Intersection Types and Domain Operators ⋆

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Abstract

We use intersection types as a tool for obtaining λ-models. Relying on the notion of *easy intersection type theory* we successfully build a λ-model in which the interpretation of an arbitrary simple easy term is any filter which can be described by a continuous predicate. This allows us to prove two results. The first gives a proof of consistency of the λ-theory where the λ-term $(\lambda x.xx)(\lambda x.xx)$ is forced to behave as the join operator. This result has interesting consequences on the algebraic structure of the lattice of λ-theories. The second result is that for any simple easy term there is a λ-model where the interpretation of the term is the *minimal* fixed point operator.

Introduction

Intersection types were introduced in the late 70's by Dezani and Coppo [8, 10, 6], to overcome the limitations of Curry's type discipline. They are a very expressive type language which allows to describe and capture various properties of λ-terms. For instance, they have been used in Pottinger [19] to give the first type theoretic characterisation of *strongly normalizable* terms and in Coppo et al. [11] to capture *persistently normalising terms* and *normalising terms*. See Dezani et al. [12], appearing also in this issue, for a more complete account of this line of research.

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Intersection types have a very significant realizability semantics with respect to applicative structures. This is a generalisation of Scott's natural semantics [20] of simple types. According to this interpretation types denote subsets of the applicative structure, an arrow type $A \to B$ denotes the sets of points which map all points belonging to the interpretation of A to points belonging to the interpretation of B, and an intersection type $A \cap B$ denotes the intersections of the interpretation of A and the interpretation of B. Building on this, intersection types have been used in Barendregt et al. [6] to give a proof of the completeness of the natural semantics of Curry's simple type assignment system in applicative structures, introduced in Scott [20].

Intersection types have also an alternative semantics based on *duality* which is related to Abramsky's *Domain Theory in Logical Form* [1]. Actually it amounts to the application of that paradigm to the special case of ω-algebraic lattice models of pure λ-calculus, see Coppo et al. [9]. Namely, types correspond to *compact elements*: the type Ω denotes the least element, intersections denote *joins* of compact elements, and arrow types denote *step functions* of compact elements. A typing judgement can then be interpreted as saying that a given term belongs to a pointed compact open set in an ω-algebraic lattice model of λ-calculus. By duality, type theories give rise to *filter λ-models*. Intersection type assignment systems can then be viewed as *finitary logical* descriptions of the interpretation of λ-terms in such models, where the meaning of a λ-term is the set of types which are deducible for it. This duality lies at the heart of the success of intersection types as a powerful tool for the analysis of λ-models, see Plotkin [18] and the references there. Section 2 of Dezani et al. [12] discusses intersection types (with partial intersection) as finitary descriptions of Scott domains.

The λ-models we build out of intersection types differ essentially in the *preorder relations* between types. In all these preorders, the equivalencies between atomic types and intersections of arrow types are crucial in order to determine the theory. In the present paper the focus is on semantic proofs of consistencies of λ-theories.

Actually, the mainstream of consistency proofs is based on the use of syntactic tools (see Kuper [15] and the references there). Instead, very little literature can be found on the application of semantic tools, we can mention the papers Zylberajch [22], Baeten and Boerboom [5], Alessi et al. [3], Alessi and Lusin [4].

In [4] Alessi and Lusin deal with the issue of easiness proofs of λ-terms from the semantic point of view (we recall that a closed term $\mathbf{e}$ is *easy* if, for any other closed term $\mathbf{t}$, the theory $\lambda\beta + \{\mathbf{t} = \mathbf{e}\}$ is consistent). Going in the direction of Alessi et al. [3], they introduced the notion of *simple easiness*: this notion, which turns out to be stronger than easiness, can be handled in a natural way by semantic tools, and allows to prove consistency results via construction of suitable filter models: given a simple easy term $\mathbf{e}$ and an arbitrary closed term $\mathbf{t}$, it is possible to build (in a canonical way) a non-trivial filter model which equates the interpretation of $\mathbf{e}$ and

$\mathbf{t}$. [4] proves in such a way the easiness of several terms.

Besides, simple easiness is interesting in itself, since it has to do with minimal sets of axioms which are needed in order to assign certain types to the easy terms.

The theoretical contribution of the present paper is to show how the relation between simple easiness and axioms on types can be put to work in order to interpret easy terms by filters described by *continuous* predicates.

This produces two nice applications. Given an arbitrary simple easy term $\mathbf{e}$ we build a filter model $\mathcal{F}^{\nabla}$ such that $\mathbf{e}$ is interpreted as the join operator: the interpretation $[\![\mathbf{e}]\!]^{\nabla}$ of $\mathbf{e}$ in $\mathcal{F}^{\nabla}$ is exactly the filter Θ such that for all filters Ξ, Υ

$$(\Theta \cdot \Xi) \cdot \Upsilon = \Xi \sqcup \Upsilon.$$

This way we prove the consistency of the λ-theory which equates $(\lambda x.xx)(\lambda x.xx)$ to the join operator. This consistency has been used in Lusin and Salibra [17] to show that there exists a sub-lattice of the lattice of λ-theories which satisfies many interesting algebraic properties.

The second result we give is the following. For an arbitrary simple easy term $\mathbf{e}$ there is a filter model $\mathcal{F}^{\nabla}$ such that the interpretation $[\![\mathbf{e}]\!]^{\nabla}$ of $\mathbf{e}$ in $\mathcal{F}^{\nabla}$ is the *minimal* fixed point operator μ (that is $\mu(f) = \bigsqcup_n f^n(\bot)$, for all continuous endofunctions f over $\mathcal{F}^{\nabla}$). This result is not trivial: easy terms can obviously be equated to an arbitrary fixed point combinator $\mathbf{Y}$, i.e. it is possible to find λ-models $\mathcal{M}$ such that $[\![\mathbf{e}]\!]^{\mathcal{M}} = [\![\mathbf{Y}]\!]^{\mathcal{M}}$. This only implies that $[\![\mathbf{e}]\!]^{\mathcal{M}}$ represents a fixed point operator, but there is no guarantee as to the *minimality*.

The present paper is organised as follows. In Section 1 we present easy intersection type theories and type assignment systems for them. In Section 2 we introduce λ-models based on spaces of filters in easy intersection type theories. Section 3 gives the main theoretical contribution of the present paper: after introducing simple easy terms, we show that each simple easy term can be interpreted as an arbitrary filter which can be described by a *continuous* predicate. In Section 4 and Section 5 we derive from our result the two above mentioned applications. Finally, Section 6 discusses similarities and differences between the present paper and Dezani et al. [12].

The consistency of the λ-theory in which the λ-term $(\lambda x.xx)(\lambda x.xx)$ behaves as the join operator was presented at WIT'02 [13].

1 Intersection Type Assignment Systems

Intersection types are syntactic objects built inductively by closing a given set $\mathbb{C}$ of *type atoms* (constants) which contains the universal type Ω under the *function type* constructor $\to$ and the *intersection* type constructor $\cap$.

Definition 1 (Intersection type language) *Let* $\mathbb{C}$ *be a countable set of constants such that* $\Omega \in \mathbb{C}$. *The* intersection type language *over* $\mathbb{C}$, *denoted by* $\mathbb{T} = \mathbb{T}(\mathbb{C})$, *is defined by the following abstract syntax:*

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}.$$

Notation Upper case Roman letters i.e. $A, B, \ldots$, will denote arbitrary types. Greek letters $\phi, \psi, \ldots$ will denote constants in $\mathbb{C}$. When writing intersection types we shall use the following conventions:

- the constructor $\cap$ takes precedence over the constructor $\to$;
- the constructor $\to$ associates to the right;
- $\bigcap_{i \in I} A_i$ with $I = \{1, \ldots, n\}$ and $n \geq 1$ is short for $((\ldots(A_1 \cap A_2)\ldots) \cap A_n)$;
- $\bigcap_{i \in I} A_i$ with $I = \emptyset$ is Ω.

Much of the expressive power of intersection type disciplines comes from the fact that types can be endowed with a *preorder relation* $\leq$ which satisfies axioms and rules $\overline{\bigtriangledown}$ of Figure 1, so inducing the structure of a meet semi-lattice with respect to $\cap$, the top element being Ω. We recall here the notion of *easy* intersection type theory as first introduced in Alessi and Lusin [4].

$(refl)$	$A \leq A$	$(trans)$	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$
(mon)	$\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$	$(idem)$	$A \leq A \cap A$
$(incl_L)$	$A \cap B \leq A$	$(incl_R)$	$A \cap B \leq B$
$(\to\cap)$	$(A \to B) \cap (A \to C) \leq A \to B \cap C$	(η)	$\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$
(Ω)	$A \leq \Omega$	$(\Omega\text{-}\eta)$	$\Omega \leq \Omega \to \Omega$

Fig. 1. The axioms and rules of $\overline{\bigtriangledown}$

Definition 2 (Easy intersection type theories) *Let* $\mathbb{T} = \mathbb{T}(\mathbb{C})$ *be an intersection type language. The* easy intersection type theory *(*eitt *for short)* $\Sigma(\mathbb{C}, \bigtriangledown)$ *over* $\mathbb{T}$ *is the set of all judgements* $A \leq B$ *derivable from* $\bigtriangledown$, *where* $\bigtriangledown$ *is a collection of axioms and rules such that (we write* $A \sim B$ *for* $A \leq B$ & $B \leq A$ *and* $\bigtriangledown^-$ *for* $\bigtriangledown \setminus \overline{\bigtriangledown}$*):*

(1) $\bigtriangledown$ *contains the set* $\overline{\bigtriangledown}$ *of axioms and rules shown in Figure 1.*

(2) $\bigtriangledown^-$ *contains axioms of the following two shapes only:*

$$\phi \leq \phi',$$
$$\phi \sim \bigcap_{h \in H}(\varphi_h \to F_h).$$

where $\phi, \phi', \varphi_h \in \mathbb{C}$*,* $F_h \in \mathbb{T}$*, and* $\phi, \phi' \not\equiv \Omega$*;*

(3) no rule is in $\bigtriangledown^-$*;*

(4) for each $\phi \not\equiv \Omega$ *there is exactly one axiom in* $\bigtriangledown^-$ *of the shape* $\phi \sim \bigcap_{h \in H}(\varphi_h \to F_h)$*;*

(5) let $\bigtriangledown^-$ *contain* $\phi \sim \bigcap_{h \in H}(\varphi_h \to F_h)$ *and* $\phi' \sim \bigcap_{k \in K}(\varphi'_k \to F'_k)$*, with* $\phi, \phi' \not\equiv \Omega$*. Then* $\bigtriangledown^-$ *contains also* $\phi \leq \phi'$ *iff for each* $k \in K$*, there exists* $h_k \in H$ *such that* ${\varphi'}_k \leq \varphi_{h_k}$ *and* $F_{h_k} \leq {F'}_k$ *are both in* $\bigtriangledown^-$*.*

Notice that:
(a) since $\Omega \sim \Omega \to \Omega \in \Sigma(\mathbb{C}, \bigtriangledown)$ by (Ω) and (Ω-η), it follows that all atoms in $\mathbb{C}$ are equivalent to suitable (intersections of) arrow types;
(b) $\cap$ (modulo $\sim$) is associative and commutative;
(c) in the last clause of the above definition F'_k and F_{h_k} must be constant types for each $k \in K$.

