Lyapunov methods in robustness—Part A

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Overview

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- Applications.

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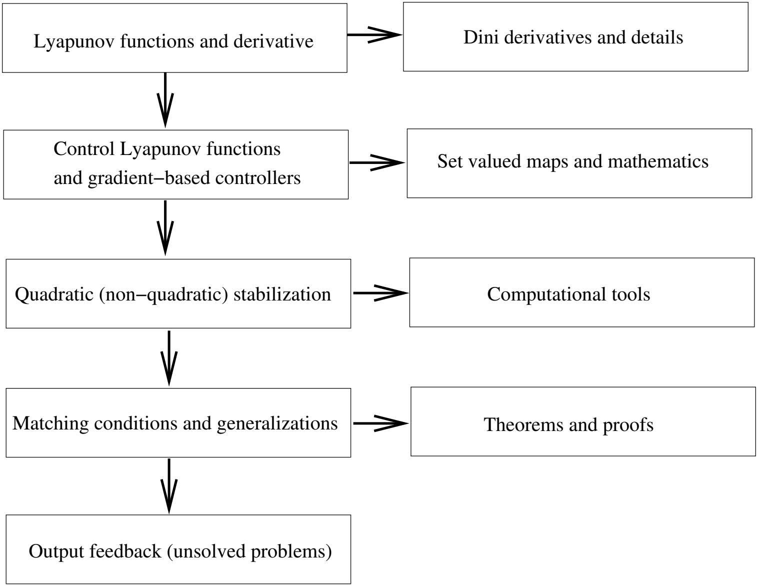
Suggestion

Identify and take home what is fundamental....first!

Lecture table



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Definition

A property $\mathscr{P}$ is said **robust** for the family $\mathscr{F}$ of dynamic systems if any member of $\mathscr{F}$ satisfies $\mathscr{P}$.

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- For instance if $\mathscr{P}$ is "stability" and $\mathscr{F}$ is a family of systems with uncertain parameters ranging in a set, we have to specify if these parameters are constant or time–varying.
- In the context of robustness the family $\mathscr{F}$ represents the uncertainty in the knowledge of the system.

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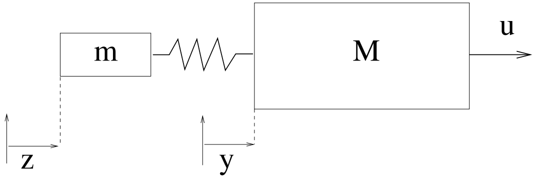
In this work we mainly consider parametric uncertainties.

Robustness 4



Example

$$
\begin{aligned}
M\ddot{y}(t) &= k(z-y)+\alpha(\dot{z}-\dot{y})+u \\
m\ddot{z}(t) &= k(y-z)+\alpha(\dot{y}-\dot{z})
\end{aligned}
$$

z, x: positions; M: known mass, u: force;
m, k: uncertain parameters.

Parametric uncertainty: Parameters m, k, and α are assumed subject to bounds

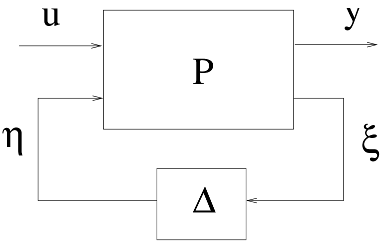
$$k^- \leq k \leq k^+, \quad m^- \leq m \leq m^+, \quad \alpha^- \leq \alpha \leq \alpha^+.$$

Non–parametric uncertainty: Define $\dot{\eta} = k(z - y) + \alpha(\dot{z} - \dot{y})$ and $\xi = y$. And write

$$\begin{aligned} Ms^2 y(s) &= u(s) - s\eta(s), \\ \xi(s) &= y(s) \\ \eta(s) &= \Delta(s)\xi(s) \end{aligned}$$

where

$$\Delta(s) = \frac{ms^2(\alpha s + k)}{ms^2 + \alpha s + k}$$



$\Delta(s)$ is a transfer function such that

$$\|\Delta\| \leq \mu$$

where $\|\cdot\|$ is a relevant norm. For instance $\|\Delta\| = \sup_{\omega \geq 0} \ |\Delta(j\omega)|$

Time–varying parameters 1

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Fact

Parameter variation may have a crucial effect on stability.

Time–varying parameters 2

Example

$$\dot{x}(t) = A(w(t))x(t)$$

$$A(w) = \left[\begin{array}{cc} 0 & 1 \\ -1+w & -a \end{array} \right],$$

$|w| \leq \bar{w}$, where $\bar{w} < 1$ is the uncertainty bound.

