KALMAN FILTER

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Teoria dei Sistemi e del Controllo
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DISCRETE-TIME KALMAN FILTER

Rudolf Emil Kalman
"A New Approach to Linear Filtering and Prediction Problems"
Transactions of the ASME, Journal of Basic Engineering, pp. 35–45,
March 1960

Problem formulation

Discrete-time linear system:

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\begin{aligned}
x(k+1) &= A(k)x(k) + B_v(k)v(k) \\
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- $x \in \mathbb{R}^n$ system state
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Looking for a recursive algorithm that yields an estimate $\hat{x}$ of the system state

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w, v white noises: zero mean, independent components

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- $E[w(k)v(h)^\top] = 0, \quad E[w(k)x(h)^\top] = 0, \quad E[v(k)x(h)^\top] = 0$

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The filter belongs to the class of Luenberger observers:

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We look for the sequence of matrices $L(k)$ that minimizes

$$E[\|x(k) - \hat{x}(k)\|^2] = trE[(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^\top]$$

$$\left(trM = \sum_i M_{ii}, \quad tr[SS^\top] = tr[S^\top S] = \sum_{ij} S_{ij}^2 \right)$$

Let us consider...

$$\ldots P(k) = E[(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^\top]$$

Hence, $P(k_0) = P_0$ and, at first, the best estimate is $\hat{x}(k_0) = \overline{x}_0$

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Given $P(k)$,

$$P(k+1) = [A(k) - L(k)C(k)]P(k)[A(k) - L(k)C(k)]^\top + B_v(k)Q(k)B_v^\top(k) + L(k)R(k)L^\top(k)$$

Minimization

We choose L so as to minimize $tr[P(k+1)]$:

$$\min_L\{tr[(A-LC)P(A-LC)^\top + B_v Q B_v^\top + LRL^\top)]\} =$$
$$\min_L\{tr[APA^\top]-2tr[APC^\top L^\top]+tr[LCPC^\top L^\top]+tr[B_v Q B_v^\top]+tr[LRL^\top]\}$$

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Differentiate with respect to L and equate to zero:

$$2L(CPC^\top + R) - 2APC^\top = 0 \implies L = APC^\top(CPC^\top + R)^{-1}$$

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Plug into the expression of $P(k+1)$,

$$(A-LC)P(A-LC)^\top + B_v Q B_v^\top + LRL^\top$$
$$= APA^\top - APC^\top(CPC^\top + R)^{-1}CPA^\top + B_v Q B_v^\top$$

Result

Optimal filter

$$\hat{x}(k+1) = [A(k) - L(k)C(k)]\hat{x}(k) + L(k)y(k)$$

$$L(k) = A(k)P(k)C^\top(k)[C(k)P(k)C^\top(k) + R(k)]^{-1}$$

$P(k)$ is recursively given by

$$P(k+1) = A(k)P(k)A^\top(k) \\ - A(k)P(k)C^\top(k)[C(k)P(k)C^\top(k) + R(k)]^{-1}C(k)P(k)A^\top(k) \\ + B_v(k)Q(k)B_v^\top(k),$$

with initial conditions $P(k_0) = P_0$ and $\hat{x}(k_0) = \overline{x}_0$

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Estimation error $x(k+1) - \hat{x}(k+1) = e(k+1) =$

$$[A(k) - L(k)C(k)]e(k) + B_v(k)[v(k) - \overline{v}(k)] - L(k)[w(k) - \overline{w}(k)]$$

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...for instance a control signal, the system becomes

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Estimation error $x(k+1) - \hat{x}(k+1) =$

$$[A(k) - L(k)C(k)][x(k) - \hat{x}(k)] + B_v(k)v(k) - L(k)w(k)$$

exactly as in the absence of u !