Notation When we consider an eitt $\Sigma(\mathbb{C}, \bigtriangledown)$, we will write $\mathbb{C}^\bigtriangledown$ for $\mathbb{C}$, $\mathbb{T}^\bigtriangledown$ for $\mathbb{T}(\mathbb{C})$ and $\Sigma^\bigtriangledown$ for $\Sigma(\mathbb{C}, \bigtriangledown)$. Moreover $A \leq_\bigtriangledown B$ will be short for $(A \leq B) \in \Sigma^\bigtriangledown$ and $A \sim_\bigtriangledown B$ for $A \leq_\bigtriangledown B \leq_\bigtriangledown A$. We will consider syntactic equivalence "$\equiv$" of types up to associativity and commutativity of $\cap$.

One can easily show that all types (not only type constants) are equivalent to suitable intersections of arrow types. This is stated in the following lemma together with a simple inequality between intersections of arrows and arrows of intersections.

Lemma 3

(1) For all $A \in \mathbb{T}^\bigtriangledown$ *there are* I*, and* $B_i, C_i \in \mathbb{T}^\bigtriangledown$ *such that*

$$A \sim_\bigtriangledown \bigcap_{i \in I}(B_i \to C_i).$$

(2) For all $J \subseteq I$*, and* $A_i, B_i \in \mathbb{T}^\bigtriangledown$*,*

$$\bigcap_{i \in I}(A_i \to B_i) \leq_\bigtriangledown (\bigcap_{i \in J} A_i) \to (\bigcap_{i \in J} B_i).$$

A nice feature of eitts is the possibility of performing smooth induction proofs based on the number of arrows in types. In view of this aim next definition and lemma work.

Definition 4 *The mapping $\# : \mathbb{T}^\triangledown \to \mathbb{N}$ is defined inductively on types as follows:*

$$\begin{array}{lll} \#(A) & = 0 & \textit{if } A \in \mathbb{C}^\triangledown; \\ \#(A \to B) & = \#(A) + 1; & \\ \#(A \cap B) & = \max\{\#(A), \#(B)\}. & \end{array}$$

Lemma 5 *For all $A \in \mathbb{T}^\triangledown$ with $\#(A) \geq 1$ there is $B \in \mathbb{T}^\triangledown$ such that $A \sim_\triangledown B$, $B \equiv \bigcap_{i\in I}(C_i \to D_i)$, and $\#(B) = \#(A)$.*

PROOF. Let $A \equiv (\bigcap_{j\in J}(C'_j \to D'_j)) \cap (\bigcap_{h\in H} \phi_h)$, where $C'_j, D'_j \in \mathbb{T}^\triangledown$, $\phi_h \in \mathbb{C}^\triangledown$. For each $h \in H$ there are $I^{(h)}$, $\varphi_i^{(h)} \in \mathbb{C}^\triangledown$, $F_i^{(h)} \in \mathbb{T}^\triangledown$, such that

$$\phi_h \sim_\triangledown \bigcap_{i\in I^{(h)}} (\varphi_i^{(h)} \to F_i^{(h)}).$$

We can choose

$$B \equiv (\bigcap_{j\in J}(C'_j \to D'_j)) \cap (\bigcap_{h\in H}(\bigcap_{i\in I^{(h)}} (\varphi_i^{(h)} \to F_i^{(h)}))).$$

Another nice feature of eitts is that the order between intersections of arrows agrees with the order between joins of step functions. This property, which is fully explained in Section 2 of [12], relies on the next theorem.

Theorem 6 *For all I, and $A_i, B_i, C, D \in \mathbb{T}^\triangledown$,*

$$\bigcap_{i\in I}(A_i \to B_i) \leq_\triangledown C \to D \textit{ iff } \bigcap_{i\in J} B_i \leq_\triangledown D \textit{ where } J = \{i \in I \mid C \leq_\triangledown A_i\}.$$

PROOF. $(\Leftarrow)$ easily follows from Lemma 3*(2)* and rule (η).

As to $(\Rightarrow)$, recall that, by Definition 2, for each constant $\phi \neq \Omega$, there is exactly one judgment in $\triangledown^-$ of the form $\phi \sim \bigcap_{h\in H}(\varphi_h \to F_h)$. One can prove the statement by induction on the definition of $\leq_\triangledown$; the only non-trivial case is when the inequality is derived using transitivity as a last step with the middle type consisting in an intersection containing some constant in $\mathbb{C}^\triangledown$. In that case use property of Definition 2*(2)*.

Notice that in the statement of Theorem 6 the set J may be empty, and in this case we get $\Omega \sim_\triangledown D$.

Before giving the crucial notion of *intersection-type assignment system*, we introduce bases and some related definitions.

Definition 7 (Basis) *A $\triangledown$-basis is a (possibly infinite) set of statements of the shape $x:A$, where $A \in \mathbb{T}^\triangledown$, with all variables distinct.*

We will use the following notation:

(1) $x \in \Gamma$ is short for $(x : A) \in \Gamma$ for some A.
(2) If Γ is a $\triangledown$-basis and $A \in \mathbb{T}^\triangledown$ then $\Gamma, x:A$ is short for $\Gamma \cup \{x:A\}$ when $x \notin \Gamma$.
(3) Let Γ and Γ' be $\triangledown$-bases. The $\triangledown$-basis $\Gamma \uplus \Gamma'$ is defined as follows:

$$\begin{aligned}\Gamma \uplus \Gamma' = {} & \{x\colon A \cap B \mid x\colon A \in \Gamma \text{ and } x\colon B \in \Gamma'\} \\ & \cup \{x\colon A \mid x\colon A \in \Gamma \text{ and } x \notin \Gamma'\} \\ & \cup \{x\colon B \mid x\colon B \in \Gamma' \text{ and } x \notin \Gamma\}.\end{aligned}$$

Definition 8 (Type assignment system) *The* intersection type assignment system *relative to the eitt $\Sigma^\triangledown$, notation $\lambda\cap^\triangledown$, is a formal system for deriving judgements of the form $\Gamma \vdash^\triangledown \mathbf{t} : A$, where the* subject $\mathbf{t}$ *is an untyped λ-term, the* predicate A *is in $\mathbb{T}^\triangledown$, and Γ is a $\triangledown$-basis. Its axioms and rules are the following:*

$$(\mathrm{Ax})\ \frac{(x:A) \in \Gamma}{\Gamma \vdash^\triangledown x:A} \qquad (\mathrm{Ax}\text{-}\Omega)\ \Gamma \vdash^\triangledown \mathbf{t} : \Omega$$

$$(\to \mathrm{I})\ \frac{\Gamma, x:A \vdash^\triangledown \mathbf{t} : B}{\Gamma \vdash^\triangledown \lambda x.\mathbf{t} : A \to B} \qquad (\to \mathrm{E})\ \frac{\Gamma \vdash^\triangledown \mathbf{t} : A \to B \quad \Gamma \vdash^\triangledown \mathbf{u} : A}{\Gamma \vdash^\triangledown \mathbf{tu} : B}$$

$$(\cap \mathrm{I})\ \frac{\Gamma \vdash^\triangledown \mathbf{t} : A \quad \Gamma \vdash^\triangledown \mathbf{t} : B}{\Gamma \vdash^\triangledown \mathbf{t} : A \cap B} \qquad (\leq_\triangledown)\ \frac{\Gamma \vdash^\triangledown \mathbf{t} : A \quad A \leq_\triangledown B}{\Gamma \vdash^\triangledown \mathbf{t} : B}$$

Example 9 *Self-application can be easily typed in all $\lambda\cap^\triangledown$, as follows.*

$$\dfrac{\dfrac{\dfrac{x : (A \to B) \cap A \vdash^\triangledown x : (A \to B) \cap A}{x : (A \to B) \cap A \vdash^\triangledown x : A \to B}\,(\leq_\triangledown) \qquad \dfrac{x : (A \to B) \cap A \vdash^\triangledown x : (A \to B) \cap A}{x : (A \to B) \cap A \vdash^\triangledown x : A}\,(\leq_\triangledown)}{x : (A \to B) \cap A \vdash^\triangledown xx : B}\,(\to \mathrm{E})}{\vdash^\triangledown \lambda x.xx : (A \to B) \cap A \to B}\,(\to \mathrm{I})$$

Notice that due to the presence of axiom $(\mathrm{Ax}\text{-}\Omega)$, one can type terms without assuming types for their free variables.

As usual we consider λ-terms modulo α-conversion. Notice that intersection elimination rules

$$(\cap \mathrm{E})\ \frac{\Gamma \vdash^\triangledown \mathbf{t} : A \cap B}{\Gamma \vdash^\triangledown \mathbf{t} : A} \qquad \frac{\Gamma \vdash^\triangledown \mathbf{t} : A \cap B}{\Gamma \vdash^\triangledown \mathbf{t} : B}$$

are derivable[1] in any $\lambda\cap^\triangledown$.

[1] Recall that a rule is *derivable* in a system if, for each instance of the rule, there is a deduction in the system of its conclusion from its premises. A rule is *admissible* in a system

Moreover, the following rules are admissible:

$$(\leq_\nabla \text{L})\ \frac{\Gamma, x : A \vdash \mathbf{t} : B \quad A' \leq_\nabla A}{\Gamma, x : A' \vdash \mathbf{t} : B}$$

$$(\text{W})\ \frac{\Gamma \vdash \mathbf{t} : B \quad x \notin \Gamma}{\Gamma, x : A \vdash \mathbf{t} : B} \qquad (\text{S})\ \frac{\Gamma, x : A \vdash \mathbf{t} : B \quad x \notin FV(\mathbf{t})}{\Gamma \vdash \mathbf{t} : B}$$

We end this section with a standard Generation Theorem.

Theorem 10 (Generation Theorem)

(1) Assume $A \not\sim_\nabla \Omega$*. Then* $\Gamma \vdash^\nabla x : A$ *iff* $(x : B) \in \Gamma$ *and* $B \leq_\nabla A$ *for some* $B \in \mathbb{T}^\nabla$.
(2) $\Gamma \vdash^\nabla \mathbf{tu} : A$ *iff* $\Gamma \vdash^\nabla \mathbf{t} : B \to A$*, and* $\Gamma \vdash^\nabla \mathbf{u} : B$ *for some* $B \in \mathbb{T}^\nabla$.
(3) $\Gamma \vdash^\nabla \lambda x.\mathbf{t} : A$ *iff* $\Gamma, x : B_i \vdash^\nabla \mathbf{t} : C_i$ *and* $\bigcap_{i \in I}(B_i \to C_i) \leq_\nabla A$*, for some* I *and* $B_i, C_i \in \mathbb{T}^\nabla$.
(4) $\Gamma \vdash^\nabla \lambda x.\mathbf{t} : B \to C$ *iff* $\Gamma, x : B \vdash^\nabla \mathbf{t} : C$.

PROOF. The proof of each ($\Leftarrow$) is easy. So we only treat ($\Rightarrow$).

(1) Easy by induction on derivations, since only the axioms (Ax), (Ax-Ω), and the rules ($\cap$I), ($\leq_\nabla$) can be applied. Notice that the condition $A \not\sim_\nabla \Omega$ implies that $\Gamma \vdash^\nabla x : A$ cannot be obtained just using axiom (Ax-Ω).

