Lyapunov methods in robustness—Part C

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Quadratic stabilization 1

$$\Psi(x) = x^T P x$$

$P > 0$ (symmetric positive). The gradient is

$$\nabla\Psi(x) = 2x^T P$$

Given

$$\dot{x}(t) = f(x(t), u(t), w(t))$$

we have

$$\dot{\Psi}(x, u, w) = 2x^T P f(x, u, w) \leq -\phi(\|x\|)$$

(Hard to check).

Quadratic stabilization 2

Case of linear systems

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$$

The derivative is

$$\dot{\Psi}(x,u,w) = x^T(A(w)^T P + PA(w))x + (u^T B(w)^T Px + x^T PB(w)u)$$

For stability analysis set $u = 0$

$$\dot{\Psi}(x,w) = x^T(A(w)^T P + PA(w))x \doteq -x^T(Q(w))x$$

The Lyapunov derivative is negative iff $Q(w) > 0$ for all $w \in \mathscr{W}$.

Quadratic stabilization 3

Technicality: $\mathscr{W}$ should be compact

$$\dot{x}(t) = -wx(t), \qquad 0 < w \leq 1$$

and $P = 1$ then $\dot{\Psi} = -2wx^2 < 0$. However, for $w(t) = e^{-t} \in (0, 1]$,

$$x(t) = x(0)e^{e^{-t}-1}.$$

Consider the case in which B is certain $B(w) = B$

$$\dot{\Psi}(x, u, w) = x^T (A(w)^T P + PA(w))x + 2x^T (PBu)x$$

The gradient–based control is linear static

$$u = -\gamma B^T Px \qquad (LIN)$$

Quadratic stabilization 4

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Theorem

If there exists a continuous control $u(x)$ such that

$$\dot{\Psi}(x,u(x),w) \leq -\alpha(\|x\|)$$

then this condition can be guaranteed by means of a control of the form (LIN). Actually, exponential stability holds.

Quadratic stabilization 5

Definition

A (system (controlled system) is said to be quadratically stable (stabilizable) if it admits a quadratic Lyapunov Function (Control Lyapunov Function).

Quadratic stabilization 6

Quadratic stabilization 6

- Non–parametric uncertainties:

$$\begin{aligned} A(\Delta) &= A_0 + D\Delta E, \\ B(\Delta) &= B_0 + D\Delta F, \quad \|\Delta\| \leq 1, \end{aligned}$$

  D, E and F are known matrices.

Quadratic stabilization 6

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  D, E and F are known matrices.
- Polytopic uncertainties:

$$
\begin{aligned}
A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\
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Quadratic stabilization 6

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  Special case: interval matrices

$$A = A_0 + A_1 p_1 + A_2 p_2, \quad p_1^- \leq p_1 \leq p_1^+, \quad p_2^- \leq p_2 \leq p_2^+,$$

$$\begin{aligned} \hat{A}_1 = A_0 + A_1 p_1^+ + A_2 p_2^+, \quad \hat{A}_2 = A_0 + A_1 p_1^+ + A_2 p_2^-, \\ \hat{A}_3 = A_0 + A_1 p_1^- + A_2 p_2^+, \quad \hat{A}_4 = A_0 + A_1 p_1^- + A_2 p_2^- \end{aligned}$$

Quadratic stabilization 7

- Connection with $\mathscr{H}_\infty$.

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  Given $W(s)$ rational stable and proper define the $\mathscr{H}_\infty$ norm as

$$\|W(s)\|_\infty = \sup_{Re(s)\geq 0} \sqrt{\sigma[W^T(s^*)W(s)]}$$

$\sigma[M]$ = maximum modulus of the eigenvalues.

