Lyapunov methods in robustness—Part B

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Control Lyapunov functions 1

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Class $\mathscr{C}$ of controllers.

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State feedback with feedforward : if $u(t) = \Phi(x(t), w(t))$:

Control Lyapunov functions 2

Definition

Given a class of controllers $\mathscr{C}$ and a locally Lipschitz positive definite function Ψ (and possibly a set $\mathscr{P}$) we say that Ψ is a global control Lyapunov function (a control Lyapunov function outside $\mathscr{P}$ or a control Lyapunov function inside $\mathscr{P}$) if there exists a controller in $\mathscr{C}$ such that:

- for each initial condition $x(0)$ there exists a solution $x(t)$, for any admissible $w(t)$, and each of such solutions is defined for all $t \geq 0$;
- the function Ψ is a global Lyapunov function (a Lyapunov function outside $\mathscr{P}$ or a Lyapunov function inside $\mathscr{P}$) for the closed–loop system.

Control Lyapunov functions 3

- Systems with control constraints

$$u(t) \in \mathcal{U}.$$

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- Systems with output (state) constraints

$$x(t) \in \mathscr{X}$$

In this case we must consider conditions of the form

$$\mathscr{N}[\Psi, \mu] \subseteq \mathscr{X}$$

the constraints can be satisfied as long as $x(0) \in \mathscr{N}[\psi, \mu]$.

$$D^+\Psi[x,u,w] \doteq \limsup_{h\to 0^+} \frac{\Psi(x+hf(x,u,w))-\Psi(x)}{h} \leq -\phi(\|x\|).$$

Associating a feedback control with a CLF 1

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- For all $x \in \mathcal{N}[\Psi,\alpha,\beta]$, there exists u such that the inequality is satisfied for all $w \in \mathcal{W}$;

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Fact

These conditions holds if Ψ is a CLF associated with controls $u=\Phi(x)$ and $u=\Phi(x,w)$ respectively.

Associating a feedback control with a CLF 2

$$D^+\Psi[x,u,w] \doteq \limsup_{h\to 0^+} \frac{\Psi(x+hf(x,u,w))-\Psi(x)}{h} \leq -\phi(\|x\|). \quad (INEQ)$$

State feedback case. Consider the set

$$\Omega(x) = \{u : \quad (INEQ) \text{ is satisfied for all } w \in \mathscr{W}\}$$

Can we find (and how) $u = \Phi(x)$ such that $\Phi(x) \in \Omega(x)$?

Associating a feedback control with a CLF 3

Assume Ψ differentiable and consider control–affine systems

$$\dot{x} = a(x,w) + b(x,w)u$$

Then

$$\nabla\Psi(x)[a(x,w) + b(x,w)u] \leq -\phi(\|x\|),$$

$$\Omega(x) = \{u : \quad \nabla\Psi(x)b(x,w)u \leq -\nabla\Psi(x)a(x,w) - \phi(\|x\|), \; \forall \quad w \in \mathscr{W}\}$$

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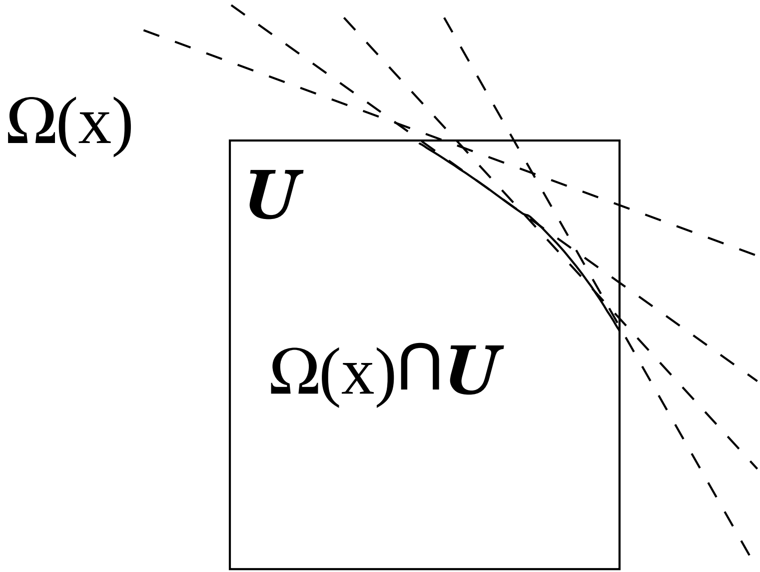
Theorem

If $\Omega(x)$ is nonempty, then there exists $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ continuous for $x \neq 0$ such that

$$\Phi(x) \in \Omega(x)$$

(Freeman and Kokotovic (1998))


Ω(x)
U
Ω(x)∩U

$\Omega(x)$ is a **set–valued map**; a function $\Phi(x) \in \Omega(x)$ is called a **selection**.

