Lyapunov methods in robustness—Part D

Franco Blanchini[1]

[1]Dipartimento di Matematica e Informatica
Universitá degli Studi di Udine

May 11, 2016

Definition

The system

$$\dot{x}(t) = f(x(t), u(t))$$

is absorbed in the system

$$\dot{x}(t) = F(x(t), u(t), w(t))$$

within domains $\mathscr{X}$ and $\mathscr{U}$, if for all $x \in \mathscr{X}$ and $u \in \mathscr{U}$

$$f(x, u) = F(x, u, w), \quad \text{for some} \quad w \in \mathscr{W}.$$

Controlling nonlinear systems 2

Consider the system

$$\dot{x}(t) = f(x(t)) + Bu(t) \qquad\qquad (NL)$$

and assume that

$$f(x) = A(w)x, \quad w = w(x), \qquad \text{with} \quad A(w) \in \mathscr{A}$$

Controlling nonlinear systems 2

Consider the system

$$\dot{x}(t) = f(x(t)) + Bu(t) \qquad\qquad (NL)$$

and assume that

$$f(x) = A(w)x, \quad w = w(x), \qquad \text{with} \quad A(w) \in \mathscr{A}$$

Fact

If $u = \Phi(x)$ *stabilizes*

$$\dot{x}(t) = A(w(t))x(t) + Bu(t)$$

for all $A(w(t)) \in \mathscr{A}$*, then it stabilizes (NL).*

Controlling nonlinear systems 2

Consider the system

$$\dot{x}(t) = f(x(t)) + Bu(t) \qquad\qquad (NL)$$

and assume that

$$f(x) = A(w)x, \quad w = w(x), \qquad \text{with} \quad A(w) \in \mathscr{A}$$

Fact

If $u = \Phi(x)$ stabilizes

$$\dot{x}(t) = A(w(t))x(t) + Bu(t)$$

for all $A(w(t)) \in \mathscr{A}$, then it stabilizes (NL).

Trading nonlinearity for linear uncertainty.

Controlling nonlinear systems 3

A simple case: $f(x) \in conv\{A_i\}\ x$

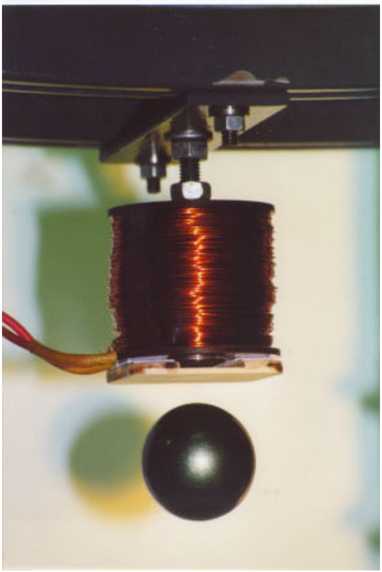
$$\mathscr{A} = \left\{ A(w) = \sum_{k=1}^{r} w_i A_i, \quad \sum_{k=1}^{r} w_i = 1, \quad w_i \geq 0 \right\}$$

$$\begin{aligned} f_i(x_1, x_2, \ldots, x_n) &= f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) + \int_{\bar{x}}^{x} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x_1, x_2, \ldots, x_n) dx_j \\ f_i(x_1, x_2, \ldots, x_n) &= \sum_{j=1}^{n} a_{ij}(x_1, x_2, \ldots, x_n)(x_j - \bar{x}_j) \end{aligned}$$

where a_{ij} is the "average". If we are able to provide bounds the absorbing system is

$$\dot{x}(t) = A(t)x(t) + Bu(t), \quad \text{with} \quad a_{ij}^{-} \leq a_{ij}(t) \leq a_{ij}^{+}$$

Controlling nonlinear systems 2



Controlling nonlinear systems 2

Example

$$\ddot{y}(t) = -k\frac{i(t)^2}{y(t)^2} + g = f(y(t), i(t))$$

We can write

$$f(y,u) = \int_{(\bar{y},\bar{i})}^{(y,i)} \left[\frac{2ki^2}{y^3}dy - \frac{2ki}{y^2}di\right]$$

$$0 < y^- \leq y \leq y^+, \quad 0 < i^- \leq i \leq i^+, \quad 0 < k^- \leq k \leq k^+.$$