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Theorem

$$A(\Delta) = A_0 + D\Delta E \qquad \|\Delta\| \leq \rho,$$

There exists a positive definite matrix P such that

$$x^T PA(\Delta)x < 0, \quad \textit{for all complex} \quad \|\Delta\| \leq \rho$$

if and only if A_0 is stable and

$$\|E(sI - A_0)^{-1}D\|_\infty < \frac{1}{\rho}$$

Theorem

$$
\begin{aligned}
\dot{x}(t) &= [A_0 + D\Delta E]x(t) + [B_0 + D\Delta F]u(t) \\
y(t) &= C_0 x(t), \qquad ||\Delta(t)|| \leq 1.
\end{aligned}
$$

Khargonekar Petersen and Zhou (1990): Control $u(s) = K(s)y(s)$ quadratically stabilizes the system iff the d-to-z transfer function of

$$
\begin{aligned}
sx(s) &= A_0 x(s) + Dd(s) + B_0 u(s) \\
z(s) &= Ex(s) + Fu(s) \\
y(s) &= C_0 x(s) \\
u(s) &= K(s)y(s)
\end{aligned}
$$

satisfies the condition

$$
\|W_{zd}(s)\|_\infty < 1
$$

Polytopic case.

$$\dot{x}(t) = A(w(t))x(t),$$

$$e.g. \quad A(w) \quad = \quad \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0,$$

The system is quadratically stable iff

$$2x^T PA(w)x = x^T(PA(w) + A(w)^T P)x = -x^T Q(w)x < 0$$

This condition is equivalent to

$$PA_i + A_i^T P = -Q_i < 0,$$

Linear Matrix Inequality (LMI) Boyd et al. (1994).

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

$$e.g. \quad [A(w)\ B(w)] \quad = \quad \sum_{i=1}^{s} w_i [A_i\ B_i], \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0,$$

Consider the linear controller $u = Kx$.The condition becomes

$$(A_i + B_i K)^T P + P(A_i + B_i K) < 0, \quad i = 1, 2, \ldots, s$$

unfortunately this condition is nonlinear. However, if we set

$$Q = P^{-1}, \quad KQ = R$$

then we get the LMI condition

$$QA_i^T + A_i Q + R^T B_i^T + B_i R < 0, \quad i = 1, 2, \ldots, s, \quad Q > 0$$

$$K = RP$$

Limitations of quadratic functions 1

Stability does not imply quadratic stability.

Example

The system

$$A(w) = \begin{bmatrix} 0 & 1 \\ -1+w(t) & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{E} w \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{F},$$

with $|w| \leq \rho$, is stable iff

$$\rho < \rho_{ST} = 1, \quad \text{(robust stability radius)}$$

it is quadratically stable iff

$$\rho < \rho_Q = \frac{\sqrt{3}}{2}, \quad \text{(quadratic stability radius)}$$

Indeed $\|D(sI - A_0)^{-1}E\|_\infty = \sqrt{3}/2$.

Limitations of quadratic functions 2

Stabilization

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \qquad w \in \rho \mathscr{W}$$

Define the following stabilizability margins

$$\begin{aligned} \rho_{ST} &= \sup\{\rho : \text{(S) is stabilizable}\} \\ \rho_Q &= \sup\{\rho : \text{(S) is quadratically stabilizable}\} \end{aligned}$$

There are systems for which $\frac{\rho_{ST}}{\rho_Q} = \infty$.

Example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} w(t) \\ 1 \end{bmatrix}, \quad \mathscr{W} = [-1, 1].$$

For this system

$$\rho_{ST} = \infty$$

$$\rho_Q = 1$$

Limitations of quadratic functions 3

Also choosing linear controllers is a restriction.

Example

The system

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} w(t) \\ 1 \end{bmatrix}, \quad \mathscr{W} = [-1, 1].$$

can be stabilized for any $w(t) \in \rho\mathscr{W}$, but for ρ large enough there does not exist a stabilizing linear state feedback of the form

$$u = k_1 x_1 + k_2 x_2$$

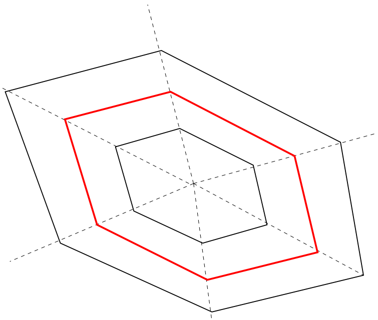
Competitors?

