Inverse Limit Models as Filter Models

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Abstract. Natural intersection type preorders are the type structures which agree with the plain intuition of intersection type constructor as set-theoretic intersection operation and arrow type constructor as set-theoretic function space constructor. In this paper we study the relation between natural intersection type preorders and natural λ-structures, i.e. ω-algebraic lattices $\mathcal{D}$ with Galois connections given by $F : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$ and $G : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}$. We prove on one hand that natural intersection type preorders induces natural λ-structures, on the other hand that natural λ-structures admits presentations through intersection type preorders. Moreover we give a concise presentations of classical D_∞ models of untyped λ-calculus through suitable natural intersection type preorders and prove that filter models induced by them are isomorphic to D_∞.

1 Introduction

Intersection type preorders can be viewed as *domain logics* for ω-algerbraic lattices (see [CDCHL84], [Abr91]). That means that ω-algebraic lattices can be defined in a syntactic way through "axioms and rules" which involve intersection type preorders. This possibility brings a nice consequence. The classical way to interpret a statement of the shape $M \models \phi$ (the program M satisfies the property ϕ) in a semantic domain $\mathcal{D}$ is to view M as a point in $\mathcal{D}$, and ϕ as a (suitable) subset Φ of $\mathcal{D}$, obtaining a membership judgment in $\mathcal{D}$: $M \models \phi$ is translated into $[\![M]\!]^{\mathcal{D}} \in \Phi$, where the interpretation function $[\![\cdot]\!]$ maps programs to elements of $\mathcal{D}$. The Stone duality perspective uses intersection type preorders in order to "reverse" this point of view. Types are taken for setting up a basis for the topology of the space (in algebraic terms: the meet-semilattice of coprime compact open sets of the lattice under consideration). Points are not the building blocks of the semantic domains, rather they are recovered as *filters* of types. Following this view $M \models \phi$ is translated in an "opposite" membership judgment $A \in [\![M]\!]^{\mathcal{D}}$, that is: "the type A (corresponding to the property ϕ and interpreted as Φ) is a member of the filter (of properties) which sets up the whole interpretation of M".

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This view is fruitful in the following sense: the interpretation of a program is fully determined when all the properties which the program satisfies are known. Since actually the syntactic way of defining lattices through intersection type preorders puts at disposal a machinery (the *type assignment system*) which allows to assign types/properties to programs in a finitary way, the gain consists in the possibility of definining program interpretations by answering the question: "which types can be assigned to programs by the type assignment system?", whose answer can in turn exploit useful technical results on type assignment system (such as, for instance, the Generation Theorem at page 12).

Since, as mentioned, Stone duality is the mathematical framework where to settle the relationship between intersection type preorders and ω-algebraic complete lattices, we now recall shortly some basics facts concerning it. A complete treatment can be found in the milestone paper [Abr91].

Let X be a topological space with topology $\Omega(X)$ (we recall that $\Omega(X)$ is a frame, that is a complete distributive lattice).

Define a *completely prime filter* over X[1] as a subset $\xi \subseteq \Omega(X)$ such that (a, b range over $\Omega(X)$):

- $X \in \xi$;
- $a \in \xi$ and $a \subseteq b$ imply $b \in \xi$;
- $a \in \xi$ and $b \in \xi$ imply $a \cap b \in \xi$;
- $\bigcup_{i \in I} a_i \in \xi$ implies $a_i \in \xi$ for some $i \in I$.

Let $\mathsf{Pt}(\Omega(X))$ be the set of all completely prime filters over $\Omega(X)$. The fundamental result is that if we work in the category **Sob** of *sober* spaces, then we have bijections

$$(\dagger) \qquad X \simeq \mathsf{Pt}(\Omega(X))$$

from which it follows an equivalence between the categories **Sob** and **Loc** (this last one is the opposite of the category of frames).

The importance of this result can be summarized as follows: given certain topological spaces (the sober ones), one can forget points, since topology allows to recover them completely.

Without entering the details of the rather involved definition of sober space (see [Joh86]), we just recall that all algebraic domains used in denotational semantics enjoy the property of being sober.