(2) If $A \sim_\nabla \Omega$ we can choose $B \sim_\nabla \Omega$. Otherwise, the proof is by induction on derivations. The only interesting case is when $A \equiv A_1 \cap A_2$ and the last applied rule is ($\cap$I):

$$(\cap\text{I})\ \frac{\Gamma \vdash^\nabla \mathbf{tu} : A_1 \quad \Gamma \vdash^\nabla \mathbf{tu} : A_2}{\Gamma \vdash^\nabla \mathbf{tu} : A_1 \cap A_2}.$$

The condition $A \not\sim_\nabla \Omega$ implies that we cannot have $A_1 \sim_\nabla A_2 \sim_\nabla \Omega$. We give the proof for $A_1 \not\sim_\nabla \Omega$ and $A_2 \not\sim_\nabla \Omega$, the other cases can be treated similarly. By induction there are B, D such that

$$\Gamma \vdash^\nabla \mathbf{t} : B \to A_1, \quad \Gamma \vdash^\nabla \mathbf{u} : B,$$
$$\Gamma \vdash^\nabla \mathbf{t} : D \to A_2, \quad \Gamma \vdash^\nabla \mathbf{u} : D.$$

Then $\Gamma \vdash^\nabla \mathbf{t} : (B \to A_1) \cap (D \to A_2)$ and by Lemma 3*(2)* and rule (η)

$$(B \to A_1) \cap (D \to A_2) \leq_\nabla B \cap D \to A_1 \cap A_2 \leq B \cap D \to A.$$

We are done, since $\Gamma \vdash^\nabla \mathbf{u} : B \cap D$ by rule ($\cap$I) .

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if, for each instance of the rule, if its premises are derivable in the system then so is its conclusion.

3*(2)* and rule

(3) The proof is very similar to the proof of Point *(2)*. It is again by induction on derivations and again the only interesting case is when the last applied rule is $(\cap \mathrm{I})$:

$$(\cap \mathrm{I})\ \frac{\Gamma \vdash^{\nabla} \lambda x.\mathbf{t} : A_1 \quad \Gamma \vdash^{\nabla} \lambda x.\mathbf{t} : A_2}{\Gamma \vdash^{\nabla} \lambda x.\mathbf{t} : A_1 \cap A_2}.$$

By induction there are I, B_i, C_i, J, D_j, G_j such that

$$\forall i \in I.\, \Gamma, x : B_i \vdash^{\nabla} \mathbf{t} : C_i, \forall j \in J.\, \Gamma, x : D_j \vdash^{\nabla} \mathbf{t} : G_j,$$
$$\bigcap_{i \in I}(B_i \to C_i) \leq_{\nabla} A_1 \quad \& \quad \bigcap_{j \in J}(D_j \to G_j) \leq_{\nabla} A_2.$$

So we are done since $(\bigcap_{i \in I}(B_i \to C_i)) \cap (\bigcap_{j \in J}(D_j \to G_j)) \leq_{\nabla} A$.

(4) The case $C \sim_{\nabla} \Omega$ is trivial. Otherwise let I, B_i, C_i as in Point *(3)*, where $A \equiv B \to C$. Then $\bigcap_{i \in I}(B_i \to C_i) \leq_{\nabla} B \to C$ implies by Theorem 6 that $\bigcap_{i \in J} C_i \leq_{\nabla} C$ where $J = \{i \in I \mid B \leq_{\nabla} B_i\}$. From $\Gamma, x : B_i \vdash^{\nabla} \mathbf{t} : C_i$ we can derive $\Gamma, x : B \vdash^{\nabla} \mathbf{t} : C_i$ by rule $(\leq_{\nabla} \mathrm{L})$, so by $(\cap \mathrm{I})$ we have $\Gamma, x : B \vdash^{\nabla} \mathbf{t} : \bigcap_{i \in J} C_i$. Finally applying rule $(\leq_{\nabla})$ we can conclude $\Gamma, x : B \vdash^{\nabla} \mathbf{t} : C$.

Note that in the point (1) of the previous theorem, we have to suppose that $A \not\sim^{\nabla} \Omega$, otherwise we could derive $\vdash^{\nabla} x : \Omega$ using axiom $(\mathrm{Ax}\text{-}\Omega)$.

2 Filter Models

In this section we discuss how to build λ-models out of type theories. We start with the definition of *filter* for eitt's. Then we show how to turn the space of filters into an applicative structure. We define continuous maps from the space of filters to the space of its continuous functions. Since the composition of these maps is the identity we get λ-models (*filter models*).

Definition 11 (Filters)

(1) A ∇-filter (or a filter over $\mathbb{T}^{\nabla}$) is a set $\Xi \subseteq \mathbb{T}^{\nabla}$ such that:
- $\Omega \in \Xi$;
- *if $A \leq_{\nabla} B$ and $A \in \Xi$, then $B \in \Xi$;*
- *if $A, B \in \Xi$, then $A \cap B \in \Xi$;*

(2) $\mathcal{F}^{\nabla}$ denotes the set of ∇-filters over $\mathbb{T}^{\nabla}$;
(3) if $\Xi \subseteq \mathbb{T}^{\nabla}$, $\uparrow^{\nabla} \Xi$ denotes the ∇-filter generated by Ξ;
(4) a ∇-filter is principal *if it is of the shape $\uparrow^{\nabla} \{A\}$, for some type A. We shall denote $\uparrow^{\nabla} \{A\}$ simply by $\uparrow^{\nabla} A$.*

It is well known that $\mathcal{F}^{\triangledown}$ is an ω-algebraic lattice, whose poset of compact (or finite) elements is isomorphic to the reversed poset obtained by quotienting the preorder on $\mathbb{T}^{\triangledown}$ by $\sim_{\triangledown}$. That means that compact elements are the filters of the form $\uparrow^{\triangledown} A$ for some type A, the top element is $\mathbb{T}^{\triangledown}$, and the bottom element is $\uparrow^{\triangledown} \Omega$. Moreover the join of two filters is the filter induced by their union and the meet of two filters is their intersection, i.e.:

$$\Xi \sqcup \Upsilon = \uparrow^{\triangledown} (\Xi \cup \Upsilon)$$

$$\Xi \sqcap \Upsilon = \Xi \cap \Upsilon.$$

The key property of $\mathcal{F}^{\triangledown}$ is to be a reflexive object in the category of ω-algebraic complete lattices and Scott-continuous functions. This become clear by endowing the space of filters with a notion of application which induces continuous maps from $\mathcal{F}^{\triangledown}$ to its function space $[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}]$ and vice versa.

Definition 12 (Application)

(1) Application $\cdot : \mathcal{F}^{\triangledown} \times \mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}$ *is defined as*

$$\Xi \cdot \Upsilon = \{B \mid \exists A \in \Upsilon . A \to B \in \Xi\}.$$

(2) The continuous maps $\mathbb{F}^{\triangledown} : \mathcal{F}^{\triangledown} \to [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}]$ *and* $\mathbb{G}^{\triangledown} : [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}$ *are defined as:*

$$\mathbb{F}^{\triangledown}(\Xi) = \lambda\!\!\lambda \Upsilon \in \mathcal{F}^{\triangledown} . \Xi \cdot \Upsilon;$$

$$\mathbb{G}^{\triangledown}(f) = \uparrow^{\triangledown} \{A \to B \mid B \in f(\uparrow^{\triangledown} A)\}.$$

Notice that previous definition is sound, since it is easy to verify that $\Xi \cdot \Upsilon$ is a $\triangledown$-filter.

We start with a useful lemma on application.

Lemma 13 *Let* $\Sigma^{\triangledown}$ *be an eitt,* $\Xi \in \mathcal{F}^{\triangledown}$ *and* $C \in \mathbb{T}^{\triangledown}$. *Then*

$$B \in \Xi \cdot \uparrow^{\triangledown} C \quad \textit{iff} \quad C \to B \in \Xi.$$

PROOF.

$$\begin{aligned} B \in \Xi \cdot \uparrow^{\triangledown} C &\Leftrightarrow \exists C' . C \leq_{\triangledown} C' \ \&\ C' \to B \in \Xi && \text{by definition of application} \\ &\Leftrightarrow C \to B \in \Xi && \text{by rule } (\eta). \end{aligned}$$

As expected, $\mathbb{F}^{\triangledown}$ and $\mathbb{G}^{\triangledown}$ are inverse to each other.

Lemma 14

$$\begin{aligned} \mathbb{F}^{\triangledown} \circ \mathbb{G}^{\triangledown} &= id_{[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}]}; \\ \mathbb{G}^{\triangledown} \circ \mathbb{F}^{\triangledown} &= id_{\mathcal{F}^{\triangledown}}. \end{aligned}$$

PROOF. It suffices to consider compact elements.

$$\begin{aligned} (\mathbb{F}^{\triangledown} \circ \mathbb{G}^{\triangledown})(f)(\uparrow^{\triangledown} A) &= \{B \mid A \to B \in \mathbb{G}^{\triangledown}(f)\} \\ &\quad \text{by definition of } \mathbb{F}^{\triangledown} \text{ and Lemma 13} \\ &= \{B \mid A \to B \in \uparrow^{\triangledown} \{A' \to B' \mid B' \in f(\uparrow^{\triangledown} A')\}\} \\ &\quad \text{by definition of } \mathbb{G}^{\triangledown} \\ &= \{B \mid \exists I, A_i, B_i.(\forall i \in I.B_i \in f(\uparrow^{\triangledown} A_i)) \\ &\qquad \& \textstyle\bigcap_{i \in I}(A_i \to B_i) \leq_{\triangledown} A \to B\} \\ &\quad \text{by definition of filter} \\ &= \{B \mid \exists I, A_i, B_i.(\forall i \in I.B_i \in f(\uparrow^{\triangledown} A_i)) \\ &\qquad \& \textstyle\bigcap_{i \in J} B_i \leq_{\triangledown} B\} \\ &\quad \text{where } J = \{i \in I \mid A \leq_{\triangledown} A_i\} \text{ by Theorem 6} \\ &= \{B \mid B \in f(\uparrow^{\triangledown} A)\} \\ &\quad \text{by the monotonicity of } f \\ &= f(\uparrow^{\triangledown} A). \end{aligned}$$

$$\begin{aligned} (\mathbb{G}^{\triangledown} \circ \mathbb{F}^{\triangledown})(\uparrow^{\triangledown} A) &= \uparrow^{\triangledown} \{B \to C \mid C \in \uparrow^{\triangledown} A \cdot \uparrow^{\triangledown} B\} && \text{by definition of } \mathbb{F}^{\triangledown} \text{ and } \mathbb{G}^{\triangledown} \\ &= \uparrow^{\triangledown} \{B \to C \mid B \to C \in \uparrow^{\triangledown} A\} && \text{by Lemma 13} \\ &= \uparrow^{\triangledown} A && \text{by Lemma 3}(1). \end{aligned}$$

Lemma 14 implies that $\mathcal{F}^{\triangledown}$ induces an extensional λ-model. Let $\mathsf{Env}_{\mathcal{F}^{\triangledown}}$ be the set of all mappings from the set of term variables to $\mathcal{F}^{\triangledown}$ and ρ range over $\mathsf{Env}_{\mathcal{F}^{\triangledown}}$. Via the maps $\mathbb{F}^{\triangledown}$ and $\mathbb{G}^{\triangledown}$ we get the standard *semantic interpretation* $[\![\ ]\!]^{\triangledown} : \Lambda \times$

$\mathsf{Env}_{\mathcal{F}^\nabla} \to \mathcal{F}^\nabla$ of λ-terms:

$$\begin{aligned} [\![x]\!]^\nabla_\rho &= \rho(x); \\ [\![\mathbf{tu}]\!]^\nabla_\rho &= \mathbb{F}^\nabla([\![\mathbf{t}]\!]^\nabla_\rho)([\![\mathbf{u}]\!]^\nabla_\rho); \\ [\![\lambda x.\mathbf{t}]\!]^\nabla_\rho &= \mathbb{G}^\nabla(\lambda\!\!\lambda \Xi \in \mathcal{F}^\nabla.[\![\mathbf{t}]\!]^\nabla_{\rho[\Xi/x]}). \end{aligned}$$

Actually, by using the Generation Theorem 10, it is easy to prove by induction on λ-terms that:

$$[\![\mathbf{t}]\!]^\nabla_\rho = \{A \in \mathbb{T}^\nabla \mid \exists \Gamma \rhd \rho.\ \Gamma \vdash^\nabla \mathbf{t} : A\},$$

where the notation $\Gamma \rhd \rho$ means that for $(x : B) \in \Gamma$ one has that $B \in \rho(x)$.