Competitors?

Question

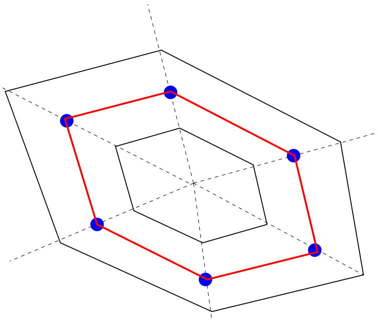
Are there other convenient classes of Lyapunov functions?

Competitors: polyhedral Lyapunov functions



$$V(x) = \max_i \quad F_i x$$

Competitors: polyhedral Lyapunov functions



$$V(x) = \min\{\sum p_i : \; p_i \geq 0, \quad x = Xp\}$$

Polyhedral Lyapunov functions 1

A (symmetrical) polyhedral function is a function of the form

$$\Psi(x) = \|Fx\|_\infty$$

where F is a full column rank matrix.

or

$$\Psi(x) = \min \ \{\|p\|_1 : \quad Xp = x\}$$

where X is a full row rank matrix.

Polyhedral Lyapunov functions 2

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The polyhedral functions are non–conservative.

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Theorem

Brayton and Tong (1980). Molchanov and Pyatnitskii (1986).

$$\dot{x}(t) = A(w(t))x(t), \quad w \in \mathscr{W}$$

is stable if and only if it admits a polyhedral Lyapunov function.

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Theorem

Blanchini (1995). The system

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t), \quad w \in \mathscr{W}$$

is stabilizable if and only if it admits a polyhedral control Lyapunov function.

$$\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t),$$

$$[A(w), B(w)] \;=\; \sum_{i=1}^{s} w_i [A_i, B_i], \;\; \sum_{i=1}^{s} w_i = 1, \;\; w_i \geq 0.$$

Lyapunov function: plane representation

$$\Psi(x) = \max_i \;\; F_i x = \max(Fx)$$

Symmetric case $\Psi(x) = \|Gx\|_\infty$ take $F = \begin{bmatrix} G \\ -G \end{bmatrix}$.

Lyapunov function: vertex representation.

$$\Psi(x) = \min\{\sum_{j=1}^{v} \alpha_j : \;\; \sum_{j=1}^{v} \alpha_j x_j = x\} = \min\{\bar{1}^T \alpha : \;\; X\alpha = x\}$$

Polyhedral Lyapunov functions 3

Definition

A square matrix M is said a $\mathscr{M}$–matrix if $M_{ij} \geq 0$ for $i \neq j$.

Theorem

$\Psi(x)$ *is a Lyapunov function for the system* $\dot{x}(t) = A(w(t))x(t)$

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Theorem

$\Psi(x)$ is a Lyapunov function for the system $\dot{x}(t) = A(w(t))x(t)$

- *iff there exist r $\mathscr{M}$–matrices H_1, H_2, $\ldots H_s$ and $\beta > 0$ s.t.*

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\begin{aligned}
FA_k &= H_k F \\
H_k \bar{1} &\leq -\beta \bar{1}
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$$\begin{aligned} A_k X &= X P_k \\ \bar{1}^T P_k &\leq -\beta \bar{1}^T \end{aligned}$$

Stabilization problem.

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Theorem

$\Psi(x)$ is a CLF for system $\dot{x}(t) = A(w(t))x(t) + B(w(t))u(t)$ iff there exist r $\mathscr{M}$–matrices P_1, P_2, $\ldots P_s$, a matrix U and $\beta > 0$ such that

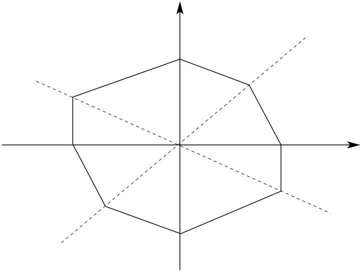
$$\begin{array}{rcl} A_k X + B_k U & = & X P_k \\ \bar{1}^T P_k & \leq & -\beta \bar{1}^T \end{array}$$