Intersection type preorders are particular structures which arise when restricting the equivalence ($\dagger$) above to the case of the category **ALG** of ω-algebraic lattices endowed with their Scott topology. In such a case, it is possible to exploit the following property of the topology of ω-algebraic lattices: $\Omega(X)$ can be completely recoverd by the subsets $\mathsf{Cpr}(\Omega(X))$ of the *coprime* compact open sets (an open set a is coprime if $a \subseteq b \cup c$ implies $a \subseteq b$ or $a \subseteq c$). The domain $\mathsf{Cpr}(\Omega(X))$ turns out to be a meet-semilattice (whence the meet-semilattice structure of

[1] Actually one can take completely prime filters over any complete lattice $\mathcal{D}$, not just topologies.

intersection types) and it satisfies

$$\mathsf{Pt}(\Omega(X)) \simeq \mathsf{Filt}(\mathsf{Cpr}(\Omega(X))),$$

where Filt is the operation of taking filters (defined by dropping the last condition in the definition above of completely prime filter). As a consequence of (†), any ω-algebraic lattice X satisfies

$$X \simeq \mathsf{Filt}(\mathsf{Cpr}(\Omega(X))).$$

A further step is to notice that $\mathsf{Filt}(\mathsf{Cpr}(\Omega(X)))$ is isomorphic to $\mathcal{K}^{op}(X)$, the subspace of compact elements of X, with the reverse ordering of X. Thus the final form which the "Stone duality" theory assumes when applied to ω-algebraic lattices is expressed by the isomorphism:

$$X \simeq \mathsf{Filt}(\mathcal{K}^{op}(X)).$$

This result is the foundation which guarantees the possibility of describing ω-algebraic lattices by means of intersection type preorders.

In the present paper we are mainly interested in a fine analysis of type preorders which agree with the intuition that arrow type constructor corresponds to the set-theoretic continuous function space constructor. We call *natural* this kind of type preorders. Our first result is to show that the semantic counterpart of natural type preorders are ω-algebraic lattices $\mathcal{D}$ endowed with pairs of continuous function $F : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$, $G : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}$ which set up Galois connection:

$$F \circ G \sqsupseteq Id_{[\mathcal{D}\to\mathcal{D}]} \qquad G \circ F \sqsubseteq Id_{\mathcal{D}}.$$

We call *natural λ-structures* this kind of lattices. We prove on one hand that the space of filter on a natural type preorders is a natural λ-structure. On the other hand natural λ-structures can be presented via natural type preorders, that is

(iso) each natural λ-structure is isomorphic, both as lattice and as applicative structure, to the space of filters of a suitable natural type preorder.

Then we turn our attention to models of untyped λ-calculus computed inside **ALG**, built through the classical inverse limit technique (see [Sco72]). As a consequence of *(iso)*, for any D_∞ it is possible to build a filter structure isomorphic to it, but the construction given in the proof of *(iso)* is not effective and uses a possibly countable amount of redundant types (since it introduces a constant type for any compact element of the domain). So we look for a more concise presentation of D_∞. Our second result is to prove that the natural type preorder which induces a filter model isomorphic to D_∞, starting from D_0, is exactly the natural type preorder *freely generated* by a type preorder which induces D_0 together with the equalities which arise from encoding the initial projections.

This second isomorphism result could be obtained by adapting the technique of [Abr91], Section 4. Our approach does not use the complex Abramsky's machinery (tailored for more general domains, the SFP's ones) and allows to get a rather quick isomorphism proof.

Finally, the organization of the paper. In Section 2 we recall some standard facts on ω-algebraic lattices, and introduce natural λ-structures. Section 3 discusses type preorders, filter structures and type assignment systems. In Section 4 we prove the two isomorphism results which relate natural intersection type preorders with natural λ-structures. Finally, in Section 5, we give the effective and "concise" presentations of D_∞ models via suitable natural intersection type preorders and show that the filter structures induced by them are isomorphic to D_∞'s.