Remark 15 *Note that any intersection type theory satisfying Theorem 6 produces a reflexive object in the category of omega-algebraic lattices, and Lemma 3(1) ensures that the retraction pair* $(\mathbb{G}, \mathbb{F})$ *consists of isomorphisms.*

We conclude this section with the formal definition of filter models.

Definition 16 (Filter models)

The extensional λ-model $\langle \mathcal{F}^\nabla, \cdot, [\![\ ]\!]^\nabla \rangle$ *is called the* filter model *over* Σ^∇.

3 Simple Easy Terms and Continuous Predicates

In this section we give the main notion of the paper, namely *simple easiness*. A term $\mathbf{e}$ is simple easy if, given an eitt Σ^∇ and a type $E \in \mathbb{T}^\nabla$, we can extend in a conservative way Σ^∇ to an eitt $\Sigma^{\nabla'}$, so that $[\![\mathbf{e}]\!]^{\nabla'} = (\uparrow^{\nabla'} E) \sqcup [\![\mathbf{e}]\!]^\nabla$. This allows to build with a uniform technique, the filter models in which the interpretation of $\mathbf{e}$ is *a filter of types induced by a continuous predicate* (see Definition 20).

First we introduce *EITT maps*: an EITT map applied to an easy intersection type theory and to a type builds a new easy intersection type theory which is a conservative extension of the original one.

Definition 17 (EITT maps)

(1) Let Σ^∇ and $\Sigma^{\nabla'}$ two eitts. We say that $\Sigma^{\nabla'}$ is a conservative extension *of Σ^∇ (notation $\Sigma^\nabla \sqsubseteq \Sigma^{\nabla'}$) iff $\mathbb{C}^\nabla \subseteq \mathbb{C}^{\nabla'}$ and for all $A, B \in \mathbb{T}^\nabla$,*

$$A \leq_\nabla B \text{ iff } A \leq_{\nabla'} B.$$

(2) A pointed *eitt is a pair (Σ^∇, E) with $E \in \mathbb{T}^\nabla$.*

(3) An EITT map *is a map* M : *PEITT* $\to$ *EITT, such that for all* (Σ^∇, E)

$$\Sigma^\nabla \sqsubseteq \mathsf{M}(\Sigma^\nabla, E),$$

where EITT and PEITT denote respectively the class of eitts and pointed eitts.

We now give the central notion of *simple easy* term.

Definition 18 (Simple easy terms) *An unsolvable term* $\mathbf{e}$ *is* simple easy *if there exists an EITT map* $\mathsf{M}_\mathbf{e}$ *such that for all pointed eitt* (Σ^∇, E),

$$\vdash^{\nabla'} \mathbf{e} : B \textit{ iff } \exists C \in \mathbb{T}^\nabla . C \cap E \leq_{\nabla'} B \ \& \ \vdash^\nabla \mathbf{e} : C,$$

where $\Sigma^{\nabla'} = \mathsf{M}_\mathbf{e}(\Sigma^\nabla, E)$.

Define $\mathbf{I} \equiv \lambda x.x$, $\mathbf{W}_2 \equiv \lambda x.xx$, $\mathbf{W}_3 \equiv \lambda x.xxx$, and $\mathbf{R}_n$ inductively as $\mathbf{R}_0 = \mathbf{W}_2\mathbf{W}_2$, $\mathbf{R}_{n+1} = \mathbf{R}_n\mathbf{R}_n$. Examples of simple easy terms are $\mathbf{W}_2\mathbf{W}_2$ (see next section), $\mathbf{W}_3\mathbf{W}_3\mathbf{I}$, and $\mathbf{R}_n$ for all n [4]. See Lusin [16] for further examples of simple easy terms.

The first key property of simple easy terms is the following.

Theorem 19 *With the same notation of previous definition, we have*

$$[\![\mathbf{e}]\!]^{\nabla'} = (\uparrow^{\nabla'} E) \sqcup [\![\mathbf{e}]\!]^\nabla.$$

PROOF. $(\supseteq)$ Taking $C = \Omega$ in Definition 18, we have $\vdash^{\nabla'} \mathbf{e} : E$. Therefore, $(\uparrow^{\nabla'} E) \subseteq [\![\mathbf{e}]\!]^{\nabla'}$. Since $[\![\mathbf{e}]\!]^\nabla \subseteq [\![\mathbf{e}]\!]^{\nabla'}$, we get $[\![\mathbf{e}]\!]^{\nabla'} \supseteq (\uparrow^{\nabla'} E) \sqcup [\![\mathbf{e}]\!]^\nabla$.
$(\subseteq)$ If $B \in [\![\mathbf{e}]\!]^{\nabla'}$, then $\vdash^{\nabla'} \mathbf{e} : B$, hence, by Definition 18, there exists $C \in \mathbb{T}^\nabla$ such that $C \in [\![\mathbf{e}]\!]^\nabla$ and $C \cap E \leq_{\nabla'} B$. We are done, since $C \cap E \in (\uparrow^{\nabla'} E) \sqcup [\![\mathbf{e}]\!]^{\nabla'}$.

Finally, we define filters by *continuous* predicates.

Definition 20 (Continuous predicates) *Let* P : *PEITT* $\to \{tt, ff\}$ *a predicate. We say that* P *is* continuous *iff (as usual* $\mathsf{P}(\Sigma^\nabla, E)$ *is short for* $\mathsf{P}(\Sigma^\nabla, E) = tt$*):*

(1) $\Sigma^\nabla \sqsubseteq \Sigma^{\nabla'} \ \& \ \mathsf{P}(\Sigma^\nabla, E) \ \Rightarrow \ \mathsf{P}(\Sigma^{\nabla'}, E)$
(2) $\mathsf{P}(\Sigma^{\nabla\infty}, E) \ \Rightarrow \ \exists n. \mathsf{P}(\Sigma^{\nabla n}, E)$

where $\Sigma^{\nabla\infty} = \Sigma(\bigcup_n \mathbb{C}^{\nabla_n}, \bigcup_n \nabla_n)$.

The ∇*-filter induced by* P *over* Σ^∇ *is the filter defined by:*

$$\Xi^\nabla_\mathsf{P} = \uparrow^\nabla \{A \in \mathbb{T}^\nabla \mid \mathsf{P}(\Sigma^\nabla, A)\}.$$

Note that if we endow *PEITT* with the Scott topology induced by the ordering $(\Sigma^{\nabla}, E) \sqsubseteq (\Sigma^{\nabla'}, E')$ iff $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$ and $E \sim_{\nabla} E'$, then continuous predicates are in one-to-one correspondence with Scott open sets in *PEITT*.

For the proof of the Main Theorem it is useful to recall some properties of Scott's λ-model D_∞ [20]. We are interested in the inverse limit λ-model D_∞, obtained starting from the two point lattice $D_0 = \{\bot, \top\}$ and the embedding $\mathsf{i}_0 : D_0 \to [D_0 \to D_0]$ defined by:

$$\mathsf{i}_0(\bot) = \bot \Rightarrow \bot$$

$$\mathsf{i}_0(\top) = \bot \Rightarrow \top$$

where $\mathsf{a} \Rightarrow \mathsf{b}$ is the step function defined by

$$\lambda \mathsf{d}.\ \textbf{if}\ \mathsf{a} \sqsubseteq \mathsf{d}\ \textbf{then}\ \mathsf{b}\ \textbf{else}\ \bot.$$

It is well-known (and first shown in Wadsworth [21]) that in this model the interpretation of all unsolvable terms is bottom. Moreover this model is isomorphic to the filter model $\langle \mathcal{F}^{\nabla_0}, \cdot, [\![\ ]\!]^{\nabla_0} \rangle$ induced by the eitt Σ^{∇_0} defined by:

$$\begin{aligned} \mathbb{C}^{\nabla_0} &= \{\Omega, \omega\}; \\ \nabla_0 &= \overline{\nabla} \cup \{\omega \sim \Omega \to \omega\}. \end{aligned}$$

This isomorphism (stated with a proof sketch in Coppo et al. [9] and fully proved in Alessi [2]) is a particular case of the duality discussed in Section 2 of [12]. Therefore we get:

Proposition 21 *In the filter model $\langle \mathcal{F}^{\nabla_0}, \cdot, [\![\ ]\!]^{\nabla_0} \rangle$ the interpretation of all unsolvable terms is $\uparrow^{\nabla_0} \Omega$.*

Theorem 22 (Main Theorem) *Let* e *be a simple easy term,* $\mathsf{P} : \mathit{PEITT} \to \{\mathit{tt}, \mathit{ff}\}$ *be a continuous predicate and* $\Xi^{\nabla}_{\mathsf{P}}$ *be the* ∇*-filter induced by* P *over* Σ^{∇}*. Then there is a filter model* $\mathcal{F}^{\nabla}$ *such that*

$$[\![\mathbf{e}]\!]^{\nabla} = \Xi^{\nabla}_{\mathsf{P}}.$$

PROOF. Let $\langle \cdot, \cdot \rangle$ denote any fixed bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ such that $\langle r, s \rangle \geq r$.

We will choose a denumerable sequence of eitts $\Sigma^{\nabla_0}, \ldots, \Sigma^{\nabla_r}, \ldots$. For each r we will consider a fixed enumeration $\langle W^{(r)}_s \rangle_{s \in \mathbb{N}}$ of the set $\{A \in \mathbb{T}^{\nabla_r} \mid \mathsf{P}(\Sigma^{\nabla_r}, A)\}$.

We can construct the model as follows.

step 0: take the eitt Σ^{∇_0} defined as above.

step $(n+1)$: if $n = \langle r, s\rangle$ we define $\Sigma^{\nabla_{n+1}} = \mathsf{M}_{\mathbf{e}}(\Sigma^{\nabla_n}, W_s^{(r)})$ (notice that $\Sigma^{\nabla_n} \sqsubseteq \Sigma^{\nabla_{n+1}}$).

final step: take $\Sigma^{\nabla_\infty} = \Sigma(\bigcup_n \mathbb{C}^{\nabla_n}, \bigcup_n \nabla_n)$.

First we prove that the model $\mathcal{F}^{\nabla_\infty}$ is non-trivial by showing that $[\![\mathbf{I}]\!]^{\nabla_\infty} \neq [\![\mathbf{K}]\!]^{\nabla_\infty}$, where $\mathbf{I} \equiv \lambda x.x$, $\mathbf{K} \equiv \lambda xy.x$. Let $D \equiv (\omega \to \omega) \to (\omega \to \omega)$. Since $\vdash^{\nabla_\infty} \mathbf{I} : D$, we have that $D \in [\![\mathbf{I}]\!]^{\nabla_\infty}$. On the other hand, if it were $D \in [\![\mathbf{K}]\!]^{\nabla_\infty}$, then there would exist n such that $D \in [\![\mathbf{K}]\!]^{\nabla_n}$. This would imply (by applying several times the Generation Theorem) $\omega \to \omega \leq_{\nabla_n} \omega$. Since we have $\Sigma^{\nabla_n} \sqsubseteq \Sigma^{\nabla_{n+1}}$ for all n, we should have $\omega \to \omega \leq_{\nabla_0} \omega$. Since $\omega \sim_{\nabla_0} \Omega \to \omega$, we should conclude by Theorem 6, $\Omega \leq_{\nabla_0} \omega$, which is a contradiction. Therefore, we cannot have $D \in [\![\mathbf{K}]\!]^{\nabla_\infty}$ and the model $\mathcal{F}^{\nabla_\infty}$ is non-trivial.