Steady-state (asymptotic) filtering algorithm

When all of the system matrices are constant, under suitable assumptions, $P(k+1)$ tends to the symmetric, positive definite solution P of the discrete-time algebraic Riccati equation

$$P = APA^\top - APC^\top (CPC^\top + R)^{-1} CPA^\top + B_v Q B_v^\top$$

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ASYMPTOTIC Kalman filter

$$\hat{x}(k+1) = (A - LC)\hat{x}(k) + Ly(k)$$

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Duality

optimal LQ control, $Q = H^\top H$	optimal observer, $Q = I$
K feedback matrix: $u(k) = Kx(k)$	L observer gain

are DUAL problems

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$$P = A^\top PA - A^\top PB(B^\top PB + R)^{-1}B^\top PA + H^\top H \text{ (LQ)}$$

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$$K = -(B^\top PB + R)^{-1} B^\top PA \qquad L = APC^\top (CPC^\top + R)^{-1}$$

$$K = -L^\top$$

Kalman filter: asymptotic stability

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P is the symmetric positive definite solution of

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which is the

discrete-time Lyapunov equation

$$(A - LC)P(A - LC)^\top - P = -B_v Q B_v^\top - LRL^\top$$

with $B_v Q B_v^\top + LRL^\top$ symmetric positive definite matrix

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$A - LC$ is asymptotically stable and the sequence $\{e_k\}$ goes to zero when $v(k),\ w(k) \equiv 0\ \forall k$

CONTINUOUS-TIME KALMAN-BUCY FILTER

Rudolf Emil Kalman and Richard Snowden Bucy
"New Results in Linear Filtering and Prediction Theory"
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choosing $P(t) = E[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^\top]$

Sampled system, Δt small enough

$$e(t+\Delta t) = [I + A\,\Delta t - LC\,\Delta t]e(t) + B_v\,\Delta t\,v(t) - L\,\Delta t\,w(t)$$

$$\left(e^{M\Delta t} = \sum_{k=0}^{+\infty} \frac{M^k(\Delta t)^k}{k!} = I + M\Delta t + \frac{1}{2!}M^2(\Delta t)^2 + \frac{1}{3!}M^3(\Delta t)^3 + \dots\right)$$

cf. Euler

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Hence, given $P(t)$,
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$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \dot{P}(t) = [A(t) - L(t)C(t)]P(t) + P(t)[A(t) - L(t)C(t)]^\top + B_v(t)Q(t)B_v^\top(t) + L(t)R(t)L^\top(t)$

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Plug into the expression of $\dot{P}(t)$,

$$AP + PA^\top - PC^\top R^{-1} CP + B_v Q B_v^\top$$

Result

Optimal filter

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$$L(t) = P(t)C^\top(t)R^{-1}(t)$$

where

$$\dot{P}(t) = A(t)P(t) + P(t)A^\top(t) - P(t)C^\top(t)R^{-1}C(t)P(t) + B_v(t)Q(t)B_v^\top(t)$$

with initial conditions $P(t_0) = P_0$ and $\hat{x}(t_0) = \overline{x}_0$

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$$A^\top P + PA - PBR^{-1}B^\top P + H^\top H = 0 \text{ (LQ)}$$

$$\overset{dual}{\leftrightarrow}$$

$$AP + PA^\top - PC^\top R^{-1} CP + B_v B_v^\top = 0 \text{ (KF)}$$

$$K = -R^{-1}B^\top P \qquad\qquad L = PC^\top R^{-1}$$

$$K = -L^\top$$

Kalman filter: asymptotic stability

$\dot{e}(t) = (A - LC)e(t) + B_v v(t) - Lw(t)$ stable?

