg\min_i \; x^T PA_i x,$$

assures the condition

$$\dot{\Psi}(x) \leq -x^T Qx$$

Example

The existence of such a stable $\tilde{A}$ is not necessary.

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1+w & -a \end{bmatrix}$$

where $a < 0$ is not stable for any value of a. If $w \in \{-\bar{w}, \bar{w}\} < 1$ is a switched parameter and a is small enough, then there exists a stabilizing strategy.

Switching and switched systems 6

Example

There exists linear switched systems which can be stabilized by a control law of the form $i = i(x)$ but do not admit convex control–Lyapunov functions. For instance

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A_2 = \begin{bmatrix} \gamma & -1 \\ 1 & \gamma \end{bmatrix}$$

is an example (Blanchini and Savorgnan 2008).

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The problem is more difficult than the switching stability case!

Stabilization problem

$$\dot{x}(t) = A_i x(t) + B_i u(t)$$

If in the convex hull there exists a stabilizable systems

$$[\tilde{A}, \tilde{B}] = [\sum_{i=1}^{s} w_i A_i, \sum_{i=1}^{s} w_i B_i], \quad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0.$$

We can consider a stabilizing feedback $u = \tilde{K}x$, so that

$$\tilde{A} + \tilde{B}\tilde{K} = \sum_{i=1}^{s} w_i [A_i + \tilde{K} B_i] = \sum_{i=1}^{s} w_i [\hat{A}_i]$$

and we can subsequently apply the strategy to the switched system.

Switched fluid system 1

Example

Convergence speed-up.

$$\begin{aligned}\dot{h}_1(t) &= -\alpha\sqrt{x_1(t)+\bar{h}_1-x_2(t)-\bar{h}_2}+u(t)+\bar{q}\\ \dot{h}_2(t) &= \alpha\sqrt{x_1(t)+\bar{h}_1-x_2(t)-\bar{h}_2}-\beta\sqrt{x_2(t)+\bar{h}_2}\end{aligned}$$

The steady state conditions

$$\bar{h}_1=\left(\frac{\bar{q}}{\alpha}\right)^2+\left(\frac{\bar{q}}{\beta}\right)^2, \qquad \bar{h}_2=\left(\frac{\bar{q}}{\beta}\right)^2.$$

The system is stable. The problem is to improve convergence.

Candidate Lyapunov function

$$\Psi(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

The derivative is

$$\begin{aligned} & \dot{\Psi}(x,u) = \\ = \quad & \underbrace{\alpha(x_1 - x_2)\left(\sqrt{x_1(t) + \bar{h}_1 - x_2(t) - \bar{h}_2} - \bar{q}\right) + x_2\left(\beta\sqrt{x_2(t) + \bar{h}_2} - \bar{q}\right)}_{\doteq \dot{\Psi}_N(x)} \\ + \quad & x_1 u \leq \underbrace{\dot{\Psi}_N(x)}_{\leq -\phi_N(\|x\|)} + x_1 u \end{aligned}$$

Note that $\dot{\Psi}_N(x) \leq -\phi_N(x_1, x_2)$, the "natural derivative", is negative definite: the system is stable.

Control input admits three values

$$u(t) \in \{-\bar{q}; 0, \bar{q}\}$$

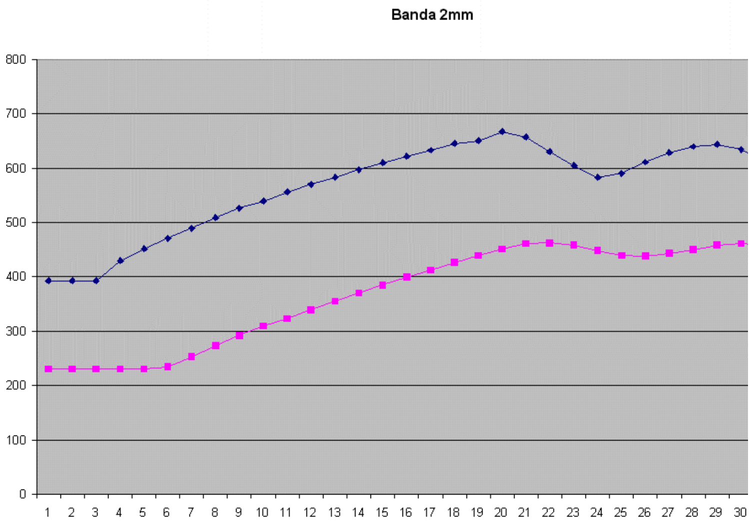
We can take the switching law that minimizes the derivative $\dot{\Psi}(x,u) = \dot{\Phi}_N(x_1,x_2) + x_1 u$

$$u = -\bar{q}\ sgn(x_1)$$

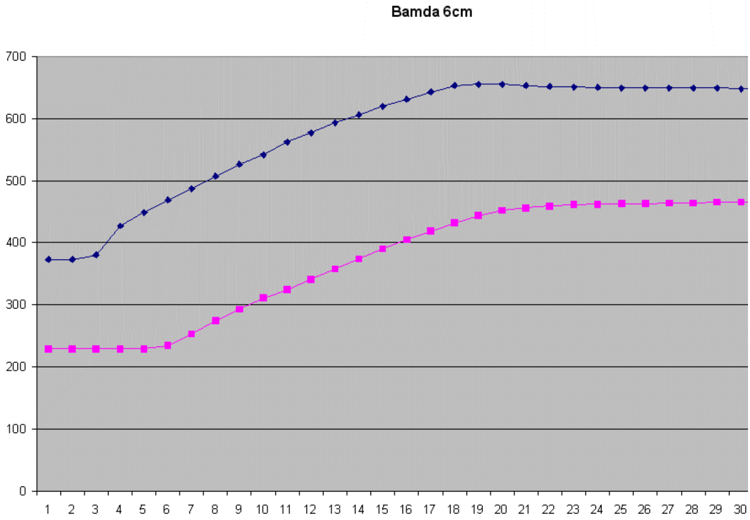
From a practical point of view, the controller has to implemented with a threshold.

$$u = \begin{cases} -\bar{q}\ sgn(x_1) & \text{if} \quad |x_1| > \varepsilon \\ 0 & \text{if} \quad |x_1| \leq \varepsilon \end{cases}$$

Switched fluid system 5



Switched fluid system 6



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- The theory basically works for state–feedback types of controls, the output feedback is still a very hard problem to be faced by means of these tools.
- The theory is conservative when we deal with constant uncertain parameter or slowly time–varying parameters.

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- The theory proposes techniques that are effective and insightful.
- For important classes of problems and special classes of functions the theory is supported by efficient numerical tools such as those based on LMIs.
- There is no valid alternative to Lyapunov theory for nonlinear uncertain systems.

Thanks

Merci!