Now we prove that $[\![\mathbf{e}]\!]^{\nabla_\infty} =\uparrow^{\nabla_\infty} \{T_s^{(r)} \mid r, s \in \mathbb{N}\}$ by showing that $[\![\mathbf{e}]\!]^{\nabla_n} =\uparrow^{\nabla_n} \{T_s^{(r)} \mid \langle r, s\rangle < n\}$ for all n. The inclusion $(\supseteq)$ is immediate by construction. We prove $(\subseteq)$ by induction on n. If $n = 0$, then $[\![\mathbf{e}]\!]^{\nabla_0} =\uparrow^{\nabla_0} \Omega$ by Proposition 21. Suppose the thesis is true for $n = \langle r_n, s_n\rangle$ and let $B \in [\![\mathbf{e}]\!]^{\nabla_{n+1}}$. Then $\vdash^{\nabla_{n+1}} \mathbf{e} : B$. This is possible only if there exists $C \in \mathbb{T}^{\nabla_n}$ such that $C \cap T_{s_n}^{(r_n)} \leq_{\nabla_{n+1}} B$ and moreover $\vdash^{\nabla_n} \mathbf{e} : C$. By induction we have $C \in\uparrow^{\nabla_n} \{T_s^{(r)} \mid \langle r, s\rangle < n\}$, hence $T_{s_1}^{(r_1)} \cap \ldots \cap T_{s_k}^{(r_k)} \leq_{\nabla_n} C$ for some $r_1, \ldots, r_k, s_1, \ldots, s_k$ with $\langle r_i, s_i\rangle < n$ $(1 \leq i \leq k)$. We derive $T_{s_1}^{(r_1)} \cap \ldots \cap T_{s_k}^{(r_k)} \cap T_{s_n}^{(r_n)} \leq_{\nabla_{n+1}} B$, i.e. $B \in\uparrow^{\nabla_{n+1}} \{T_s^{(r)} \mid \langle r, s\rangle < n+1\}$.

Finally we show that

$$A \in \mathbb{T}^{\nabla_\infty} \;\&\; \mathsf{P}(\Sigma^{\nabla_\infty}, A) \quad \Leftrightarrow \quad \exists r, s.\, A \equiv T_s^{(r)}.$$

$(\Leftarrow)$ is immediate by Definition 20*(1)*.

We prove $(\Rightarrow)$. If $A \in \mathbb{T}^{\nabla_\infty}$ and $\mathsf{P}(\Sigma^{\nabla_\infty}, A)$, then by definition of Σ^{∇_∞} and the continuity of P, it follows that there is r such that $A \in \mathbb{T}^{\nabla_r}$ and $\mathsf{P}(\Sigma^{\nabla_r}, A)$. Therefore by definition of $T_s^{(r)}$, there is s such that $A \equiv T_s^{(r)}$.

So we can conclude $[\![\mathbf{e}]\!]^{\nabla_\infty} =\uparrow^{\nabla_\infty} \{A \in \mathbb{T}^{\nabla_\infty} \mid \mathsf{P}(\Sigma^{\nabla_\infty}, A)\}$, i.e. $[\![\mathbf{e}]\!]^{\nabla_\infty} = \Xi_{\mathsf{P}}^{\nabla_\infty}$.

4 Consistency of λ-theories

We introduce now a λ-theory whose consistency has been first proved using a suitable filter model in Dezani and Lusin [13]. We obtain the same model here as a consequence of Theorem 22.

Definition 23 (Theory $\mathcal{J}$) *The λ-theory $\mathcal{J}$ is axiomatized by*

$$\mathbf{W}_4xx = x; \quad \mathbf{W}_4xy = \mathbf{W}_4yx; \quad \mathbf{W}_4x(\mathbf{W}_4yz) = \mathbf{W}_4(\mathbf{W}_4xy)z$$

where $\mathbf{W}_2 \equiv \lambda x.xx$ *and* $\mathbf{W}_4 \equiv \mathbf{W}_2\mathbf{W}_2$.

It is clear that the previous equations hold if the interpretation of $\mathbf{W}_4$ is the join operator on filters. In order to use Theorem 22 we need:

- the join operator on filters to be a filter generated by a continuous predicate;
- $\mathbf{W}_4$ to be simple easy.

For the first condition it is easy to check that the join relative to $\mathcal{F}^\nabla$ is represented by the filter:

$$\Theta = \uparrow^\nabla \{A \to B \to A \cap B \mid A, B \in \mathbb{T}^\nabla\}.$$

In fact, by easy calculation we have:

$$(\Theta \cdot \Xi) \cdot \Upsilon \; = \; \Xi \sqcup \Upsilon$$

for all filters Ξ, Υ in $\mathcal{F}^\nabla$. Therefore, the required predicate is

$$\mathsf{P}(\Sigma^\nabla, C) \quad \Leftrightarrow \quad C \equiv A \to B \to A \cap B.$$

It is trivial that the predicate above is continuous.

To show that $\mathbf{W}_4$ is simple easy we give a lemma which characterises the types derivable for $\mathbf{W}_2$ and $\mathbf{W}_4$.

Lemma 24

(1) $\vdash^\nabla \mathbf{W}_2 : A \to B$ *iff* $A \leq_\nabla A \to B$*;*
(2) $\vdash^\nabla \mathbf{W}_4 : B$ *iff* $A \leq_\nabla A \to B$ *for some* $A \in \mathbb{T}^\nabla$ *such that* $\vdash^\nabla \mathbf{W}_2 : A$*.*
(3) If $\vdash^\nabla \mathbf{W}_4 : B$ *then there exists* $A \in \mathbb{T}^\nabla$ *such that* $\#(A) = 0$*,* $A \leq_\nabla A \to B$ *and* $\vdash^\nabla \mathbf{W}_2 : A$*.*

PROOF. *(1)* By a straightforward computation, $A \leq_\nabla A \to B$ implies $\vdash^\nabla \mathbf{W}_2 : A \to B$. Conversely, suppose $\vdash^\nabla \mathbf{W}_2 : A \to B$. If $B \sim_\nabla \Omega$, then by axioms (Ω), $(\Omega$-$\eta)$, and rules (η), $(trans)$, we have $A \leq_\nabla A \to B$. Otherwise, by Theorem 10*(4)* it follows $x : A \vdash^\nabla xx : B$. By Theorem 10*(2)* there exists a type $C \in \mathbb{T}^\nabla$ such that $x : A \vdash^\nabla x : C \to B$ and $x : A \vdash^\nabla x : C$. Notice that $B \not\sim_\nabla \Omega$ implies $C \to B \not\sim_\nabla \Omega$, since from $C \to B \sim_\nabla \Omega$ we get $C \to B \sim_\nabla \Omega \to \Omega$ by axiom $(\Omega$-$\eta)$ and rule $(trans)$ and this implies $B \sim_\nabla \Omega$ by Theorem 6. So by Theorem 10*(1)*, we get $A \leq_\nabla C \to B$. We have $A \leq_\nabla C$ either by Theorem 10*(1)* if $C \not\sim_\nabla \Omega$ or by axiom (Ω) and rule $(trans)$ if $C \sim_\nabla \Omega$. From $A \leq_\nabla C \to B$ and $A \leq_\nabla C$ by rule (η) it follows $A \leq_\nabla A \to B$.

(2) The case $B{\sim_\triangledown}\Omega$ is trivial. Otherwise, if $\vdash^\triangledown \mathbf{W}_4 : B$, by Theorem 10*(2)* it follows that there exists $A \in \mathbb{T}^\triangledown$ such that $\vdash^\triangledown \mathbf{W}_2 : A$ and $\vdash^\triangledown \mathbf{W}_2 : A \to B$. We conclude by Point *(1)*.

(3) Let $\vdash^\triangledown \mathbf{W}_4 : B$. Then, by Point *(2)*, the set

$$\Pi = \{A \in \mathbb{T}^\triangledown \mid \vdash^\triangledown \mathbf{W}_2 : A \text{ and } A \leq_\triangledown A \to B\}$$

is non-empty. Let $\overline{A} \in \Pi$ be such that $\#(\overline{A})$ is minimal. We are done if we prove $\#(\overline{A}) = 0$. By contradiction, if it is not the case, by applying Lemma 5, we obtain a type A' such that $A'{\sim_\triangledown}\overline{A}$, $A' \equiv \bigcap_{i\in I}(C_i \to D_i)$ and $\#(A') = \#(\overline{A})$. From $\overline{A} \leq_\triangledown \overline{A} \to B$ we have $\bigcap_{i\in I}(C_i \to D_i) \leq_\triangledown \overline{A} \to B$, hence, by Theorem 6, $\bigcap_{i\in J} D_i \leq_\triangledown B$ where $J = \{i \in I \mid \overline{A} \leq_\triangledown C_i\}$. Since $\vdash^\triangledown \mathbf{W}_2 : \overline{A}$, by $(\leq_\triangledown)$ it follows $\vdash^\triangledown \mathbf{W}_2 : C_i \to D_i$ for all $i \in J$ and $\vdash^\triangledown \mathbf{W}_2 : \bigcap_{i\in J} C_i$. By Point *(1)* it follows $\forall i \in J. C_i \leq_\triangledown C_i \to D_i$. By axiom $(\to \cap)$ and rule (η) we get $C \leq_\triangledown C \to \bigcap_{i\in J} D_i$, and also $C \leq_\triangledown C \to B$, where $C \equiv \bigcap_{i\in J} C_i$. We have obtained:

$$\vdash^\triangledown \mathbf{W}_2 : C;$$

$$C \leq_\triangledown C \to B;$$

$$\#(C) < \#(A') = \#(\overline{A}).$$

This is a contradiction, since $C \in \Pi$ contradicts the minimality of $\#(\overline{A})$.

The crucial step for proving simple easiness of $\mathbf{W}_4$ is to find its EITT map.

Definition 25 *Let $(\Sigma^\triangledown, E)$ a pointed eitt and $\chi \notin \mathbb{C}^\triangledown$. We define*

$$\mathsf{M}_{\mathbf{W}_4}(\Sigma^\triangledown, E) = \Sigma^{\triangledown'},$$

where:

- $\mathbb{C}^{\triangledown'} = \mathbb{C}^\triangledown \cup \{\chi\}$ *;*
- $\triangledown' = \triangledown \cup \{\chi \sim \chi \to E\}$.

First we notice that $\mathsf{M}_{\mathbf{W}_4}$ is an EITT map. In fact $\mathsf{M}_{\mathbf{W}_4}(\Sigma^\triangledown, E)$ is an easy intersection type theory by Definition 25. Moreover, it is straightforward to show by induction on derivations that $\Sigma^\triangledown \sqsubseteq \mathsf{M}_{\mathbf{W}_4}(\Sigma^\triangledown, E)$.

Now we prove that $\mathsf{M}_{\mathbf{W}_4}$ is an EITT map for $\mathbf{W}_4$.