2 Natural λ-structures

We start with a standard definition:

Definition 1. *1. If $\mathcal{D}$ is an ω-algebraic complete lattice, $[\mathcal{D} \to \mathcal{D}]$ denotes the set of continuous functions from $\mathcal{D}$ to $\mathcal{D}$ and $\mathcal{K}(\mathcal{D})$ the set of compact elements of $\mathcal{D}$.*

2. If $a, b \in \mathcal{D}$, then $a \Rightarrow b$ is the step function defined by

$$a \Rightarrow b\ (d) = \ \textit{if } a \sqsubseteq d \textit{ then } b \textit{ else } \bot.$$

Recall that the compact elements in the domain of continuous functions are exactly the sups of finite sets of step functions between compact elements. Moreover we restate some well know properties of continuous functions [GHK+80]. Let I be a finite set.

Proposition 1. *1. $c \Rightarrow d \sqsubseteq \bigsqcup_{i \in I}(a_i \Rightarrow b_i)$ iff $d \sqsubseteq \bigsqcup_{i \in J} b_i$ where $J = \{i \in I \mid a_i \sqsubseteq c\}$.*

2. Each continuous function f is the sup of the step functions between compact elements which are under f, i.e.

$$\begin{aligned} f &= \bigsqcup\{a \Rightarrow b \mid a \Rightarrow b \sqsubseteq f, a \textit{ and } b \textit{ compact}\} \\ &= \bigsqcup\{a \Rightarrow b \mid b \sqsubseteq f(a), a \textit{ and } b \textit{ compact}\}. \end{aligned}$$

Next definition introduces *natural λ-structures.* Natural λ-structures set up a bridge between domain theoretic λ-models and *filter structures*: more precisely, they are the semantic counterpart of those intersection type preorders (the *natural* ones, see Definition 6) whose axioms agree with the intuition that the arrow type constructor corresponds to the set-theoretic function space constructor.

Definition 2 (Natural λ-structure). *A* natural λ-structure *is a triple $\langle \mathcal{D}, F_\mathcal{D}, G_\mathcal{D} \rangle$, where $\mathcal{D}$ is an ω-algebraic complete lattice, and $F_\mathcal{D} : \mathcal{D} \to [\mathcal{D} \to \mathcal{D}]$, $G_\mathcal{D} : [\mathcal{D} \to \mathcal{D}] \to \mathcal{D}$ are Scott continuous functions such that $\langle F_\mathcal{D}, G_\mathcal{D} \rangle$ sets up a Galois connection, i.e.:*

1. $F_{\mathcal{D}} \circ G_{\mathcal{D}} \sqsupseteq Id_{[\mathcal{D}\to\mathcal{D}]}$;
2. $G_{\mathcal{D}} \circ F_{\mathcal{D}} \sqsubseteq Id_{\mathcal{D}}$.

Given a natural λ-structure $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ and $a, b \in \mathcal{D}$, we will often write $a \cdot b$ as short for $F_{\mathcal{D}}(a)(b)$.

Example 1. An example of a natural λ-structure is $\langle \mathcal{D}^{\spadesuit}, F_{\spadesuit}, G_{\spadesuit} \rangle$, where

- $\mathcal{D}^{\spadesuit}$ is $\mathbb{N} \cup \{\bot, \top\}$, endowed with the order which is flat on natural numbers, and moreover $\bot = m \sqcap n$, $\top = m \sqcup n$, for any $m, n \in \mathbb{N}$, $m \neq n$;
- $F(a) = (\bot \Rightarrow a)$ for any $a \in \mathcal{D}^{\spadesuit}$;
- $G(f) = f(\top)$ for any $f \in [D^{\spadesuit} \to \mathcal{D}^{\spadesuit}]$.

$\langle \mathcal{D}^{\spadesuit}, F_{\spadesuit}, G_{\spadesuit} \rangle$ is a natural λ-structure. In fact

- $G_{\spadesuit}(F_{\spadesuit}(a)) = G_{\spadesuit}(\bot \Rightarrow a) = (\bot \Rightarrow a)(\top) = a$;
- $F_{\spadesuit}(G_{\spadesuit}(f)) = (\bot \Rightarrow f(\top)) \sqsupseteq f$,

hence $F_{\spadesuit}$ and $G_{\spadesuit}$ set up a Galois connection.

Natural λ-structures are λ-structures as defined in [Plo93], Section 3.