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- For any constant $\bar{w} < 1$ and $a > 0$, the corresponding time-invariant system is stable.

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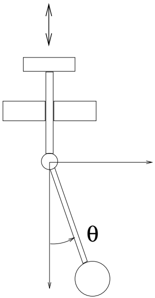
$|w| \leq \bar{w}$, where $\bar{w} < 1$ is the uncertainty bound.

- For any constant $\bar{w} < 1$ and $a > 0$, the corresponding time-invariant system is stable.
- For a small enough and time-varying $w(t)$, with $|w(t)| \leq \bar{w}$, the system can be unstable.

Time–varying parameters 3



$$
\begin{aligned}
J\ddot{\theta}(t) &= -(g+b(t))\sin(\theta(t)) - a(t)\dot{\theta}(t) \\
J\ddot{\theta}(t) &\simeq -(g+b(t))\theta(t) - a(t)\dot{\theta}(t)
\end{aligned}
$$

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- The parameter are unknown and not measurable on–line.

Fact

With some exceptions, the possibility of measuring the parameter on–line is an advantage, but it implies a cost.

Models 1

Continuous–time

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), u(t)) && (1) \\ y(t) &= h(x(t), w(t)) && (2) \end{aligned}$$

Discrete–time

$$\begin{aligned} x(t+1) &= f(x(t), w(t), u(t)) && (3) \\ y(t) &= h(x(t), w(t)) && (4) \end{aligned}$$

We have that $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^q$ is an external input, $u(t) \in \mathbb{R}^m$ is a control input and $y(t)$ is the output. Assume

$$f(0,0,0) = 0$$

Models 2

We assume that the compensator has the form

$$\begin{aligned} \dot{x}_c(t) &= f_c(x_c(t), w(t), y(t)) && (5) \\ u(t) &= h_c(x_c(t), w(t), y(t)) && (6) \end{aligned}$$

or, it has the form,

$$\begin{aligned} x_c(t+1) &= f_c(x_c(t), w(t), y(t)) && (7) \\ u(t) &= h_c(x_c(t), w(t), y(t)) && (8) \end{aligned}$$

Model 3

Consider the augmented state

$$z(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \tag{9}$$

A dynamic feedback can be always regarded as the static feedback

$$\begin{aligned} v(t) &= f_c(x_c(t), w(t), y(t)) \\ u(t) &= h_c(x_c(t), w(t), y(t)) \end{aligned}$$

for the augmented system

$$\dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} f(x(t), w(t), u(t)) \\ v(t) \end{bmatrix}$$

$$y(t) = h(x(t), w(t))$$

Constraints

Model 4

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- Input constraints

$$u(t) \in \mathcal{U} \tag{10}$$

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- Output constraints
$$y(t) \in \mathcal{Y} \tag{11}$$
- Uncertainty bounds
$$w(t) \in \mathcal{W} \tag{12}$$

Compensator nature

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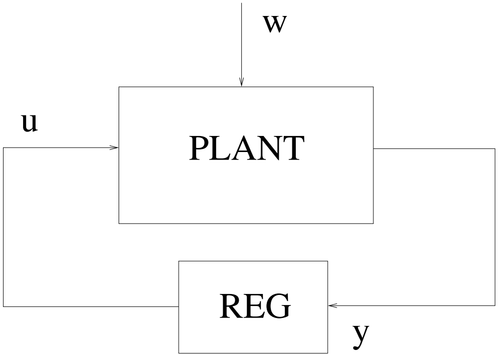
Dynamic game interpretation : game between two individuals u, the “good guy”, and w, the “bad guy”.


w
u
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y

Notations

Sub-level set

$$\mathcal{N}[\Psi,\beta] = \{x : \ \Psi(x) \leq \beta\}.$$

Between–level set

$$\mathcal{N}[\Psi,\alpha,\beta] = \{x : \ \alpha \leq \Psi(x) \leq \beta\}.$$

Gradient

$$\nabla\Psi(x) = \ [\ \frac{\partial\Psi}{\partial x_1} \ \frac{\partial\Psi}{\partial x_2} \ \dots \ \frac{\partial\Psi}{\partial x_n} \ ]$$

Positive (negative) (semi–)definiteness

$$P \ > (\geq, <, \leq) \ 0 \quad \Leftrightarrow \quad x^T P x \ > (\geq, <, \leq) \ 0, \ \ \forall x \neq 0.$$

κ–function. $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, st. increasing $\phi(0) = 0$.