Lemma 26 *Let $\Sigma^{\triangledown'} = \mathsf{M}_{\mathbf{W}_4}(\Sigma^\triangledown, E)$. Then*

$$\vdash^{\triangledown'} \mathbf{W}_4 : B \textit{ iff } \exists C \in \mathbb{T}^\triangledown. C \cap E \leq_{\triangledown'} B \;\&\; \vdash^\triangledown \mathbf{W}_4 : C.$$

PROOF. Throughout the proof we use the Generation Theorem and Theorem 6 without explicitly mentioning them each time.

($\Rightarrow$) Let $\vdash^{\nabla'} \mathbf{W}_4 : B$. First we show that there is a type $D \in \mathbb{T}^{\nabla'}$ such that

(a) $D \equiv \bigcap_{i\in I}(\varphi_i \to F_i) \cap \chi$;

(b) $\forall i \in I.\varphi_i \in \mathbb{C}^{\nabla}$ & $F_i \in \mathbb{T}^{\nabla}$;

(c) $D \leq_{\nabla'} D \to B$;

(d) $\vdash^{\nabla'} \mathbf{W}_2 : D$.

By Lemma 24*(3)* $\vdash^{\nabla'} \mathbf{W}_4 : B$ implies that there exists $T \in \mathbb{T}^{\nabla'}$ such that the following three properties hold:

(i) $\#(T) = 0$;

(ii) $T \leq_{\nabla'} T \to B$;

(iii) $\vdash^{\nabla'} \mathbf{W}_2 : T$.

If we consider $T' \equiv T \cap \chi$, it is easy to check that T' satisfies (i) and (iii) above. Moreover, (ii) holds for T' since by Lemma 3*(2)* and rules (mon), (η):

$$T \cap \chi \leq_{\nabla'} (T \to B) \cap (\chi \to E) \leq_{\nabla'} T \cap \chi \to B \cap E \leq_{\nabla'} T \cap \chi \to B.$$

It must be $T' \sim_{\nabla'} (\bigcap_{k\in K} \phi_k) \cap \chi$, with $\phi_k \in \mathbb{C}^{\nabla}$, $\phi_k \not\sim_{\nabla'} \Omega$ for all $k \in K$, since the unique possible shape for T' is an intersection of constants containing χ. Next, since for each $k \in K$, we have, from the axioms of ∇, $\phi_k \sim_{\nabla} \bigcap_{l\in L^{(k)}}(\varphi_l^{(k)} \to F_l^{(k)})$, we can define $D \equiv \bigcap_{k\in K}(\bigcap_{l\in L^{(k)}}(\varphi_l^{(k)} \to F_l^{(k)}))\cap\chi$. Then, by reindexing the types and using a unique intersection, we get the syntactic shape for D required by conditions (a) and (b). Moreover, conditions (c) and (d) hold since by construction $D \sim_{\nabla} T'$.

Considering (a), (d), rule $(\leq_{\nabla'})$ and Lemma 24*(1)*, we have that for all $i \in I$, $\varphi_i \leq_{\nabla'} \varphi_i \to F_i$. Since $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$ and for each $i \in I$, $\varphi_i, F_i \in \mathbb{T}^{\nabla}$, it follows that $\varphi_i \leq_{\nabla} \varphi_i \to F_i$, for all $i \in I$. By applying Lemma 24*(1)* and rule $(\cap\mathrm{I})$, we get $\vdash^{\nabla} \mathbf{W}_2 : \bigcap_{i\in I}(\varphi_i \to F_i)$. This implies by Lemma 3*(2)* and rule $(\leq_{\nabla})$

$$\vdash^{\nabla} \mathbf{W}_2 : (\bigcap_{i\in I'} \varphi_i) \to (\bigcap_{i\in I'} F_i) \text{ for all } I' \subseteq I. \tag{1}$$

Because of (c), $(\bigcap_{i\in J} F_i) \cap E \leq_{\nabla'} B$ where $J = \{i \in I \mid D \leq_{\nabla'} \varphi_i)\}$. Because of (d) and rule $(\leq_{\nabla'})$, it follows $\vdash^{\nabla'} \mathbf{W}_2 : \bigcap_{i\in J} \varphi_i$. Let $\varphi_i \equiv \bigcap_{h\in H^{(i)}}(\zeta_h^{(i)} \to D_h^{(i)})$. Then by rule $(\leq_{\nabla'})$, we have $\vdash^{\nabla'} \mathbf{W}_2 : \zeta_h^{(i)} \to D_h^{(i)}$ for each $i \in J$ and $h \in H^{(i)}$. By

Lemma 24*(1)* it follows, for each $i \in J$ and $h \in H^{(i)}$, $\zeta_h^{(i)} \leq_{\nabla'} \zeta_h^{(i)} \to D_h^{(i)}$. Using again $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$, we have, for each $i \in J$ and $h \in H^{(i)}$, $\zeta_h^{(i)} \leq_{\nabla} \zeta_h^{(i)} \to D_h^{(i)}$, hence, by Lemma 24*(1)*, $\vdash^{\nabla} \mathbf{W}_2 : \zeta_h^{(i)} \to D_h^{(i)}$, for each $i \in J$ and $h \in H^{(i)}$. Therefore, by rule $(\cap\mathrm{I})$, we have $\vdash^{\nabla} \mathbf{W}_2 : \bigcap_{i\in J}(\bigcap_{h\in H^{(i)}}(\zeta_h^{(i)} \to D_h^{(i)}))$, that is

$$\vdash^{\nabla} \mathbf{W}_2 : \bigcap_{i\in J} \varphi_i. \tag{2}$$

Applying rule $(\to \mathrm{E})$ to (1) with $I' = J$ and (2), we obtain $\vdash^{\nabla} \mathbf{W}_4 : \bigcap_{i\in J} F_i$. Since we have proven $(\bigcap_{i\in J} F_i) \cap E \leq_{\nabla'} B$, we are done, by choosing $C \equiv \bigcap_{i\in J} F_i$.

$(\Leftarrow)$ By Theorem 19 we have that $\vdash^{\nabla'} \mathbf{W}_4 : E$. Since by hypothesis $\vdash^{\nabla} \mathbf{W}_4 : C$ and moreover $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$, we obtain $\vdash^{\nabla'} \mathbf{W}_4 : C$. By applying rule $(\leq_{\nabla'})$ we have $\vdash^{\nabla'} \mathbf{W}_4 : B$.

The previous lemma yields the second crucial step in the construction of the model.

Theorem 27 $\mathbf{W}_4$ *is simple easy.*

We can conclude:

Theorem 28 (Consistency of $\mathcal{J}$) *The λ-theory $\mathcal{J}$ is consistent.*

Remark 29 *The set of all λ-theories is naturally equipped with a structure of complete lattice (see Barendregt [7], Chapter 4), with meet $\sqcap$ defined as set theoretic intersection. The join $\sqcup$ of two λ-theories $\mathcal{T}$ and $\mathcal{S}$ is the least equivalence relation including $\mathcal{T} \cup \mathcal{S}$. Lusin and Salibra [17] consider the set $[\mathcal{J})$ of all λ-theories extending $\mathcal{J}$: this is a sublattice of the lattice of λ-theories. They prove that this sublattice has many interesting algebraic properties, due to the validity of the equations defining $\mathcal{J}$ (see Definition 23). In particular $[\mathcal{J})$ satisfies a restricted form of distributivity, called meet semidistributivity, i.e. for all λ-theories $\mathcal{T}, \mathcal{S}, \mathcal{G} \in [\mathcal{J})$, if $\mathcal{T} \sqcap \mathcal{S} = \mathcal{T} \sqcap \mathcal{G}$, then $\mathcal{T} \sqcap \mathcal{S} = \mathcal{T} \sqcap (\mathcal{S} \sqcup \mathcal{G})$.*

5 Minimal Fixed Point Operators

In this section we prove that for all simple easy terms e there are filter models $\mathcal{F}^{\nabla}$ such that $[\![\mathbf{e}]\!]^{\nabla}$ represents the minimal fixed point operator μ.

Actually, since $[\![\mathbf{e}]\!]^{\nabla} \in \mathcal{F}^{\nabla}$, while $\mu \in [[\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}] \to \mathcal{F}^{\nabla}]$, the identification between $[\![\mathbf{e}]\!]^{\nabla}$ and μ is possible via the "canonical embedding" of $[[\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}] \to \mathcal{F}^{\nabla}]$ in $\mathcal{F}^{\nabla}$.

Lemma 14 implies that every "higher order" space can be embedded in a canonical way in $\mathcal{F}^{\triangledown}$, by defining standard appropriate mappings via $\mathbb{F}^{\triangledown}$ and $\mathbb{G}^{\triangledown}$. For instance, in order to embed the space of μ, namely $[[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]$, in $\mathcal{F}^{\triangledown}$, we consider the pair of mappings $\mathbb{H}^{\triangledown} : \mathcal{F}^{\triangledown} \to [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]$ and $\mathbb{K}^{\triangledown} : [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}$ defined as follows:

$$
\begin{aligned}
\mathbb{H}^{\triangledown}(\Xi) &= \mathbb{F}^{\triangledown}(\Xi) \circ \mathbb{G}^{\triangledown}, \\
\mathbb{K}^{\triangledown}(\mathsf{H}) &= \mathbb{G}^{\triangledown} \circ \lambda\!\!\lambda\Xi.(\mathsf{H} \circ \mathbb{F}^{\triangledown})(\Xi).
\end{aligned}
$$

It is easy to check that

$$
\mathbb{H}^{\triangledown} \circ \mathbb{K}^{\triangledown} = id_{[[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}] \to [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]}. \tag{3}
$$

We say that a filter Ξ *represents* an operator $\mathsf{H} \in [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]$ if $\mathbb{H}^{\triangledown}(\Xi) = \mathsf{H}$. The equality (3) guarantees that each H is represented by $\mathbb{K}^{\triangledown}(\mathsf{H})$.

So the problem of finding a filter model $\mathcal{F}^{\triangledown}$, such that $[\![\mathbf{e}]\!]^{\triangledown}$ represents μ, can be solved if we find $\mathcal{F}^{\triangledown}$ such that $[\![\mathbf{e}]\!]^{\triangledown} = \mathbb{K}^{\triangledown}(\mu)$.