The following properties of natural λ-structures follow easily from their definitions. Although they are almost immediate consequence of the fact that, from a categorical point of view, $F_{\mathcal{D}}$ is left adjoint of $G_{\mathcal{D}}$, we will recall the direct proof.

Proposition 2. *Let $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ be a natural λ-structure.*

1. $G_{\mathcal{D}}$ maps always compact elements into compact elements.
2. $F_{\mathcal{D}}$ determines $G_{\mathcal{D}}$ by

$$G_{\mathcal{D}}(f) = \sqcap\{d \mid f \sqsubseteq F_{\mathcal{D}}(d)\}$$

for all continuous functions f.
3. $G_{\mathcal{D}}$ is additive, $G_{\mathcal{D}}(f \sqcup g) = G_{\mathcal{D}}(f) \sqcup G_{\mathcal{D}}(g)$.

Proof. Notice that, by condition (2) of Definition 2,

$$(*) \quad G_{\mathcal{D}}(f) \sqsubseteq G_{\mathcal{D}}(F_{\mathcal{D}}(d)) \text{ imply } G_{\mathcal{D}}(f) \sqsubseteq d.$$

1. We show that if f is compact then $G_{\mathcal{D}}(f)$ is compact, that is if $G_{\mathcal{D}}(f) \sqsubseteq \bigsqcup_{z\in Z} z$, where Z is directed, then $G_{\mathcal{D}}(f) \sqsubseteq z$ for some $z \in Z$.

$$\begin{aligned} G_{\mathcal{D}}(f) \sqsubseteq \textstyle\bigsqcup_{z\in Z} z &\Rightarrow F_{\mathcal{D}}(G_{\mathcal{D}}(f)) \sqsubseteq \textstyle\bigsqcup_{z\in Z} F_{\mathcal{D}}(z) \\ &\qquad \text{since } F_{\mathcal{D}} \text{ is continuous} \\ &\Rightarrow f \sqsubseteq \textstyle\bigsqcup_{z\in Z} F_{\mathcal{D}}(z) \\ &\qquad \text{by condition (1) of Definition 2} \\ &\Rightarrow \exists z \in Z. f \sqsubseteq F_{\mathcal{D}}(z) \\ &\qquad \text{since } f \text{ is compact and } \{F_{\mathcal{D}}(z) \mid z \in Z\} \text{ is directed} \\ &\Rightarrow \exists z \in Z. G_{\mathcal{D}}(f) \sqsubseteq G_{\mathcal{D}}(F_{\mathcal{D}}(z)) \\ &\qquad \text{since } G_{\mathcal{D}} \text{ is monotone} \\ &\Rightarrow \exists z \in Z. G_{\mathcal{D}}(f) \sqsubseteq z \\ &\qquad \text{by (*)}. \end{aligned}$$

2. It sufficies to show that $G_{\mathcal{D}}(f) \sqsubseteq d$ iff $f \sqsubseteq F_{\mathcal{D}}(d)$.

$$\begin{array}{rll} G_{\mathcal{D}}(f) \sqsubseteq d & \Rightarrow F_{\mathcal{D}}(G_{\mathcal{D}}(f)) \sqsubseteq F_{\mathcal{D}}(d) & \text{since } F_{\mathcal{D}} \text{ is monotone} \\ & \Rightarrow f \sqsubseteq F_{\mathcal{D}}(d) & \text{by condition (1) of Definition 2} \end{array}$$

$$\begin{array}{rll} f \sqsubseteq F_{\mathcal{D}}(d) & \Rightarrow G_{\mathcal{D}}(f) \sqsubseteq G_{\mathcal{D}}(F_{\mathcal{D}}(d)) & \text{since } G_{\mathcal{D}} \text{ is monotone} \\ & \Rightarrow G_{\mathcal{D}}(f) \sqsubseteq d & \text{by (*).} \end{array}$$