If the condition is satisfied then the exists a control $u = \Phi(x)$ such that

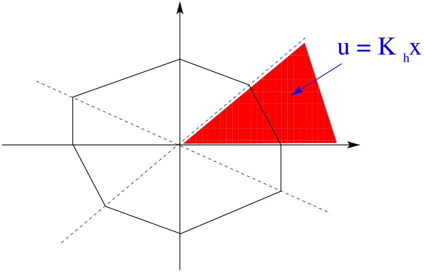
$$\|x(t)\| \leq C \|x(0)\| e^{-\beta t}, \quad C > 0.$$

Polyhedral control Lyapunov functions 2

In general a controlled–invariant polyhedron does not admit linear controllers,



In general a controlled–invariant polyhedron does not admit linear controllers, but a piecewise–linear control.



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Polyhedral control Lyapunov functions 3

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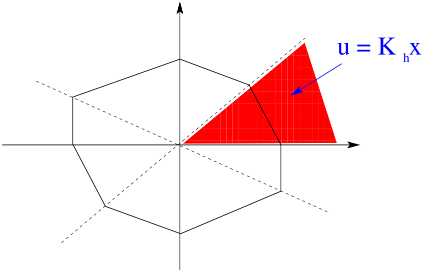
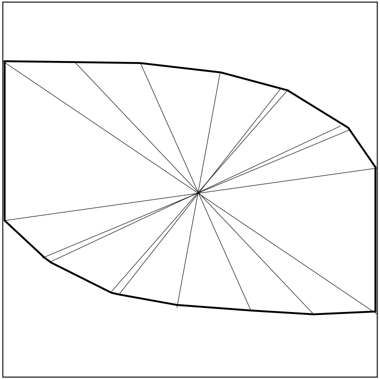


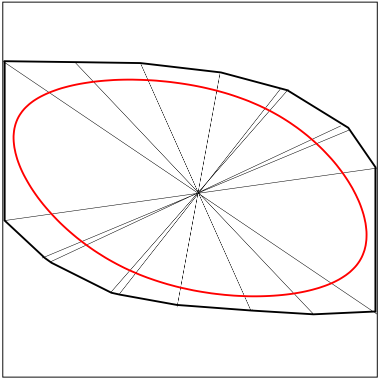
Gutman and Cwikel (1986)

... yes ... but how many?

Polyhedral control Lyapunov functions 5



Polyhedral control Lyapunov functions 5

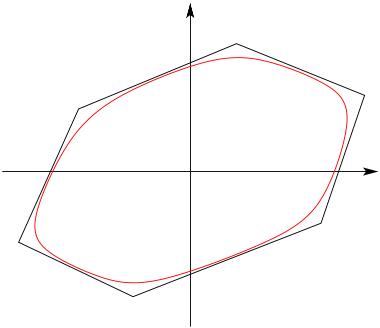


Ellipsoidal approximation.

Smoothed polyhedral Lyapunov functions

Polyhedral Lyapunov functions are non–smooth... .... but we can adopt the smoothed version. In the symmetric case

$$V(x) = \|Fx\|_\infty \approx \|Fx\|_{2p}$$



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- Merged (Balestrino, Caiti, Grammatico).

Quadratic stabilizability—discrete–time

$$
\begin{aligned}
x(t+1) &= [A_0 + D\Delta E]x(t) + [B_0 + D\Delta F]u(t) \\
y(t) &= C_0 x(t), \qquad \|\Delta(t)\| \leq 1.
\end{aligned}
$$

The stabilizing control $u(z) = K(z)y(z)$ is robustly quadratically stabilizing iff the d-to-e transfer function of the loop

$$
\begin{aligned}
zx(z) &= A_0 x(z) + Dd(z) + B_0 u(z) \\
e(z) &= Ex(z) + Fu(z) \\
y(z) &= C_0 x(z) \\
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\end{aligned}
$$

satisfies the condition

$$
\|W_{zd}(z)\|_\infty \doteq \sup_{|z|\geq 1} \sqrt{\sigma[W_{zd}(z)^T W_{zd}(z)]} \leq 1
$$