Remark 30 *We point out that, given a simple easy term* $\mathbf{e}$*, the existence of a model* $\mathcal{F}^{\triangledown}$ *where* $[\![\mathbf{e}]\!]^{\triangledown}$ *represents* μ *is not trivial at all. A simple easy term* $\mathbf{e}$*, being easy, can obviously be equated to an arbitrary fixed point combinator* $\mathbf{Y}$*. This could be useful in view of identifying* $[\![\mathbf{e}]\!]$ *with* μ*, provided that there exists a fixed point combinator* $\tilde{\mathbf{Y}}$ *such that* $[\![\tilde{\mathbf{Y}}]\!]^{\triangledown}$ *represents* μ *in* each *filter model* $\mathcal{F}^{\triangledown}$*. In fact, if there were such a* $\tilde{\mathbf{Y}}$*, then it would be possible to find a filter model* $\mathcal{F}^{\triangledown'}$ *such that* $[\![\tilde{\mathbf{Y}}]\!]^{\triangledown'} = [\![\mathbf{e}]\!]^{\triangledown'}$*, following the technique of Alessi and Lusin [4]. Therefore we would obtain* $\mathbb{H}^{\triangledown'}([\![\mathbf{e}]\!]^{\triangledown'}) = \mu$*. Unfortunately, such a* $\tilde{\mathbf{Y}}$ *does not exist. In fact, consider the filter model* $\mathcal{F}^{Park}$ *isomorphic to the Park* λ*-model* $\mathcal{D}^{Park}$ *of* λ*-calculus (see Honsell and Ronchi [14]). As proven in [14], for all closed* λ*-terms* $\mathbf{t}$*,* $[\![\mathbf{t}]\!]^{Park}$ *is above a certain compact element* $\mathbf{c}$ *different from the bottom element. In particular, for all fixed point combinators* $\mathbf{Y}$*,* $[\![\mathbf{YI}]\!]^{Park}$ *is above* $\mathbf{c}$*, where* $\mathbf{I}$ *is the identity combinator. Since* $\mu(\lambda\!\!\lambda X.X)$ *is obviously the bottom element, we have that it is not possible that* $\mathbb{H}^{Park}([\![\mathbf{Y}]\!]^{Park})$ *represents* μ*, since*

$$
\begin{aligned}
\mathbb{H}^{Park}([\![\mathbf{Y}]\!]^{Park})(\lambda\!\!\lambda X.X) &= (\lambda\!\!\lambda g.(\mathbb{F}^{Park}([\![\mathbf{Y}]\!]^{Park}))(\mathbb{G}^{Park}(g)))(\lambda\!\!\lambda X.X) \\
&= \mathbb{F}^{Park}([\![\mathbf{Y}]\!]^{Park})(\mathbb{G}^{Park}(\lambda\!\!\lambda X.X)) \\
&= [\![\mathbf{Y}]\!]^{Park} \cdot [\![\mathbf{I}]\!]^{Park} \\
&= [\![\mathbf{YI}]\!]^{Park} \\
&\sqsupseteq \mathbf{c}
\end{aligned}
$$

where we have used the fact that $[\![\mathbf{I}]\!]^{\triangledown} = \mathbb{G}^{\triangledown}(\lambda\!\!\lambda X.X)$ *for all* $\Sigma^{\triangledown}$*.*

We intend to prove the desired result using Theorem 22 as follows:

- given an eitt Σ^{∇}, we characterise the filter $\mathbb{K}^{\nabla}(\mu)$,
- we notice that $\mathbb{K}^{\nabla}(\mu)$ can be defined as a filter of types which satisfies a continuous predicate,
- finally, we apply Theorem 22.

In the following definition we introduce the sets Φ_n^{∇}, the filters Ψ_n^{∇} and the filter Ψ_{∞}^{∇}. Later on we shall prove that Ψ_{∞}^{∇} coincide with $\mathbb{K}^{\nabla}(\mu)$.

Definition 31 (The filter Ψ_{∞}^{∇}) *Let Σ^{∇} an eitt. For all integers n, the sets Φ_n^{∇} and the filters Ψ_n^{∇} are defined by mutual induction as follows:*

$$\begin{array}{llll} \Phi_0^{\nabla} & = \{\Omega\} & \Psi_0^{\nabla} & = \uparrow^{\nabla} \Phi_0^{\nabla}; \\ \Phi_{n+1}^{\nabla} & = \{C \to B \mid \exists B'.C \to B' \in \Psi_n^{\nabla} \ \& \ C \leq_{\nabla} B' \to B\} & \Psi_{n+1}^{\nabla} & = \uparrow^{\nabla} \Phi_{n+1}^{\nabla}. \end{array}$$

We define $\Psi_{\infty}^{\nabla} = \bigcup_n \Psi_n^{\nabla}$.

For instance $A \to \Omega \in \Phi_1^{\nabla}$, $(\Omega \to A) \to A \in \Phi_2^{\nabla}$, $(\Omega \to A_0) \cap (A_0 \to A_1) \to A_1 \in \Phi_3^{\nabla}$, and $(\Omega \to A_0) \cap (A_0 \to A_1) \cap \ldots \cap (A_{n-1} \to A_n) \to A_n \in \Phi_{n+2}^{\nabla}$ for all $A, A_0, \ldots, A_n$.

Now we prove two useful lemmata on Φ_n^{∇}, Ψ_n^{∇}. The first lemma shows that $(\Psi_n^{\nabla})_n$ is a chain: the second lemma shows that Φ_n^{∇} and Ψ_n^{∇} contain the same arrow types for $n > 0$.

Lemma 32 *For all $n \geq 0$ we have $\Psi_n^{\nabla} \subseteq \Psi_{n+1}^{\nabla}$.*

PROOF. We prove the thesis by induction on n. By definition Ψ_0^{∇} is the bottom element of $\mathcal{F}^{\nabla}$, hence $\Psi_0^{\nabla} \subseteq \Psi_1^{\nabla}$.
Suppose $\Psi_n^{\nabla} \subseteq \Psi_{n+1}^{\nabla}$. It is enough to prove $\Phi_{n+1}^{\nabla} \subseteq \Psi_{n+2}^{\nabla}$. Let $C \to B \in \Phi_{n+1}^{\nabla}$. Then there exists B' such that $C \to B' \in \Psi_n^{\nabla}$ and $C \leq_{\nabla} B' \to B$. By induction we have $\Psi_n^{\nabla} \subseteq \Psi_{n+1}^{\nabla}$, hence by definition $C \to B \in \Phi_{n+2}^{\nabla} \subseteq \Psi_{n+2}^{\nabla}$.

Lemma 33 *For all $n > 0$ we have $C \to B \in \Phi_n^{\nabla} \quad \Leftrightarrow \quad C \to B \in \Psi_n^{\nabla}$.*

PROOF. $(\Rightarrow)$ is obvious by definition.
For $(\Leftarrow)$ let $C \to B \in \Psi_n^{\nabla}$ (with $B \not\sim_{\nabla} \Omega$, otherwise the thesis is trivial). Then there are I and D_i, G_i such that for all $i \in I$, $G_i \to D_i \in \Phi_n^{\nabla}$ and $\bigcap_{i \in I}(G_i \to D_i) \leq_{\nabla} C \to B$. By Theorem 6:

$$\bigcap_{i \in J} D_i \leq_{\nabla} B \text{ where } J = \{i \in I \mid C \leq_{\nabla} G_i\}. \tag{4}$$

Moreover, by definition of $\Phi_n^{\triangledown}$, we get that there are D'_i, for all $i \in I$, such that $G_i \to D'_i \in \Psi_{n-1}^{\triangledown}$ and $G_i \leq_{\triangledown} D'_i \to D_i$. From this last judgements and (4) above, by Lemma 3*(2)* we get $C \leq_{\triangledown} (\bigcap_{i\in J} D'_i) \to (\bigcap_{i\in J} D_i)$. This together with (4) gives:

$$C \leq_{\triangledown} \tilde{D} \to B, \tag{5}$$

where $\tilde{D} = \bigcap_{i\in J} D'_i$. Since $\Psi_{n-1}^{\triangledown}$ is a filter and $G_i \to D'_i \in \Psi_{n-1}^{\triangledown}$, we have that $\bigcap_{i\in J}(G_i \to D'_i) \in \Psi_{n-1}^{\triangledown}$, hence by Lemma 3*(2)* and again the fact that $\Psi_{n-1}^{\triangledown}$ is a filter, we have $(\bigcap_{i\in J} G_i) \to \tilde{D} \in \Psi_{n-1}^{\triangledown}$. Since $C \leq_{\triangledown} (\bigcap_{i\in J} G_i)$, by rule (η) we get $(\bigcap_{i\in J} G_i) \to \tilde{D} \leq_{\triangledown} C \to \tilde{D}$, hence $C \to \tilde{D} \in \Psi_{n-1}^{\triangledown}$. This last fact together with (5) implies $C \to B \in \Phi_n^{\triangledown}$.

As a consequence of previous lemma, the filter $\Psi_\infty^{\triangledown}$ is generated by the union of $\Phi_n^{\triangledown}$'s.

Lemma 34 *Let $\Sigma^{\triangledown}$ an eitt. Then $\Psi_\infty^{\triangledown}$ represents a fixed point operator:*

$$\forall f \in [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}].\mathbb{H}^{\triangledown}(\Psi_\infty^{\triangledown})(f) = (f \circ (\mathbb{H}^{\triangledown}(\Psi_\infty^{\triangledown}))(f).$$

PROOF. By definition of $\mathbb{H}^{\triangledown}$ we have to prove:

$$\forall f \in [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}].(\mathbb{F}^{\triangledown}(\Psi_\infty^{\triangledown}) \circ \mathbb{G}^{\triangledown})(f) = (f \circ \mathbb{F}^{\triangledown}(\Psi_\infty^{\triangledown}) \circ \mathbb{G}^{\triangledown})(f).$$

Since $\mathbb{F}^{\triangledown}$ is surjective onto $[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}]$, we can take $f = \mathbb{F}^{\triangledown}(\Xi)$ and we get using Lemma 14 and the definition of $\mathbb{F}^{\triangledown}$:

$$\forall \Xi \in \mathcal{F}^{\triangledown}.\Psi_\infty^{\triangledown} \cdot \Xi = \Xi \cdot (\Psi_\infty^{\triangledown} \cdot \Xi).$$

As usual, we only consider compact filters, i.e. we will prove that:

$$\forall C \in \mathbb{T}^{\triangledown}.\Psi_\infty^{\triangledown} \cdot \uparrow^{\triangledown} C = \uparrow^{\triangledown} C \cdot (\Psi_\infty^{\triangledown} \cdot \uparrow^{\triangledown} C).$$

For all $B \in \mathbb{T}^{\triangledown}$ we have:

$$
\begin{aligned}
B \in \Psi^{\triangledown}_{\infty} \cdot \uparrow^{\triangledown} C &\Leftrightarrow C \to B \in \Psi^{\triangledown}_{\infty} && \text{by Lemma 13} \\
&\Leftrightarrow \exists n.\, C \to B \in \Psi^{\triangledown}_{n+1} && \text{by definition of } \Psi^{\triangledown}_{\infty} \\
&\Leftrightarrow \exists n.\, C \to B \in \Phi^{\triangledown}_{n+1} && \text{by Lemma 33} \\
&\Leftrightarrow \exists n, B'.\, C \to B' \in \Psi^{\triangledown}_{n}\ \& \\
&\qquad C \leq_{\triangledown} B' \to B && \text{by definition of } \Phi^{\triangledown}_{n+1} \\
&\Leftrightarrow \exists n, B'.\, B' \in \Psi^{\triangledown}_{n} \cdot \uparrow^{\triangledown} C\ \& \\
&\qquad C \leq_{\triangledown} B' \to B && \text{by Lemma 13} \\
&\Leftrightarrow \exists n.\, B \in \uparrow^{\triangledown} C \cdot (\Psi^{\triangledown}_{n} \cdot \uparrow^{\triangledown} C) && \text{by Lemma 13} \\
&\Leftrightarrow B \in \uparrow^{\triangledown} C \cdot (\Psi^{\triangledown}_{\infty} \cdot \uparrow^{\triangledown} C) && \text{by definition of } \Psi^{\triangledown}_{\infty}.
\end{aligned}
$$

An operator $\mathsf{H} \in [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]$ is *pre-fixed point operator* iff:

$$\forall f \in [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}].\mathsf{H}(f) \subseteq (f \circ \mathsf{H})(f).$$

Clearly all fixed point operators are pre-fixed point operators, but not vice versa.