The model becomes

$$\ddot{y}(t) = a(t)(y(t) - \bar{y}) - b(t)(i(t) - \bar{i})$$

$$\frac{2k^- {i^-}^2}{{y^+}^3} \leq a(t) \leq \frac{2k^+ {i^+}^2}{{y^-}^3}, \quad \frac{2k^- i^-}{{y^+}^2} \leq b(t) \leq \frac{2k^+ i^+}{{y^-}^2}$$

Observers for nonlinear systems 2

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

Observer of the form

$$\dot{z} = f(z(t)) + L(y - Cz) + Bu$$

Error equation

$$\dot{e} = [f(e+x) - f(x)] + LCe$$

If we can write

$$[f(e+x) - f(x)] = Ae, \quad A \in \mathscr{A},$$

with

$$\frac{\partial f}{\partial x} \in \mathscr{A}$$

Observers for nonlinear systems 3

We get

$$\dot{e} = (A(t) - LC)e, \qquad A(t) \in \mathscr{A}$$

Observers for nonlinear systems 3

We get

$$\dot{e} = (A(t) - LC)e, \qquad A(t) \in \mathscr{A}$$

This can be handled by means of LMIs if $\mathscr{A}$ has a suitable structure.

We get

$$\dot{e} = (A(t) - LC)e, \qquad A(t) \in \mathscr{A}$$

This can be handled by means of LMIs if $\mathscr{A}$ has a suitable structure.

Remark

This is the dual equation of

$$\dot{x}(t) = (A(t) - BK)x(t)$$

The substantial difference is that no uncertainties are tolerated in the original plant for the observer to work properly.

Observers for nonlinear systems 3

Interesting case

$$\begin{aligned}\dot{x}_1(t) &= f_{11}(x_1(t)) + f_{12}(x_1(t))x_2(t) + g_1(x_1(t))u(t)\\ \dot{x}_2(t) &= f_{21}(x_1(t)) + f_{22}(x_1(t))x_2(t) + g_2(x_1(t))u(t)\\ y(t) &= x_1(t)\end{aligned}$$

where $x_1 \in \mathbb{R}^{n_1}$ $x_2 \in \mathbb{R}^{n_2}$, $x_1 + x_2 = n$. To achieve a reduced–order observer

$$w(t) = x_2(t) - Lx_1(t)$$

$$\begin{aligned}\dot{w} &= (f_{22}(x_1) - Lf_{12})x_2 + f_{21}(x_1) - Lf_{11}(x_1) + (g_2(x_1) - Lg_1(x_1))u\\ &\pm (Lf_{12}(x_1)L - f_{22}(x_1)L)\\ &= (f_{22}(x_1) - Lf_{12}(x_1))w\\ &+ \underbrace{[f_{21}(x_1) - Lf_{11}(x_1) + f_{22}(x_1)L - Lf_{12}(x_1)L] + [g_2(x_1) - Lg_1(x_1)]u}_{\doteq \rho(x_1,u)}\end{aligned}$$

Observers for nonlinear systems 3

For the resulting system

$$\dot{w} = (f_{22}(x_1) - Lf_{12}(x_1))w + \rho(x_1, u)$$

we consider the reduced observer

$$\dot{z}_w = (f_{22}(x_1) - Lf_{12}(x_1))z_w + \rho(x_1, u)$$

The error is $e = z_w - w = z_2 - x_2$ and has equation

$$\dot{e} = (f_{22}(x_1) - Lf_{12}(x_1))e$$

Example

Consider the system (e.g. the levitator)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, u)\end{aligned}$$

and define $w = x_2 - Lx_1$. Then

$$\dot{w} = -Lw + f(x_1, u) - L^2 x_1$$

for which we can construct the observer

$$\dot{z}_w = -Lz_w + f(x_1, u) - L^2 x_1$$

The error equation is scalar

$$\dot{e} = -Le$$

Observers for nonlinear systems 5

Example

If there are error in the model $\dot{x}_2 = f + \Delta$, the equation becomes

$$\dot{e} = -Le - \Delta$$

If is bounded as $\|\Delta\| \leq \delta$ then by taking L large we can arbitrarily reduce the asymptotic error.

Observers for nonlinear systems 5

Example

If there are error in the model $\dot{x}_2 = f + \Delta$, the equation becomes

$$\dot{e} = -Le - \Delta$$

If is bounded as $\|\Delta\| \leq \delta$ then by taking L large we can arbitrarily reduce the asymptotic error.