Quadratic stabilizability—discrete–time 1

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t),$$

$$\begin{aligned} e.g. \quad A(w) &= \sum_{i=1}^{s} w_i A_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0, \\ B(w) &= \sum_{i=1}^{s} w_i B_i, \qquad \sum_{i=1}^{s} w_i = 1, \quad w_i \geq 0. \end{aligned}$$

The conditions for quadratic stability becomes

$$A^T(w)PA(w) - P < 0, \quad \text{for all } w \in \mathscr{W}$$

This condition is true if and only if

$$A_i^T PA_i - P < 0, \quad \text{for all } i, \quad P > 0$$

To stabilize the system via linear control we have to find a positive definite matrix P such that

$$(A_i + B_i K)^T P (A_i + B_i K) - P < 0$$

Pre and post multiply by $Q = P^{-1}$ and let $KQ = R$. We get

$$(QA_i^T + R^T B_i^T) Q^{-1} (A_i Q + B_i R) - Q < 0$$

which is know equivalent to

$$\left[\begin{array}{cc} Q & QA_i^T + RB_i \\ A_i Q + B_i R & Q \end{array} \right] > 0, \qquad Q > 0, \quad i = 1, 2, \ldots, s,$$

which turns out to be a set of LMIs.

Theorem

Function

$$\Psi = \max_i F_i x = \min\{\bar{1}^T \alpha : \quad X\alpha = x\}$$

is a LF for $x(k+1) = A(w(k))x(k)$

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is a LF for $x(k+1) = A(w(k))x(k)$

- *iff there exist r nonnegative matrices* $H_1, H_2, \ldots H_r$

$$\begin{aligned} FA_k &= H_k F \\ H_k \bar{1} &\leq \lambda \bar{1} \end{aligned}$$

Polyhedral stabilizability—discrete–time 1

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$$\begin{array}{rcl} A_k X & = & X P_k \\ \bar{1}^T P_k & \leq & \lambda \bar{1}^T \end{array}$$

Polyhedral stabilizability—discrete–time 2

Theorem

Function $\Psi = \min\{\bar{1}^T\alpha : \quad X\alpha = x\}$ *is a CLF for system*

$$x(k+1) = A(w(k))x(k) + B(w(k))u(k)$$

iff there exist r nonnegative matrices P_1, P_2, $\dots P_s$ *and a matrix* U *such that*

$$\begin{array}{rcl} A_kX + B_kU & = & XP_k \\ \bar{1}^TP_k & \leq & \lambda\bar{1}^T \end{array}$$

for some positive constant $\lambda < 1$.

The matrices can be found by means of an iterative procedure Blanchini (1994).

Polyhedral function generation—recursive procedure.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the constraints

$$\mathscr{P} = \{x : \quad \|x\|_\infty \leq 2\}, \qquad |u| \leq 1,$$

The procedure computes a sequence of sets $\mathscr{X}^{(k)}$ that converges to the unit ball of $\Psi(x)$.

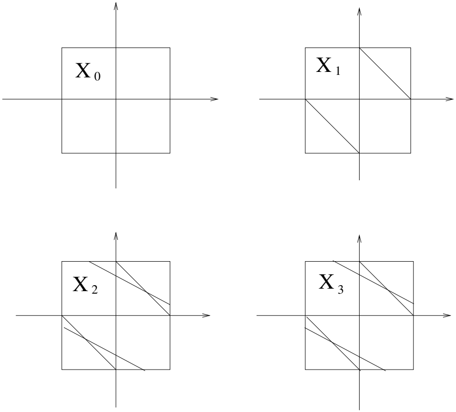

Figure : Sequence of the sets $\mathscr{X}_k$

Matching conditions 1

Consider the system

$$\dot{x}(t) = f(x(t), w(t), u(t)) = F(x(t)) + B[u(t) + g(x(t), w(t))]$$

where the uncertain term is bounded as

$$\|g(x, w)\| \leq \alpha \|x\| + \beta$$

Basic assumption: the nominal systems

$$\dot{x} = F(x)$$

is stable with a known Lyapunov function $\Psi(x)$

$$\dot{\Psi}_{nom}(x(t)) \doteq \nabla\Psi(x) F(x) \leq -\phi(\|x\|)$$

for some κ–function ϕ.