Kalman filter: asymptotic stability

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continuous-time Lyapunov equation

$$[A - LC]P + P[A - LC]^\top = -B_v Q B_v^\top - LRL^\top$$

with $B_v Q B_v^\top + LRL^\top$ symmetric positive definite matrix

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with $B_v Q B_v^\top + LRL^\top$ symmetric positive definite matrix

$A - LC$ is asymptotically stable and $e(t)$ tends to zero when $v(t)$, $w(t) \equiv 0 \; \forall t$

EXAMPLES AND APPLICATIONS

With a known input
Model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B[u(t) + v(t)]\\ y(t) &= Cx(t) + w(t)\end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

mechanical system $m\ddot{q} = F$, con $x_1 = q$, $x_2 = \dot{q}$, $u = F/m$, with disturbances v and w.

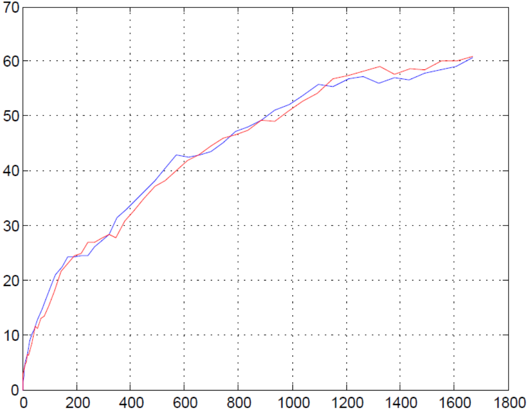
Sampled system $\longrightarrow$ discrete-time system matrices $\longrightarrow$ discrete-time Kalman filter
$\hat{x}(k+1) = (A - LC)\hat{x}(k) + Ly(k) + Bu(k)$.

With a known input

Simulation

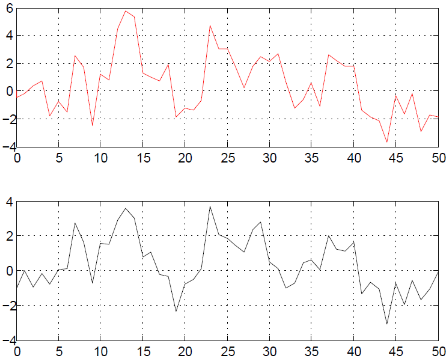
State space trajectory: x blue, $\hat{x}$ red

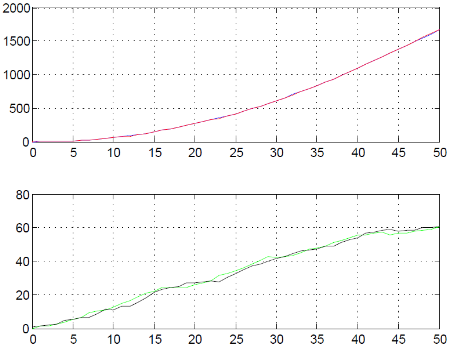
$$u = 1,\ T_s = 1,\ Q = 1,\ R = 2$$



With a known input
Simulation

$$u = 1, \; T_s = 1, \; Q = 1, \; R = 2$$





Error $x - \hat{x}$ for the two components

Comparison: $x(1)$ blue, $\hat{x}(1)$ red; $x(2)$ green, $\hat{x}(2)$ black

With a known input

Model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t) + v(t)] \\ y(t) &= Cx(t) + w(t) \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

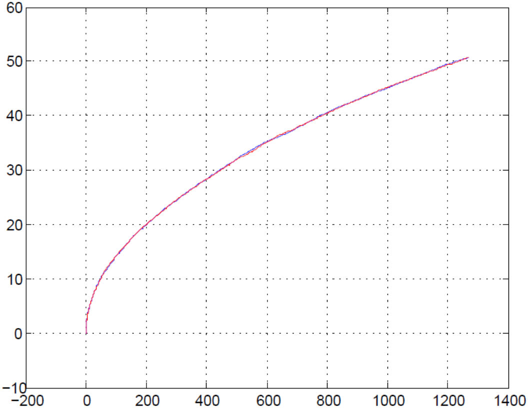
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Now rewrite the system approximating the derivative with Euler's method: $x(t+\tau) = [I + \tau A]x(t) + B\tau[u(t) + v(t)]$ and compute the continuous-time Kalman filter
$\hat{x}(t+\tau) = (I + A\tau - LC\tau)\hat{x}(t) + L\tau y(t) + B\tau u(t)$.