Remark

When dealing with a nonlinear system it is often preferable to look to the specific problem than trying to involve general theories.

Domain of attraction 1

Consider the system

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

Domain of attraction 1

Consider the system

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

Definition

We say that $\mathscr{S}$ $(0 \in int\mathscr{S})$ is a domain of attraction iff for $x(0) \in \mathscr{S}$, $x(t) \in \mathscr{S}$, $t \geq 0$ and

$$\lim_{t \to \infty} x(t) = 0.$$

We call Domain of asymptotic stability the set $\mathscr{S}_{max}$ of all $x(0)$ for the condition holds.

Domain of attraction 1

Consider the system

$$\dot{x}(t) = f(x(t)), \quad f(0) = 0$$

Definition

We say that $\mathscr{S}$ $(0 \in int\mathscr{S})$ is a domain of attraction iff for $x(0) \in \mathscr{S}$, $x(t) \in \mathscr{S}$, $t \geq 0$ and

$$\lim_{t \to \infty} x(t) = 0.$$

We call Domain of asymptotic stability the set $\mathscr{S}_{max}$ of all $x(0)$ for the condition holds.

A survey: Genesio Tartaglia and Vicino (1985)

Domain of attraction 2

Given a local Lyapunov function $\Psi(x)$, there exists k such that $\mathcal{N}[\Psi, \kappa]$ is a domain of attraction.

$$\max\{\kappa : \mathcal{N}[\Psi(x), \kappa] \subset \mathcal{N}[\dot{\Psi}(x), 0]\}$$

Domain of attraction 2

Given a local Lyapunov function $\Psi(x)$, there exists k such that $\mathcal{N}[\Psi, \kappa]$ is a domain of attraction.

$$\max\{\kappa : \mathcal{N}[\Psi(x), \kappa] \subset \mathcal{N}[\dot{\Psi}(x), 0]\}$$

A possible way to compute k is

$$\kappa = \min\{\Psi(x) \geq \delta : \ \dot{\Psi}(x) = 0\} - \varepsilon$$

($\varepsilon > 0$ and $\delta > 0$ eliminate the trivial solutions).

Domain of attraction 2

Given a local Lyapunov function $\Psi(x)$, there exists k such that $\mathcal{N}[\Psi,\kappa]$ is a domain of attraction.

$$\max\{\kappa : \mathcal{N}[\Psi(x),\kappa] \subset \mathcal{N}[\dot{\Psi}(x),0]\}$$

A possible way to compute k is

$$\kappa = \min\{\Psi(x) \geq \delta : \ \dot{\Psi}(x) = 0\} - \varepsilon$$

($\varepsilon > 0$ and $\delta > 0$ eliminate the trivial solutions).

Under uncertainties, we can replace $\Psi(x)$ by

$$\dot{\Psi}_{max}(x) = \max_{w \in \mathcal{W}} \dot{\Psi}(x,w).$$

$$\kappa = \min\{\Psi(x) \geq \delta : \ \dot{\Psi}_{max}(x) = 0\} - \varepsilon$$

Domain of attraction 3

Example

$$\begin{aligned}\dot{x}_1(t) &= -[x_1(t) - x_1^3(t)] - x_2(t)\\ \dot{x}_2(t) &= x_1(t) - x_2(t)\end{aligned}$$

and the Lyapunov function

$$\Psi(x) = x_1^2 + x_2^2$$

so that

$$\dot{\Psi}(x) = -2x_1^2 - 2x_2^2 + 2x_1^4$$

We have $\varepsilon \to 0$ $\kappa \to 1$.