Matching conditions 2

Consider the control

$$u = -\gamma(x)\frac{B^T \nabla\Psi(x)^T}{\|B^T \nabla\Psi(x)^T\|}$$

where we assume that γ is any function such that

$$\gamma(x) \geq \alpha\|x\| + \beta$$

The derivative with respect the perturbed dynamics is

$$\begin{aligned}
\dot{\Psi}_\Delta(x(t)) &= \nabla\Psi(x)[F(x) + B(u + g(x,w))] \\
&= \Psi_{nom}(x(t)) - \gamma(x)\|\nabla\Psi(x)B\| + \nabla\Psi(x)Bg(x,w) \leq \\
&\leq -\phi(\|x\|) - \gamma(x)\|\nabla\Psi(x)B\| + \|\nabla\Psi(x)B\|\|g(x,w)\| \leq \\
&\leq -\phi(\|x\|) - \|\nabla\Psi(x)B\|(\gamma(x) - \alpha\|x\| - \beta) \leq -\phi(\|x\|)
\end{aligned}$$

Therefore the control is stabilizing.

Matching conditions 3

A continuous control.

$$u = -\frac{B^T \nabla\Psi(x)^T \gamma^2(x)}{\|B^T \nabla\Psi(x)^T\| \gamma(x) + \delta}$$

The derivative is

$$\begin{aligned}
\dot{\Psi}_\Delta(x(t)) &= \nabla\Psi(x)[F(x) + Bg(x,w) + Bu] \leq \\
&\leq \nabla\Psi(x)F(x) + \|\nabla\Psi(x)\| \|g(x,w)\| + \nabla\Psi(x)Bu \leq \\
&\leq \nabla\Psi(x)F(x) + \|\nabla\Psi(x)B\|\gamma(x) - \frac{\|\nabla\Psi(x)B\|^2\gamma^2(x)}{\|\nabla\Psi(x)B\|\gamma(x) + \delta} \\
&\leq -\phi(\|x\|) + \delta \frac{\|\nabla\Psi(x)B\|\gamma(x)}{\|\nabla\Psi(x)B\|\gamma(x) + \delta} \leq -\phi(\|x\|) + \delta
\end{aligned}$$

Take $\kappa(\delta): \quad \min_{\phi(\|x\|) \geq \delta} \Psi(x)$. The state is confined in the set

$$\mathscr{S} = \{x : \quad \Psi(x) \leq \kappa\}.$$

Since $\kappa(\delta) \to 0$ as $\delta \to 0$, this control assures **practical stability**.

Backstepping 1

The system

$$\dot{x}(t) = F(x(t), w(t)) + G(x(t), w(t))u(t)$$

is in the **strict feedback form** if

$$\begin{aligned}
F(x,w) &= \begin{bmatrix} f_{11} & f_{12} & 0 & \dots & 0 \\ f_{21} & f_{22} & f_{23} & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & \dots & f_{n-1,n} \\ f_{n,1} & f_{n,2} & f_{n,3} & \dots & f_{n,n} \end{bmatrix} x + F(0,w) \\
f_{i,j} &= f_{i,j}(x_1, x_2, \dots, x_i, w) \\
G(x,w) &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_{n,n+1} \end{bmatrix} \\
f_{i,i+1} &\neq 0.
\end{aligned}$$

Backstepping 2

As a simple example consider the system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)F(x_1(t))w(t)+x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned}$$

With $|w| \leq 1$, $|F(x_1)| \leq m$. Consider the first subsitem and let us consider the "virtual" control x_2.

$$\begin{aligned}x_2 &= S(x_1)x_1\\ \dot{x}_1 &= x_1[F(x_1)w(t)+S(x_1)]\end{aligned}$$

This system is stable if

$$[F(x_1)w(t)+S(x_1)] < 0$$

Backstepping 3

Unfortunately, x_2 is **not** a control variable !! Then we control the first equation

$$\dot{x}_2(t) = u(t)$$

in such a way that x_2 is "close" to $S(x_1)x_1$

$$\dot{x}_2 = u = -k[x_2 - S(x_1)x_1]$$

This is basically the idea of backstepping.