In the gain scheduling (full–information) case we have

$$\Omega(x,w) = \{u : \quad \nabla\Psi(x)b(x,w)u \leq -\nabla\Psi(x)a(x,w) - \phi(\|x\|)\}$$

If $\Omega(x,w)$ is not empty then a control exists of the form.

$$\Phi(x,w) \in \Omega(x,w)$$

Minimum–effort control. Full information.

$$\Phi_{ME}(x,w) = \arg \min_{u \in \Omega(x,w)} \|u\|_2$$

($\|\cdot\|_2$ is the euclidean norm) Peteresen and Barmish (1987).
Consider the inequality

$$\nabla\Psi(x)b(x,w)u \leq -\nabla\Psi(x)a(x,w) - \phi(\|x\|) \doteq -c(x,w)$$

Then

$$\Phi_{ME}(x,w) = \begin{cases} -\frac{b(x,w)^T \nabla\Psi(x)^T}{\|\nabla\Psi(x)b(x,w)\|^2}\, c(x,w), & \text{if } c(x,w) > 0 \\ 0, & \text{if } c(x,w) \leq 0 \end{cases}$$

Minimum–effort control. State–feedback. Assume that the term b does not depend on w

$$\dot{x}(t) = a(x(t), w(t)) + b(x(t))u(t)$$

$$\nabla\Psi(x)b(x)u \leq -\nabla\Psi(x)a(x,w) - \phi(\|x\|) \doteq -c(x,w)$$

Define $\hat{c}(x) = \max_{w \in \mathscr{W}} c(x,w)$ and the inequality becomes

$$\nabla\Psi(x)b(x)u \leq -\hat{c}(x)$$

$$\Phi_{ME}(x) = \begin{cases} -\frac{b(x)^T \nabla\Psi(x)^T}{\|\nabla\Psi(x)b(x)\|^2}\,\hat{c}(x), & \text{if } \hat{c}(x) > 0 \\ 0, & \text{if } \hat{c}(x) \leq 0 \end{cases}$$

Associating a feedback control with a CLF 8

The minimum effort control belongs to the class of
gradient–based controllers

$$u(t) = -\gamma(x) b(x)^T \nabla\Psi(x)^T$$

where $\gamma(x) \geq \max\left\{ \frac{\hat{c}(x)}{\|\nabla\Psi(x) b(x)\|^2}, 0 \right\}$

Associating a feedback control with a CLF 8

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Fact

The controllers of the form $\gamma(x)\nabla\Psi(x)b(x)$ have infinite positive gain margin i.e. if $-\hat{\gamma}(x)b(x)^T\nabla\Psi(x)^T$ is a stabilizing control then $\gamma(x)b(x)^T\nabla\Psi(x)^T$ is stabilizing for $\gamma(x) \geq \hat{\gamma}(x)$.

Case of $b(x,w)$ **uncertain** The case becomes involved.

- state feedback inequality:

$$\min_{u\in\mathcal{U}}\max_{w\in\mathcal{W}}\{\nabla\Psi(x)[a(x,w)+b(x,w)u]\}\le-\phi(\|x\|),$$

- full information inequality:

$$\max_{w\in\mathcal{W}}\min_{u\in\mathcal{U}}\{\nabla\Psi(x)[a(x,w)+b(x,w)u]\}\le-\phi(\|x\|),$$

The "min" and the "max" are reversed.
There are cases (e.g. Meilakhs (1979), Blanchini (2000)) for which the two conditions are equivalent.