With a known input
Simulation

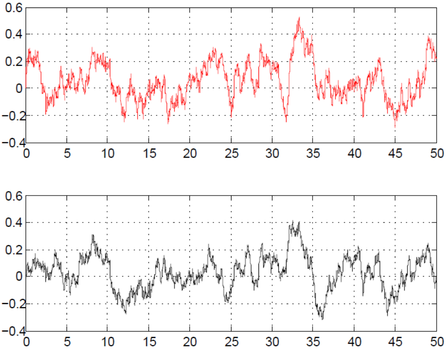
State space trajectory: x blue, $\hat{x}$ red

$$u = 1, \; \tau = 0.01, \; Q = 1, \; R = 2$$

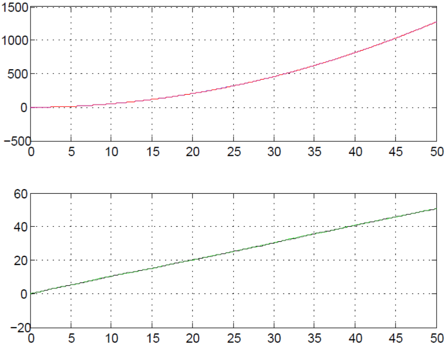


With a known input
Simulation

$$u = 1, \ \tau = 0.01, \ Q = 1, \ R = 2$$



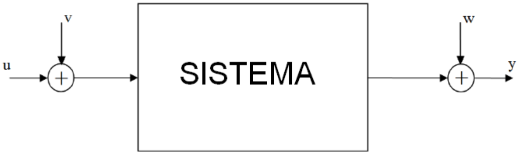
Error $x - \hat{x}$ for the two components



Comparison: $x(1)$ blue, $\hat{x}(1)$ red; $x(2)$ green, $\hat{x}(2)$ black

With a known input

Marginally stable system with disturbances: the model



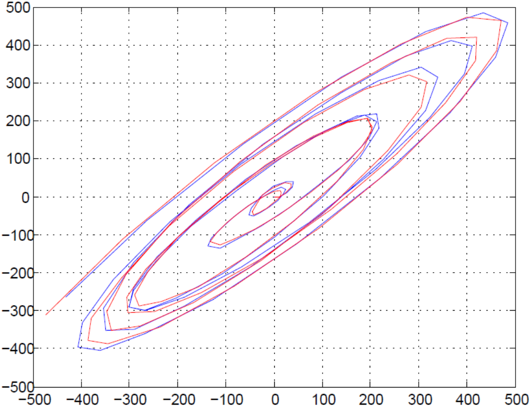
Transfer function $\frac{1}{s^2+\omega^2}$, with $\omega = 2$.

Discrete-time system version, discrete-time Kalman filter.

With a known input

Marginally stable system with disturbances: simulation

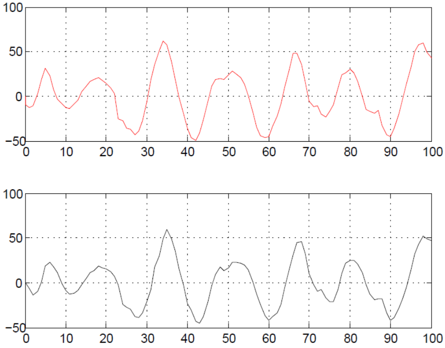
State space trajectory: x blue, $\hat{x}$ red