Consider the change of variables

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2 - S(x_1)x_1 \end{cases} \qquad \begin{cases} x_1 = z_1 \\ x_2 = z_2 - S(z_1)z_1 \end{cases}$$

and the candidate Lyapunov function

$$\Psi(z_1, z_2) = z_1^2 + z_2^2$$

we have

$$\begin{aligned} \dot{z}_1(t) &= z_1 F(z_1)w + z_2 + S(z_1)z_1 \\ \dot{z}_2(t) &= S(z_1)[z_1 F(z_1)w + z_2 + S(z_1)z_1] + S'(z_1) + u \end{aligned}$$

Backstepping 5

The control becomes $u = -kz_2$. Then

$$\begin{aligned}\dot{\Psi} &= [F(x_1)w + S(z_1)]z_1^2 + [1 + S'(z_1) + S(z_1)F(x_1)w + S(z_1)]z_1 z_2 + \\ &+ [S(z_1) - k]z_2^2\end{aligned}$$

In any bounded domain we can assume

$$\begin{aligned}&[1 + S'(z_1) + S(z_1)F(x_1)w + S(z_1)]| \leq b \\ &|F(x_1)w + S(z_1)| \leq -a \\ &|S(z_1)| \leq c\end{aligned}$$

We obtain

$$\dot{\Psi} \leq -az_1^2 + b|z1||z2| - [k - c]z_2^2 < 0, \quad \text{for} \quad (z_1, z_2) \neq 0.$$

Therefore the system is stable.

Generalization based on results by Barmish (1981). Assume that $|f_{i,j}| \leq M_{i,j}$, that $f_{i,i+1} \geq N_i$ and that $F(0,w) = 0$. Consider the following "immersion"

$$\dot{x} = F(x,w)x + G(x,w)u \in \{Ax + Bu, \quad A \in \mathscr{A}, \quad \text{and} \quad B \in \mathscr{B}\}$$

where

$$A = \begin{bmatrix} a_{11} & \bar{a}_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & \bar{a}_{23} & \dots & \vdots \\ \dots & \dots & \dots & \dots & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,n-1} & \dots & \bar{a}_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{a}_{n,n+1} \end{bmatrix}$$

such that

$$|a_{i,j}| \leq M_{i,j}, \quad \text{and} \quad b_{i,i+1} \geq N_i$$

(the $\bar{a}_{i,i+1}$ have a bar to remind that they are non–zero).

Consider the change of variable

$$z = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ k_1 & 1 & 0 & \dots & 0 \\ k_1 k_2 & k_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_1 \dots k_{n-1} & \dots & \dots & k_{n-1} & 1 \end{bmatrix} x,$$

$$x = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ -k_1 & 1 & 0 & \dots & 0 \\ 0 & -k_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & 1 & \vdots \\ 0 & \dots & \dots & -k_{n-1} & 1 \end{bmatrix} z$$

where k_i are design parameters.

After the change of variables B remains unchanged and by means of the feedback

$$u = -k_n z_n(t)$$

matrix $A_{CL}(k)$ becomes

$$\begin{bmatrix} \tilde{a}_{11} - \bar{a}_{12}k_1 & \bar{a}_{12} & 0 & \cdot & 0 \\ \tilde{a}_{21}(k_1) & \tilde{a}_{22}(k_1) - \bar{a}_{23}k_2 & \bar{a}_{23} & \cdot & 0 \\ \tilde{a}_{3,1}(k_1, k_2) & \tilde{a}_{3,2}(k_1, k_2) & \tilde{a}_{3,3}(k_1, k_2) - \bar{a}_{3,3}k_3 & \cdot & \vdots \\ \dots & \dots & \dots & \cdot & \bar{a}_{n-1,n} \\ \tilde{a}_{n,1}(k_1, .., k_{n-1}) & \tilde{a}_{n,2}(k_1, .., k_{n-1}) & \dots & \cdot & \tilde{a}_{n,n}(k_1, .., k_{n-1}) - \bar{a}_{n,n}k_n \end{bmatrix}$$