Associating a feedback control with a CLF 10

Limited effort controllers Consider the case $\|u(t)\| \leq 1$.

$$\min_{\|u\|\leq 1} \nabla\Psi(x)[a(x,w)+b(x)u]$$

Case of 2–norm:

$$u = \begin{cases} -\frac{b(x)^T\nabla\Psi(x)^T}{\|b(x)^T\nabla\Psi(x)^T\|} & \text{if} \quad \nabla\Psi(x)b(x) \neq 0, \\ 0 & \text{if} \quad \nabla\Psi(x)b(x) = 0 \end{cases}$$

Case of infinity norm:

$$u = -sgn[b(x)^T\nabla\Psi(x)^T]$$

Both controls can be approximated continuous controllers

$$u = -\frac{b(x)^T\nabla\Psi(x)^T}{\|b(x)^T\nabla\Psi(x)^T\|+\delta} \qquad u = -sat[\kappa b(x)^T\nabla\Psi(x)^T]$$

for δ small and κ large.

Output feedback CLF 1

In principle the extension is easy

$$\begin{aligned}\dot{x}(t) &= f(x(t), w(t), u(t)) \\ y(t) &= h(x(t), w(t))\end{aligned}$$

Consider the set

$$g^{-1}(y) = \{(x,w), \quad x \in \mathbb{R}^n, \ w \in \mathcal{W} : \quad y = h(x,w) \ \}$$

$$\Omega(y) = \{u : \quad D^+(x,w,u) \leq -\Phi(\|x\|), \quad \text{for all} \ \ (x,w) \in g^{-1}(y)\}$$

A suitable control of the form $u = \Phi(y)$ must be such that

$$\Phi(y) \in \Omega(y)$$

The representation of these sets extremely complex Freeman and Kokotovic (1996), Qu (1998).

Discrete–time systems 1

$$x(t+1) = f(x(t), w(t))$$

We must consider the Lyapunov difference

$$\Delta\psi(t) = \Psi(f(x(t), w(t))) - \Psi(x(t)) \doteq \Delta\Psi(x(t), w(t))$$

The condition becomes

$$\Delta\Psi(x, w) \leq -\phi(\|x(t)\|)$$

Theorem

If the system admits a Lyapunov function Ψ, Then it is globally uniformly stable.

Theorem

If the system admits a positive definite locally Lipschitz function Ψ *such that*

$$\alpha\|x\|^p \leq \Psi(x) \leq \beta\|x\|^p, \quad \textit{for all} \;\; x \in \mathbb{R}^n,$$

and

$$\Delta\Psi(x,w) \leq \beta\Psi(x)$$

$0 < \beta < 1$, *then*

$$\|x(t)\| \leq \mu\|x(0)\|\lambda^t$$

with $\lambda = 1 - \beta$.

Coefficient λ is the **discrete–time convergence speed**.

Discrete–time control Lyapunov functions 1

$$x(k+1) = f(x(k), u(k), w(k))$$

The relevant condition is

$$\Delta\Psi(x,u,w) \leq -\phi(\|x(t)\|) \qquad (INEQ)$$

$$\Omega(x) = \{u : \quad (INEQ) \text{ is satisfied for all } w \in \mathscr{W}\}$$

Then the control can be chosen as the discrete–time selection

$$\Phi(x) \in \Omega(x)$$

Regularity of $\Phi(x)$ is not essential (from the mathematical point of view).

Continuous vs discrete–time.

Example

$$\dot{x}(t) = f(x(t), w(t)) + u(t), \quad |w| \leq 1.$$

Assume $|f(x(t), w(t))| \leq \xi |x|$ and let

$$\Psi(x) = x^2/2$$

The gradient–based control is $u = -\kappa x$. Then for $\kappa > \xi$

$$\Rightarrow x[f(x,w) + u] \leq -(\kappa - \xi)x^2 < 0, \quad x \neq 0$$

Discrete–time control Lyapunov functions 3

There is no gradient–based type of control in discrete–time

Example

$$x(t+1) = f(x(t), w(t)) + u(t), \quad |w| \leq 1.$$

The control must be such that

$$u(x) \in \Omega(x) = \{x : \ [f(x,w) + u(x)]^2 \leq x^2 - \phi(|x|), \ \text{for all } |w| \leq 1 \ \}$$

Set $\Omega(x)$ can be empty. For instance

$$x(t+1) = [a + bw(t)]x + u(t), \quad |w| \leq 1.$$

We must have $|b| < 1$ for stabilizability.

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