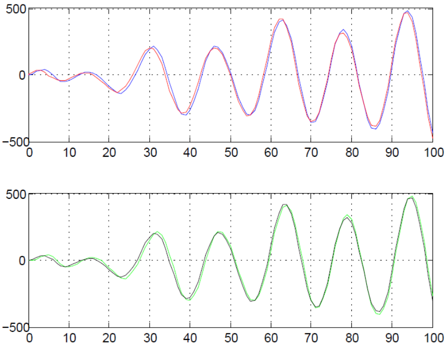
With a known input

Marginally stable system with disturbances: simulation

$$T_s = 0.2, \; Q = 1, \; R = 1$$



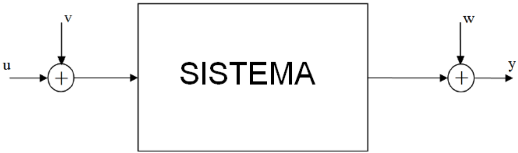
Error $x - \hat{x}$



Comparison: $x(1)$ blue, $\hat{x}(1)$ red; $x(2)$ green, $\hat{x}(2)$ black

With a known input

Stable system with disturbances: model

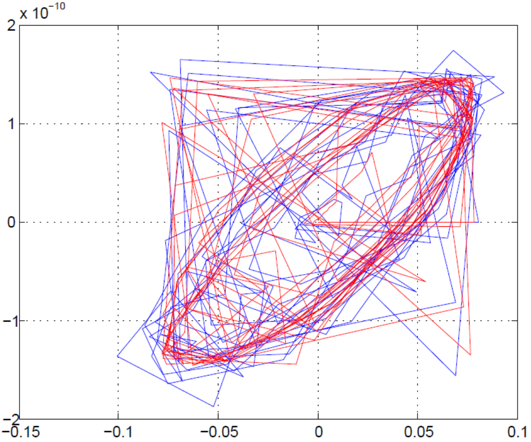


Transfer function $\frac{1}{s^2+300s+20000}$.

Discrete-time version of the system, discrete-time Kalman filter.

With a known input
Stable system with disturbances: simulation

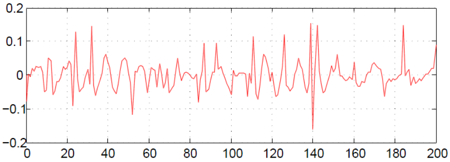
State space trajectory: x blue, $\hat{x}$ red

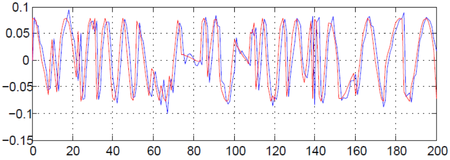


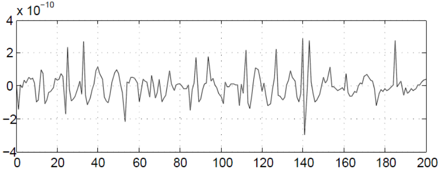
With a known input

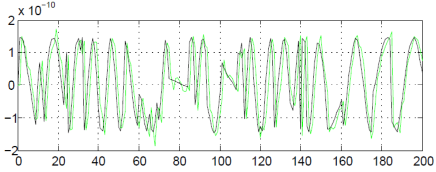
Stable system with disturbances: simulation

$$T_s = 0.2,\ Q = 1,\ R = 1$$









Error $x - \hat{x}$

Comparison: $x(1)$ blue, $\hat{x}(1)$ red; $x(2)$ green, $\hat{x}(2)$ black

Output estimate

For a system with disturbances

$$\begin{aligned} x(k+1) &= Ax(k) + Bv(k) \\ z(k) &= Hx(k) \\ y(k) &= Cx(k) + Dw(k) \end{aligned}$$

Output estimate

For a system with disturbances

$$\begin{aligned} x(k+1) &= Ax(k)+Bv(k) \\ z(k) &= Hx(k) \\ y(k) &= Cx(k)+Dw(k) \end{aligned}$$

we want to estimate output z based on measured output y:

Output estimate

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$$\begin{aligned} x(k+1) &= Ax(k)+Bv(k) \\ z(k) &= Hx(k) \\ y(k) &= Cx(k)+Dw(k) \end{aligned}$$

we want to estimate output z based on measured output y:
Kalman filter:

$$\begin{aligned} \hat{x}(k+1) &= (A-LC)\hat{x}(k)+Ly(k) \\ \hat{z}(k) &= H\hat{x}(k) \end{aligned}$$

where: $L = APC^\top(CPC^\top+R)^{-1}$
$P = APA^\top + APC^\top(CPC^\top+R)^{-1}CPA^\top + BQB^\top.$

Output estimate

For a system with disturbances . . . and a known input

$$\begin{aligned} x(k+1) &= Ax(k) + Bv(k) + Eu(k) \\ z(k) &= Hx(k) \\ y(k) &= Cx(k) + Dw(k) \end{aligned}$$

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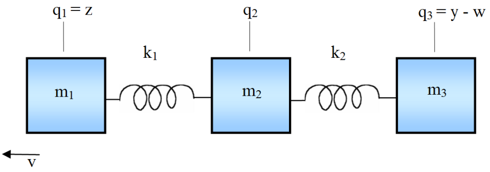
$P = APA^\top + APC^\top(CPC^\top + R)^{-1}CPA^\top + BQB^\top.$

With disturbances only

Masses and springs system: model



$$\begin{aligned} \dot{x} &= Ax + Bv \\ z &= C_z x \\ y &= C_y x + w \end{aligned}$$

$$k_1 = k_2 = 1,\ m_1 = m_2 = m_3 = 1$$

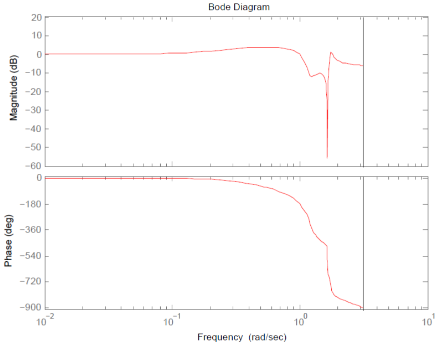
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

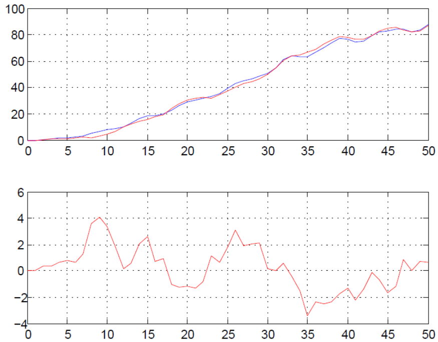
$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C_y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

With disturbances only

Masses and springs system: simulation

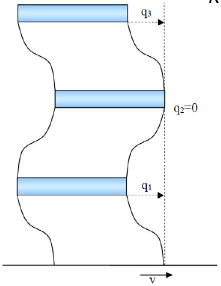
$$T_s = 1,\ Q = 1,\ R = 1$$





With disturbances only

Three-floor building: model



$k_1 = 2,\ k_{12} = 1,\ k_{23} = 1,\ \varphi = 0.01$

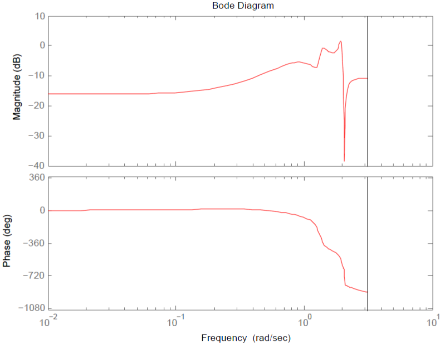
$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -(k_1 + k_{12}) & k_{12} & 0 & -\varphi & 0 & 0 \\ k_{12} & -(k_{12} + k_{23}) & k_{23} & 0 & -\varphi & 0 \\ 0 & k_{23} & -k_{23} & 0 & 0 & -\varphi \end{bmatrix}$$