3. We have

$$\begin{array}{rll} G_{\mathcal{D}}(f \sqcup g) & \sqsubseteq G_{\mathcal{D}}(F_{\mathcal{D}}(G_{\mathcal{D}}(f)) \sqcup F_{\mathcal{D}}(G_{\mathcal{D}}(g))) & \text{by condition (1) of Definition 2} \\ & \sqsubseteq G_{\mathcal{D}}(F_{\mathcal{D}}(G_{\mathcal{D}}(f) \sqcup G_{\mathcal{D}}(g))) & \text{since } F_{\mathcal{D}} \text{ is continuous} \\ & \sqsubseteq G_{\mathcal{D}}(f) \sqcup G_{\mathcal{D}}(g) & \text{by condition (2) of Definition 2.} \end{array}$$

Natural λ-structures provide interpretation to terms of λ-calculus in a standard way: interpretation of application is obtained by applying $F_{\mathcal{D}}$ to the interpretation of the term M (in function position) in (MN); interpretation of abstraction is obtained by applying $G_{\mathcal{D}}$ to the function induced by $\lambda x.M$. Notice that the possibility of interpreting λ-terms just relies on the existence of $F_{\mathcal{D}}$ and $G_{\mathcal{D}}$, independently from the fact they set up a Galois connection.

In the following Λ denotes the set of λ-terms, Env_D denotes the set of functions $Var \to \mathcal{D}$ from term variables to $\mathcal{D}$ (term environments).

Definition 3. *Let $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ be a natural λ-structure. The interpretation $[\![\ ]\!]^{\mathcal{D}} : \Lambda \times \mathsf{Env}_{\mathcal{D}} \to \mathcal{D}$ is defined inductively on λ-terms as follows:*

$$\begin{array}{ll} [\![x]\!]^{\mathcal{D}}_{\rho} & = \rho(x); \\ [\![MN]\!]^{\mathcal{D}}_{\rho} & = F_{\mathcal{D}}([\![M]\!]^{\mathcal{D}}_{\rho})([\![N]\!]^{\mathcal{D}}_{\rho}); \\ [\![\lambda x.M]\!]^{\mathcal{D}}_{\rho} & = G_{\mathcal{D}}(\boldsymbol{\lambda} d \in \mathcal{D}.[\![M]\!]^{\mathcal{D}}_{\rho[x:=d]}) \end{array}$$

where ρ ranges over the set of term environments Env_D.

Example 2. Consider the natural λ-structure $\mathcal{D}^{\spadesuit}$ defined in Example 1. Then for any $M \in \Lambda$, $[\![(\lambda x.x)M]\!]^{\mathcal{D}^{\spadesuit}}_{\rho} = \top$. In fact

$$\begin{array}{rl} [\![(\lambda x.x)]\!]^{\mathcal{D}^{\spadesuit}}_{\rho} & = G_{\spadesuit}(\boldsymbol{\lambda} d \in \mathcal{D}^{\spadesuit}.d) \\ & = G_{\spadesuit}(\bigsqcup\{a \Rightarrow a \mid a \in D^{\spadesuit}\}) \\ & = (\bigsqcup\{a \Rightarrow a \mid a \in D^{\spadesuit}\})(\top) \\ & = \top \end{array}$$

Therefore

$$\begin{array}{rl} [\![(\lambda x.x)M]\!]^{\mathcal{D}^{\spadesuit}}_{\rho} & = F_{\spadesuit}(\top)([\![M]\!]^{\mathcal{D}^{\spadesuit}}_{\rho}) \\ & = (\bot \Rightarrow \top)([\![M]\!]^{\mathcal{D}^{\spadesuit}}_{\rho}) \\ & = \top \end{array}$$

By the way notice that this proves that $\langle \mathcal{D}^{\spadesuit}, F_{\spadesuit}, G_{\spadesuit} \rangle$ is not a λ-model, since, for any y, ρ such that $\rho(y) = \bot$, it follows

$$\begin{array}{rl} [\![(\lambda x.x)y]\!]^{\mathcal{D}^{\spadesuit}}_{\rho} & = \top \\ & \neq \bot \\ & = [\![y]\!]^{\mathcal{D}^{\spadesuit}}_{\rho}. \end{array}$$

As well known, whenever $F_{\mathcal{D}} \circ G_{\mathcal{D}} = Id_{[\mathcal{D}\to\mathcal{D}]}$, the λ-structure $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ is a λ-model, being a reflexive object in the cartesian closed category of ω-algebraic lattice and continuous functions.