Lyapunov derivative 1

Given a differentiable function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and a derivable trajectory $x(t)$ $x : \mathbb{R} \to \mathbb{R}^n$, we can always consider the composed function $\Psi(x(t))$ whose derivative is

$$\dot{\Psi}(x(t)) = \nabla\Psi(x(t))\dot{x}(t)$$

If $\dot{x}(t) = f(x(t), w(t))$ then for $x(t) = x$ and $w(t) = w$

$$\dot{\Psi}(x(t))|_{x(t)=x} = \nabla\Psi(x)\dot{x} = \nabla\Psi(x)f(x, w) \doteq \dot{\Psi}(x, w)$$

Lyapunov derivative 2

If $x(t)$ is not derivable or $\Psi(x)$ is non–differentiable then we can consider the (upper–right) Dini derivative of $x(t)$

$$D^+\psi(t) \doteq \lim_{h\to 0^+}\sup \frac{\psi(t+h)-\psi(t)}{h}$$

and the upper directional derivative of $\Psi(x)$

$$D^+\Psi(x,w) \doteq \lim_{h\to 0^+}\sup \frac{\Psi(x+hf(x,w))-\Psi(x)}{h}$$

If $\Psi(x):\mathbb{R}^n\to\mathbb{R}$ is Locally Lipschitz function and $x:\mathbb{R}\to\mathbb{R}^n$ is absolutely continuous, then we can consider the following expression for the derivative of the composed function

$$D^+\psi(t) = D^+\Psi(x(t),w(t)) \tag{13}$$

valid almost everywhere.

Lyapunov derivative 3

Theorem

If the absolutely continuous function $x(t)$ is a solution of the differential equation $\dot{x} = f(x, w)$, $\Psi : \mathbb{R}^n \to \mathbb{R}$ a locally Lipschitz function and we define $\psi(t) = \Psi(x(t))$ then we have for all $0 \leq t_1 \leq t_2$

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} D^+\psi(\sigma)\ d\sigma = \int_{t_1}^{t_2} D^+\Psi(x(\sigma), w(\sigma))\ d\sigma \tag{14}$$

The main idea

Example

$$\dot{x}_1(t) = -a(t)x_1(t) + \bar{b}x_1(t)x_2^2(t)$$
$$\dot{x}_2(t) = -\bar{c}x_1(t)^2 x_2(t) - d(t)x_2(t)$$

with $\bar{a}$, $\bar{b}$, $c(t) \geq \bar{c} > 0$, $d(t) \geq \bar{d} > 0$. Choose the *candidate Lyapunov function*

$$\Psi(x_1, x_2) = (\bar{c}x_1^2 + \bar{b}x_2^2)/2$$

$$\dot{\Psi}(x_1, x_2) = \begin{bmatrix} \bar{c}x_1 & \bar{b}x_2 \end{bmatrix} \begin{bmatrix} -ax_1 + \bar{b}x_1^2 x_2 \\ -\bar{c}x_1 x_2^2 - dx_2(t) \end{bmatrix}$$

$$= -a\bar{c}x_1^2 - d\bar{b}x_2^2 \leq -\bar{a}\bar{c}x_1^2 - \bar{d}\bar{b}x_2^2 < 0, \quad \text{for } (x_1, x_2) \neq 0$$

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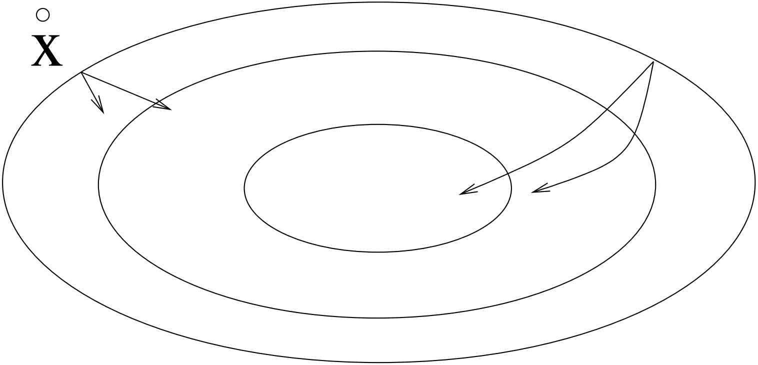

Definition

We say that a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is

The definition admits a "local" version if we replace the conditions "for all x" by "for all $x \in \mathscr{S}$".

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Positive semi-definite : if $\Psi(0) = 0$ and $\Psi(x) \geq 0$ for all x.