Lemma 35 *Let $\Sigma^{\triangledown}$ an eitt. Then $\Psi^{\triangledown}_{\infty}$ represents the minimal pre-fixed point operator: for all $\mathsf{H} \in [[\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}] \to \mathcal{F}^{\triangledown}]$ pre-fixed point operators and $f \in [\mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}]$,*

$$\mathbb{H}^{\triangledown}(\Psi^{\triangledown}_{\infty})(f) \subseteq \mathsf{H}(f).$$

PROOF. Reasoning as in the proof of previous lemma it is easy to check that we only need to show:

$$\forall \Xi \in \mathcal{F}^{\triangledown}.\Psi^{\triangledown}_{\infty} \cdot \Xi \subseteq (\mathsf{H} \circ \mathbb{F}^{\triangledown})(\Xi),$$

i.e.

$$\forall C \in \mathbb{T}^{\triangledown}.\Psi^{\triangledown}_{\infty} \cdot \uparrow^{\triangledown} C \subseteq \mathsf{H}(g),$$

where $g = \mathbb{F}^{\triangledown}(\uparrow^{\triangledown} C)$.

We first prove by induction on n that

$$\forall C \in \mathbb{T}^{\triangledown}.\Psi^{\triangledown}_{n} \cdot \uparrow^{\triangledown} C \subseteq \mathsf{H}(g). \tag{6}$$

If $n = 0$, then

$$
\begin{aligned}
\Psi^{\triangledown}_{0} \cdot \uparrow^{\triangledown} C &= \uparrow^{\triangledown} \Omega \\
&\subseteq \mathsf{H}(g).
\end{aligned}
$$

Case $n+1$.

$$\begin{array}{rll} B \in \Psi^{\triangledown}_{n+1} \cdot \uparrow^{\triangledown} C & \Leftrightarrow C \to B \in \Psi^{\triangledown}_{n+1} & \text{by Lemma 13} \\ & \Leftrightarrow C \to B \in \Phi^{\triangledown}_{n+1} & \text{by Lemma 33} \\ & \Leftrightarrow \exists B'.\, C \to B' \in \Psi^{\triangledown}_{n}\ \& & \\ & \qquad C \leq_{\triangledown} B' \to B & \text{by definition of } \Phi^{\triangledown}_{n+1} \\ & \Leftrightarrow \exists B'.\, B' \in \Psi^{\triangledown}_{n} \cdot \uparrow^{\triangledown} C\ \& & \\ & \qquad C \leq_{\triangledown} B' \to B & \text{by Lemma 13} \\ & \Rightarrow \exists B'.\, B' \in \mathsf{H}(g)\ \& & \\ & \qquad C \leq_{\triangledown} B' \to B & \text{by induction} \\ & \Leftrightarrow B \in g(\mathsf{H}(g)) & \text{by Lemma 13} \\ & & \text{being } g = \mathbb{F}^{\triangledown}(\uparrow^{\triangledown} C) \\ & \Rightarrow B \in \mathsf{H}(g) & \text{since } \mathsf{H} \text{ is a pre-fixed} \\ & & \text{point operator.} \end{array}$$

This completes the proof of (6). We now perform the final step.

$$\begin{array}{rll} B \in \Psi^{\triangledown}_{\infty} \cdot \uparrow^{\triangledown} C & \Leftrightarrow B \in (\bigcup_n \Psi^{\triangledown}_{n}) \cdot \uparrow^{\triangledown} C & \text{by definition of } \Psi^{\triangledown}_{\infty} \\ & \Leftrightarrow \exists n.\, B \in \Psi^{\triangledown}_{n} \cdot \uparrow^{\triangledown} C & \text{since the application is continuous} \\ & \Rightarrow B \in \mathsf{H}(g) & \text{by } (6). \end{array}$$

By Lemmata 34 and 35 we get that $\Psi^{\triangledown}_{\infty}$ is $\mathbb{K}^{\triangledown}(\mu)$, i.e. the filter which represents μ.

Theorem 36 *Let $\Sigma^{\triangledown}$ an eitt. Then $\Psi^{\triangledown}_{\infty}$ represents the minimal fixed point operator:*

$$\Psi^{\triangledown}_{\infty} = \mathbb{K}^{\triangledown}(\mu).$$

We can provide now the desired filter model.

Theorem 37 *Let e be a simple easy term. Then there exists a filter model $\mathcal{F}^{\triangledown}$ such that the interpretation of e is the minimal fixed point operator.*

PROOF. The predicate $\mathsf{P}(\Sigma^{\triangledown}, E) \quad \Leftrightarrow \quad E \in \Psi^{\triangledown}_{\infty}$ is trivially continuous. By Theorem 22, there exists a filter model $\mathcal{F}^{\triangledown\infty}$ such that $[\![\mathsf{e}]\!]^{\triangledown\infty}$ is the filter induced by P, that is $\Psi^{\triangledown}_{\infty}$. Finally, by Theorem 36, $\Psi^{\triangledown}_{\infty}$ represents μ.

6 Relations between the present paper and [12]

Since [12] appears in this same journal issue, we think it is worthwhile to point out some common features, as well as some fundamental differences between these two papers.

First of all, both papers use *intersection types theories to build λ-models*: this common approach is widely discussed in Section 2 of [12].

In this section we adopt the convention that definitions, theorems and any other result appearing in [12] will be typed with a final asterisk.

The first difference is the language of intersection types itself. In this paper the *intersection type constructor* is a *total function* from pairs of types to types (Definition 1), while in [12] it is a *partial function* from pairs of types to types (Definitions* 10 and 12). From the viewpoint of the domain descriptions the gain is notable: with the language of the present paper we can represent ω-algebraic lattices, while with the language of [12] we can represent Scott domains. A smaller difference is that here we deal with a class of intersection type languages, so the set of constant types is a parameter, while [12] takes into account only two intersection type languages with fixed type constants.

The *type preorders* in the two papers (Definition 2 and Definition* 14) share the first nine axioms and rules, which are standard properties of joins and step functions plus the axiom making Ω the top. Since in the present paper we allow to build intersection types starting from an arbitrary set of constants, Definition 2 only gives the shape the axioms on constants must have, while Definition* 14 gives the common axioms for the two languages considered there. The peculiar axioms of the two preorders in [12] are given in Definition* 15. We remark that all axioms in Definitions* 14 and 15 are of the shape required by Definition 2. Notably axiom (Ω-η) holds for all the preorders considered here, but only for the second preorder of [12]. The first preorder of [12] satisfies the weaker axiom ($\Omega\ \to$): this is the key for representing a lifted domain. So the first type theory of [12] is not an eitt according to Definition 2 since axiom (Ω-η) is missing, and the second type theory of [12] is not an eitt according to Definition 2 since the intersection type constructor is partial.

The definitions of filters (Definition 11 and Definition* 17), of bases (Definition 7 and Definition* 20), of type assignment systems (Definition 8 and Definition* 22) and the Generation Theorems (Theorem 10 and Theorem* 25) are exactly the same in both papers (the proof is given only here). This way both papers build λ-models, but with different aims. [12] gives two models which are isomorphic to two inverse limit models and uses them to show properties of these last models. Instead, the present paper allows to define infinitely many models, but not all of them have corresponding inverse limit models, the aim being that of finding models where we force the interpretations of suitable λ-terms.

7 Conclusion

The relation between the notions of simple easiness and easiness requires further investigation. While it is clear that simple easiness implies easiness, the question whether easiness implies simple easiness remains open.

The contribution of the present paper is to show that each simple easy term can be interpreted as an arbitrary domain operator which can be represented as a filter of types defined by a continuous predicate.

Research directions which we plan to follow are:

- the characterisation of the λ-theories whose consistency can be shown using the present approach;
- the characterisation of the operators which can be equated to simple easy terms.

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References

[1] S. Abramsky. Domain theory in logical form. *Ann. Pure Appl. Logic*, 51(1-2):1–77, 1991.

[2] F. Alessi. *Strutture di tipi, teoria dei domini e modelli del lambda calcolo*. PhD thesis, Torino University, 1991.

[3] F. Alessi, M. Dezani-Ciancaglini, and F. Honsell. Filter models and easy terms. In A.Restivo, S.Ronchi, and L.Roversi, editors, *Italian Conference on Theoretical Computer Science*, volume 2202 of *Lecture Notes in Computer Science*, pages 17–37. Springer-Verlag, 2001.

[4] F. Alessi and S. Lusin. Simple easy terms. In S. van Bakel, editor, *Intersection Types and Related Systems*, volume 70 of *Electronic Lecture Notes in Theoretical Computer Science*. Elsevier, 2002.

[5] J. Baeten and B. Boerboom. ω can be anything it shouldn't be. *Indag.Math.*, 41:111–120, 1979.

[6] H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *J. Symbolic Logic*, 48(4):931–940 (1984), 1983.

[7] H. P. Barendregt. *The Lambda Calculus: its Syntax and Semantics*. North-Holland, Amsterdam, revised edition, 1984.

[8] M. Coppo and M. Dezani-Ciancaglini. An extension of the basic functionality theory for the λ-calculus. *Notre Dame J. Formal Logic*, 21(4):685–693, 1980.

[9] M. Coppo, M. Dezani-Ciancaglini, F. Honsell, and G. Longo. Extended type structures and filter lambda models. In G.Lolli, G.Longo, and A.Marcja, editors, *Logic colloquium '82*, pages 241–262, Amsterdam, 1984. North-Holland.

[10] M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Principal type schemes and λ-calculus semantics. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 535–560. Academic Press, London, 1980.

[11] M. Coppo, M. Dezani-Ciancaglini, and M. Zacchi. Type theories, normal forms, and D_∞-lambda-models. *Inform. and Comput.*, 72(2):85–116, 1987.

[12] M. Dezani-Ciancaglini, S. Ghilezan, and S. Likavec. Behavioural inverse limit models. *Theoret. Comput. Sci.*, 2003. This issue.

[13] M. Dezani-Ciancaglini and S. Lusin. Intersection types and lambda theories. In S. Soloviev, R. D. Cosmo, and G. Longo, editors, *International Workshop on Isomorphisms of Types*, 2002. http://www.irit.fr/zeno/WIT2002/proceedings.shtml.

[14] F. Honsell and S. Ronchi Della Rocca. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *J. Comput. System Sci.*, 45(1):49–75, 1992.

[15] J. Kuper. On the Jacopini technique. *Inform. and Comput.*, 138:101–123, 1997.

[16] S. Lusin. *Intersection Types, Lambda Abstraction Algebras and Lambda Theories*. PhD thesis, Venice University, 2002. ftp://ftp.cs.unibo.it/pub/techreports/2003-07.pdf.

[17] S. Lusin and A. Salibra. The lattice of lambda theories. *J. of Logic and Computation*, 2003. to appear.

[18] G. D. Plotkin. Set-theoretical and other elementary models of the λ-calculus. *Theoret. Comput. Sci.*, 121(1-2):351–409, 1993.

[19] G. Pottinger. A type assignment for the strongly normalizable λ-terms. In J. Seldin and J. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 561–577. Academic Press, London, 1980.

[20] D. S. Scott. Continuous lattices. In F. Lawvere, editor, *Toposes, Algebraic*

Geometry and Logic, volume 274 of *Lecture Notes in Mathematics*, pages 97–136, Berlin, 1972. Springer.

[21] C. P. Wadsworth. The relation between computational and denotational properties for Scott's D_∞-models of the lambda-calculus. *SIAM J. Comput.*, 5(3):488–521, 1976.

[22] C. Zylberajch. *Syntaxe et Semantique de la Facilité en Lambda-calcul*. PhD thesis, Université Paris VII, 1991.