Domain of attraction 4

Example

Different approach: write the nonlinearity as

$$-[x_1(t) - x_1^3(t)] = -[1 - w]x_1, \quad \text{with} \quad w = x_1^2$$

Impose bounds

$$|x_1| \leq \bar{x}_1$$

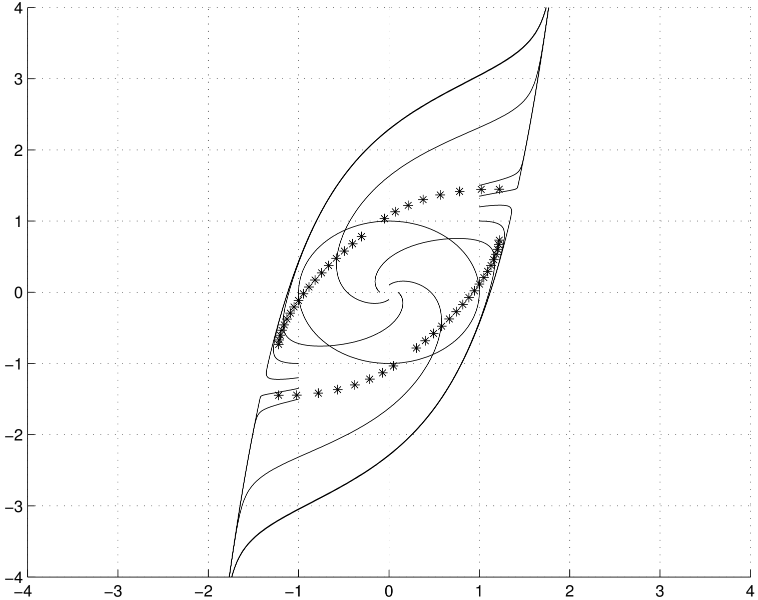
then

$$|w| \leq \bar{w} = \sqrt{\bar{x}_1}$$

We can consider the absorbing system $\dot{x} = A(w)x$ where

$$A(w) = \begin{bmatrix} -[1 - w] & -1 \\ 1 & -1 \end{bmatrix}$$

Domain of attraction 4



High gain adaptive control 1

Consider the uncertain system

$$\dot{x}(t) = f(x(t), w(t)) + Bu(t), \quad w(t) \in \mathcal{W}$$

assume that a smooth control Lyapunov function Ψ is given.
A suitable controller is

$$u(t) = -\gamma B^T \nabla \Psi(x) \tag{1}$$

with $\gamma > 0$ "large".

High gain adaptive control 1

Consider the uncertain system

$$\dot{x}(t) = f(x(t), w(t)) + Bu(t), \quad w(t) \in \mathcal{W}$$

assume that a smooth control Lyapunov function Ψ is given.
A suitable controller is

$$u(t) = -\gamma B^T \nabla \Psi(x) \tag{1}$$

with $\gamma > 0$ "large".

What does large mean?

High gain adaptive control 2

Consider the control

$$\begin{aligned} u(t) &= -\gamma(t) B^T \nabla \Psi(x) \\ \dot{\gamma}(t) &= \mu \sigma_\lambda(\Psi(x(t))) \\ \gamma(0) &= \gamma_0 \geq 0 \end{aligned}$$

$\sigma_\lambda(\xi)$, with $\xi \geq 0$ is the threshold function

$$\sigma_\lambda(\xi) = \begin{cases} 0 & \text{if} \quad 0 \leq \xi \leq \lambda \\ \xi - \lambda & \text{if} \quad \xi \geq \lambda \end{cases}$$

High gain adaptive control 2

Consider the control

$$\begin{aligned} u(t) &= -\gamma(t) B^T \nabla \Psi(x) \\ \dot{\gamma}(t) &= \mu \sigma_\lambda(\Psi(x(t))) \\ \gamma(0) &= \gamma_0 \geq 0 \end{aligned}$$

$\sigma_\lambda(\xi)$, with $\xi \geq 0$ is the threshold function

$$\sigma_\lambda(\xi) = \begin{cases} 0 & \text{if} \quad 0 \leq \xi \leq \lambda \\ \xi - \lambda & \text{if} \quad \xi \geq \lambda \end{cases}$$

Fact

No adaptation as soon as

$$\Psi(x(t)) \leq \lambda$$

where λ represents a small tolerance.

High gain adaptive control 3

Example

Consider the following system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= w(t)sin(x_1(t)) + u(t)\end{aligned}$$