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Positive semi-definite : if $\Psi(0) = 0$ and $\Psi(x) \geq 0$ for all x.

Negative semi-definite : if $\Psi(0) = 0$ and $\Psi(x) \leq 0$ for all x.

Radially unbounded : if

$$\lim_{\|x\| \to \infty} |\Psi(x)| = \infty.$$

The definition admits a "local" version if we replace the conditions "for all x" by "for all $x \in \mathscr{S}$".

$$\dot{x}(t) = f(x(t), w(t)), \quad w(t) \in \mathcal{W}$$

Definition

We say that the system is Globally Uniformly Asymptotically Stable if it is

- **Locally Stable**: for all $\nu > 0$ there exists $\delta > 0$ such that if $\|x(0)\| \leq \delta$ then

$$\|x(t)\| \leq \nu, \quad \text{for all } t \geq 0$$

- **Globally Actractive**: for all $\mu > 0$ and $\varepsilon > 0$ there exist $T(\mu, \varepsilon) > 0$ such that if $\|x(0)\| \leq \mu$ then

$$\|x(t)\| \leq \varepsilon, \quad \text{for all } t \geq T(\mu, \varepsilon);$$

for all functions $w(t) \in \mathcal{W}$.

Definition

We say that a locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a Global Lyapunov Function (GLF) for the system if it is positive definite, radially unbounded and there exists a κ–function Φ such that

$$D^+\Psi(x,w) \leq -\phi(\|x(t)\|)$$

Theorem

Assume that the system

$$\dot{x}(t) = f(x(t), w(t)), \quad w(t) \in \mathscr{W}$$

admits a Lyapunov function Ψ. Then it is globally uniformly asymptotically stable.

Definition

We say that system $\dot{x} = f(x, w)$ is Globally Exponentially Stable if there exist $\mu, \gamma > 0$ such that

$$\|x(t)\| \leq \mu \|x(0)\| e^{-\gamma t}$$

Coefficient γ is the **convergence speed**.

Theorem

Assume that there exists a locally Lipschitz function Ψ such that

$$\alpha \|x\|^p \leq \Psi(x) \leq \beta \|x\|^p, \quad \text{for all} \quad x \in \mathbb{R}^n,$$

and

$$D^+\Psi(x, w) \leq -\gamma \Psi(x)$$

for $\gamma > 0$. Then the system it is Globally Exponentially Stable.

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- requiring convergence from arbitrary initial conditions can be too restrictive;

Lyapunov stability 4

Global stability can be somewhat a too ambitious because

- requiring convergence from arbitrary initial conditions can be too restrictive;
- in practice persistent disturbances can prevent the system from approaching the origin.

Definition

Let $\mathscr{S}$ be a neighborhood of the origin. We say that the system is Uniformly Locally Asymptotically Stable with basin of attraction $\mathscr{S}$ if

- **Local Stability**: For all $\mu > 0$ there exists $\delta > 0$ such that $\|x(0)\| \leq \delta$ implies $\|x(t)\| \leq \mu$ for all $t \geq 0$.
- **Local Uniform Convergence** For all $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that if $x(0) \in \mathscr{S}$, then $\|x(t)\| \leq \varepsilon$, for all $t \geq T(\varepsilon)$;

Lyapunov stability 6

Definition

Let $\mathscr{S}$ be a neighborhood of the origin. We say that the locally Lipschitz positive definite function is a Lyapunov function inside $\mathscr{S}$ if there exists $\nu > 0$ such that

$$\mathscr{S} \subseteq \mathscr{N}[\Psi, \nu]$$

and for all $x \in \mathscr{N}[\Psi, \nu]$ the inequality

$$D^+\Psi(x, w) \leq -\phi(\|x(t)\|)$$

holds for some κ–function ϕ.

Theorem

If the system has a Lyapunov function Ψ inside $\mathscr{S}$, then it is Locally Stable with basin of attraction $\mathscr{S}$.

Lyapunov stability 7

Definition

Let $\mathscr{S}$ be a neighborhood of the origin. We say that the system is Uniformly Ultimately Bounded in $\mathscr{S}$ for all $\mu > 0$ there exists $T(\mu) > 0$ such that for $\|x(0)\| \leq \mu$

$$x(t) \in \mathscr{S}$$

for all $t \geq T(\mu)$ and all functions $w(t) \in \mathscr{W}$.

Theorem

If the system has a Lyapunov function Ψ outside $\mathscr{S}$, then it is uniformly ultimately bounded in $\mathscr{S}$.