Note that

$$\tilde{a}_{i,j} = \tilde{a}_{i,j}(k_1, k_2, \dots, k_{i-1})$$

Backstepping 9

Take $\Psi(z) = z^T z$, then the Lyapunov derivative is

$$-z^T Q(k_1, \dots, k_n) z$$

where $Q(k_1, \dots, k_n)$ is the following matrix

$$\begin{bmatrix} \tilde{q}_{11} - 2\bar{a}_{12} k_1 & q_{12} & q_{13}(k_1, k_2) & \cdot & q_{1n}(k_1, , \dots, k_{n-1}) \\ \tilde{q}_{21}(k_1) & \tilde{q}_{22}(k_1) - 2\bar{a}_{23} k_2 & \bar{q}_{23} & \cdot & q_{1n}(k_1, \dots, k_{n-1}) \\ \tilde{a}_{3,1}(k_1, k_2) & \tilde{a}_{3,2}(k_1, k_2) & \tilde{q}_{3,3}(k_1, k_2) - 2\bar{a}_{3,4} k_3 & \cdot & \vdots \\ \vdots & \vdots & \dots & \cdot & \vdots \\ \tilde{q}_{n,1}(k_1, \dots, k_{n-1}) & \tilde{q}_{n,2}(k_1, \dots, k_{n-1}) & \dots & \cdot & \tilde{q}_{n,n}(k_1, \dots, k_{n-1}) - 2\bar{a}_{n-1,n} k_n \end{bmatrix}$$

To make $Q(k_1, \dots, k_n)$ positive definite, take k_1 in such a way that $\tilde{q}_{11} - 2\bar{a}_{12} k_1$ is positive, take k_2 in such a way that the first (2×2) principal matrix has a positive determinant, take k_2 in such a way that the first (3×3) matrix has a positive determinant...

Output feedback 1

Handling output feedback problems feedback via Lyapunov methods presents difficulties that can be explained as follows.

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Handling output feedback problems feedback via Lyapunov methods presents difficulties that can be explained as follows.

- The approach based on control Lyapunov functions basically requires the knowledge (or the estimation with an error) of the state.
- Any "observer" must replicate the system dynamics, and thus state estimation for an uncertain system is a hard problem.

Output feedback 2

Consider the following plant

$$\begin{aligned}\dot{x}(t) &= A(w(t))x(t) + B(w(t))u(t)\\ y(t) &= Cx(t)\end{aligned}$$

Observer

$$\begin{aligned}\dot{z}(t) &= (A_0 - LC)z(t) + B_0u(t) + Ly(t)\\ y(t) &= Cx(t)\end{aligned}$$

Define the error $e = z - x$ to obtain

$$\dot{e}(t) = (A_0 + LC)e(t) + (A_0 - A(w(t)))x(t) + (B_0 - B(w(t)))u(t)$$

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Claim

The observer principle is fragile.

Output feedback 3

In the gain–scheduling problem (w measured on–line) we can consider the observer

$$\begin{aligned}\dot{z}(t) &= (A(w(t)) - L(w(t))C)z(t) + B(w(t))u(t) + L(w(t))y(t) \\ y(t) &= Cx(t)\end{aligned}$$

The error equation is

$$\dot{e}(t) = (A(w(t)) + L(w(t))C)e(t)$$

For polytopic systems we get an LMI problem

$$PA_k + PL_kC + A_k^TP + C^TL_k^TP = PA_k + A_k^TP + PS_k + C^TS_k^T < 0, \;\; P > 0$$

and $L = \sum \alpha_i L_i$ where $S_k \doteq PL_k$.