with

$$|w| \leq \bar{w}$$

and the control–Lyapunov function $\Psi(x_1, x_2) = \frac{1}{2}\ x^T Px$ with

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

the corresponding control is

$$u(t) = -\gamma(t)B^T Px = -\gamma(t)(x_1 + 2x_2)$$

High gain adaptive control 4

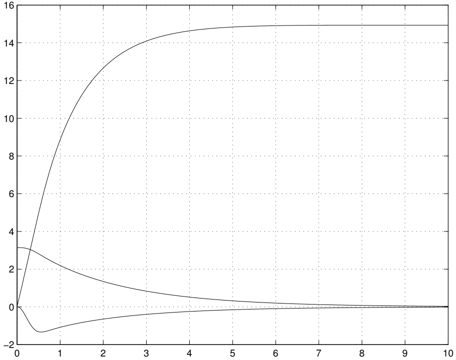
Example

$$
\begin{aligned}
\dot{\Psi}(x,\gamma) &= -\gamma x_1^2 + (2-4\gamma)x_1x_2 - (4\gamma-1)x_2^2 + (x_1+2x_2)wsin(x_1) \\
&\leq -\gamma x_1^2 + (2-4\gamma)|x_1||x_2| - (4\gamma-1)x_2^2 + \bar{w}x_1^2 + \bar{w}|x_1||x_2| \\
&= -(\gamma-\bar{w})x_1^2 + 2(1+\bar{w}-2\gamma)|x_1||x_2| - (4\gamma-1)x_2^2 < 0
\end{aligned}
$$

for $\gamma >$ large enough.
We selected $\mu = 1$ and $\lambda = 0.01$ and we achieved the limit value $\kappa_\infty = 14.93$.

Convergence proof are due to Ryan and Ilchman (1993)

High gain adaptive control 5



High gain adaptive control 5

Remark

The advantage of the scheme is

- *It solves the problem due to the ignorance of* $\bar{\gamma}$.

High gain adaptive control 5

Remark

The advantage of the scheme is

- *It solves the problem due to the ignorance of* $\bar{\gamma}$.
- *It avoids too large values of* γ *(the adaptation stops if* $\Psi(x) \leq \lambda$*).*

High gain adaptive control 5

Remark

The advantage of the scheme is

- *It solves the problem due to the ignorance of $\bar{\gamma}$.*
- *It avoids too large values of γ (the adaptation stops if $\Psi(x) \leq \lambda$).*
- *Avoids excessive control exploitation.*

High gain adaptive control 5

Remark

The advantage of the scheme is

- *It solves the problem due to the ignorance of* $\bar{\gamma}$.
- *It avoids too large values of* γ *(the adaptation stops if* $\Psi(x) \leq \lambda$*).*
- *Avoids excessive control exploitation.*
- *High values of* γ *can excite high–frequency neglected dynamics and cause instability.*

$$x(t) \in \mathscr{X}, \qquad u(t) \in \mathscr{U},$$

Consider the control–admissible set

$$\mathscr{X}_u = \{x : \ \Phi(x) \in \mathscr{U}\}$$

Consider the control Φ. The constraints are satisfied for

$$\mathscr{N}[\Psi, \kappa] \subseteq \mathscr{X}_u \bigcap \mathscr{X}$$

Constrained control 2

Case of linear systems

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

$$\begin{aligned} A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \end{aligned}$$

Assume that there exists a linear stabilizing controller $u = Kx$. The control admissible set is

$$\mathscr{X}_u = \{x : \quad \|Kx\|_\infty \leq 1\}$$

It is easy to compute an ellipsoidal set of the form

$$\mathscr{E} = \{x : \ x^T P x \leq 1\}$$

which is invariant and included in $\mathscr{X}_u$ (Boyd et al. (1994)).

Constrained control 3

Indeed denoting by $Q = P^{-1}$ we have that $\mathscr{E} \subseteq \mathscr{X}_u$ if and only if

$$k_i Q k_i^T \leq 1$$

Let $\hat{A}_i = A_i + B_i K$, then the invariance condition for $\mathscr{E}$ is

$$Q \hat{A}_i^T + \hat{A}_i Q < 0$$

which is, again, an LMI.

Indeed denoting by $Q = P^{-1}$ we have that $\mathscr{E} \subseteq \mathscr{X}_u$ if and only if

$$k_i Q k_i^T \leq 1$$

Let $\hat{A}_i = A_i + B_i K$, then the invariance condition for $\mathscr{E}$ is

$$Q\hat{A}_i^T + \hat{A}_i Q < 0$$

which is, again, an LMI.

Remark

There are several methods methods which involve non–quadratic functions (piecewise-quadratic, composite, polyhedral).

Constrained control 4

Example

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -b \end{bmatrix}$$

with

$$1507 \leq a \leq 2177, \qquad 17.8 \leq b \leq 28$$

We considered the control action

$$u = 120(0.025x_1 + 0.022x_2)$$

with $|u| \leq 0.5$. A domain of attraction is the polygon having vertices $\pm X$ where

$$X = \begin{bmatrix} 0.005 & 0.005 \\ 0 & -0.398 \end{bmatrix}$$

Constrained control 5