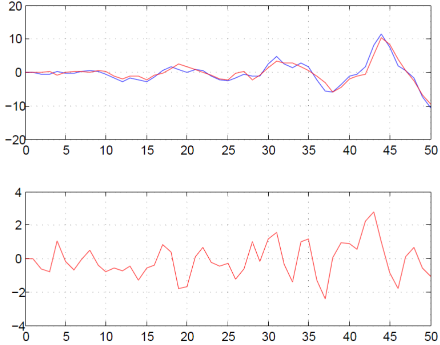
$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ C_y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

With disturbances only

Three-floor building: simulation

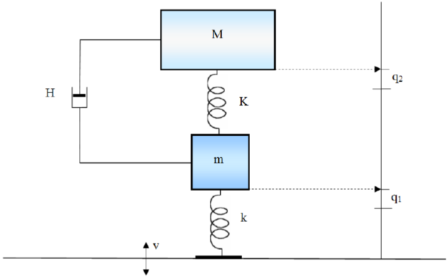
$$T_s = 1,\ Q = 1,\ R = 1$$





With disturbances only

Car damper: model



$$k_1 = 2,\ k_2 = 3,\ h = 0.1$$

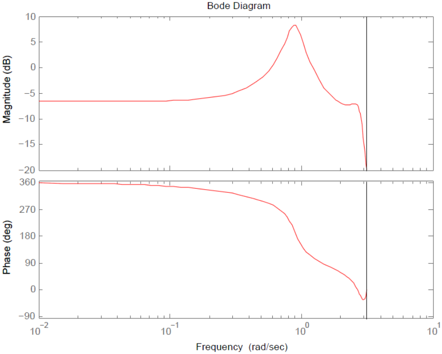
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1 + k_2) & k_2 & -h & h \\ k_2 & -k_2 & h & -h \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ k_1 \\ 0 \end{bmatrix},$$

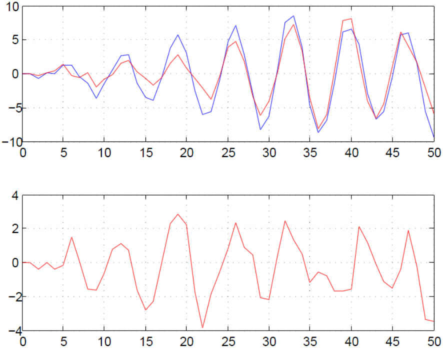
$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ C_y = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}.$$

With disturbances only

Car damper: simulation

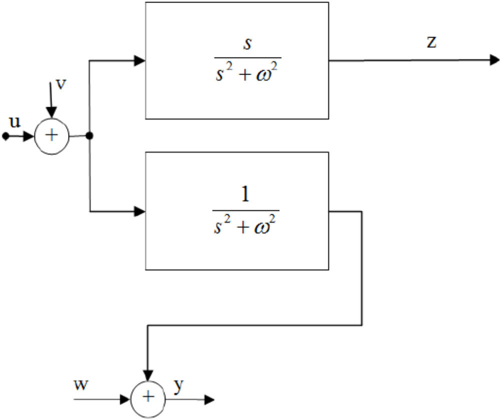
$$T_s = 1,\ Q = 1,\ R = 1$$





With a known input

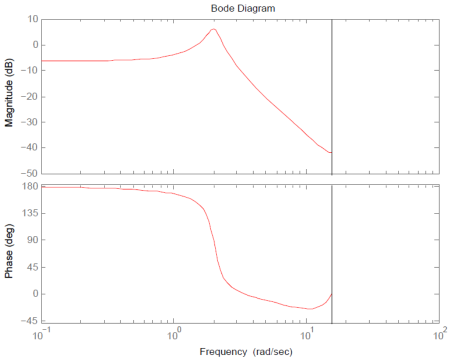
Second-order system: model

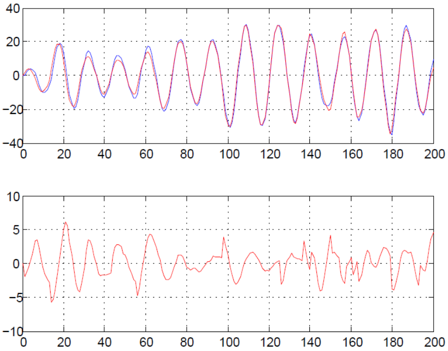


With a known input

Second-order system: simulation

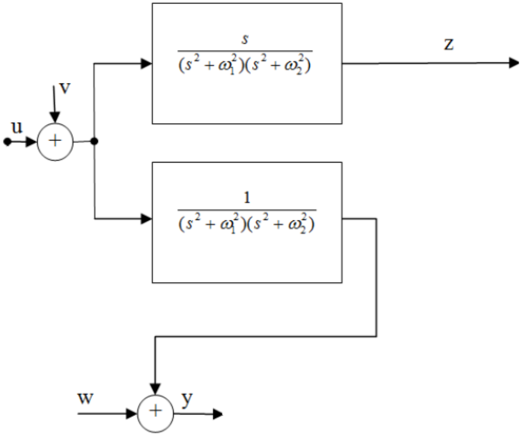
$$\omega = 2,\ T_s = 0.2,\ Q = 1,\ R = 1$$





With a known input

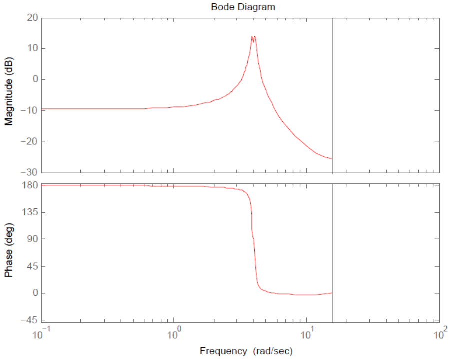
Fourth-order system: model

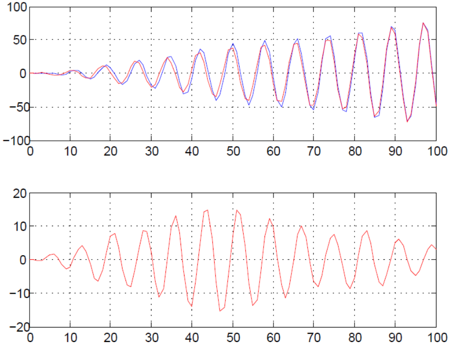


With a known input

Fourth-order system: simulation

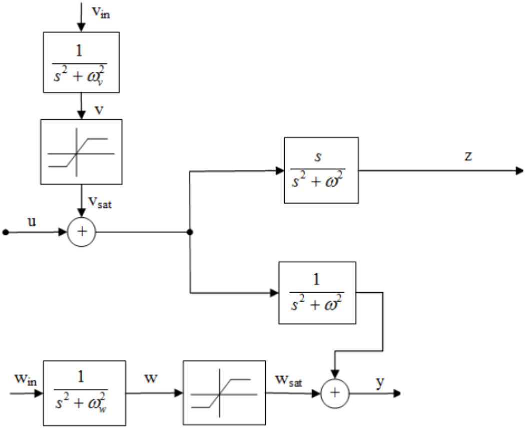
$$\omega_1 = 4,\ \omega_2 = 4,\ T_s = 0.2,\ Q = 1,\ R = 1$$





With a known input

System with selective prefilters: model



With a known input

System with selective prefilters: simulation

$$\omega = 4,\ \omega_v = 4,\ \omega_w = 2,\ T_s = 0.2,\ \beta = 1,\ \gamma = 1,\ Q = 1,\ R = 1$$