The notion of isomorphism between λ-structures is as expected: a lattice isomorphism which "commutes" with F and G.

Definition 4 (Isomorphism of natural λ-structures). *Two natural λ-structures $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ and $\langle \mathcal{E}, F_{\mathcal{E}}, G_{\mathcal{E}} \rangle$ are isomorphic if there exists a lattice isomorphism $\mathsf{m} : \mathcal{D} \to \mathcal{E}$ such that for any $d \in \mathcal{D}$ and $f \in [\mathcal{D} \to \mathcal{D}]$:*

1. $F_{\mathcal{E}}(\mathsf{m}(d)) = \mathsf{m} \circ F_{\mathcal{D}}(d) \circ \mathsf{m}^{-1}$,
2. $\mathsf{m}(G_{\mathcal{D}}(f)) = G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1})$.

It is easy to show that previous definition can be simplified.

Proposition 3. *Two natural λ-structures $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ and $\langle \mathcal{E}, F_{\mathcal{E}}, G_{\mathcal{E}} \rangle$ are isomorphic iff there exists a lattice isomorphism $\mathsf{m} : \mathcal{D} \to \mathcal{E}$ such that*

$$\forall d, d' \in \mathcal{D}. \mathsf{m}(d \cdot d') = \mathsf{m}(d) \cdot \mathsf{m}(d').$$

Proof. First notice that condition (1) of Definition 4 is equivalent to the condition of Proposition 3. So it is enough to prove that condition (1) of Definition 4 implies condition (2) of the same definition.

Proof of $G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1}) \sqsubseteq \mathsf{m}(G_{\mathcal{D}}(f))$.

$$\begin{array}{rll} G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1}) & \sqsubseteq G_{\mathcal{E}}(\mathsf{m} \circ (F_{\mathcal{D}}(G_{\mathcal{D}}(f)) \circ \mathsf{m}^{-1}) & \text{by condition (1) of Definition 2} \\ & = G_{\mathcal{E}}(F_{\mathcal{E}}(\mathsf{m}(G_{\mathcal{D}}(f)))) & \text{by condition (1) of Definition 4} \\ & \sqsubseteq \mathsf{m}(G_{\mathcal{D}}(f)) & \text{by condition (2) of Definition 2.} \end{array}$$

Before proving the other inequality, notice that in a symmetric way we can show

$$(\natural) \quad G_{\mathcal{D}}(\mathsf{m}^{-1} \circ f \circ \mathsf{m}) \sqsubseteq \mathsf{m}^{-1}(G_{\mathcal{E}}(f)).$$

Proof of $G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1}) \sqsupseteq \mathsf{m}(G_{\mathcal{D}}(f))$.

$$\begin{array}{rll} \mathsf{m}(G_{\mathcal{D}}(f)) & = \mathsf{m}(G_{\mathcal{D}}(\mathsf{m}^{-1} \circ \mathsf{m} \circ f \circ \mathsf{m}^{-1} \circ \mathsf{m})) & \\ & \sqsubseteq \mathsf{m}(G_{\mathcal{D}}(\mathsf{m}^{-1} \circ (F_{\mathcal{E}}(G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1})) \circ \mathsf{m})) & \text{by condition (1) of Definition 2} \\ & = \mathsf{m}(\mathsf{m}^{-1}(G_{\mathcal{E}}(F_{\mathcal{E}}(G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1}))))) & \text{by } (\natural) \\ & = G_{\mathcal{E}}(\mathsf{m} \circ f \circ \mathsf{m}^{-1}) & \text{by condition (2) of Definition 2.} \end{array}$$

3 Natural filter structures

Intersection types, the building blocks for the filter models, are syntactical objects built by closing a given set $\mathbb{C}$ of *type atoms* (constants), which contains the universal type Ω, under the *function type* constructor $\to$ and the *intersection type* constructor $\cap$.

Definition 5 (Intersection type language). *The* intersection type language *over $\mathbb{C}$, denoted by $\mathbb{T} = \mathbb{T}(\mathbb{C})$ is defined by the following abstract syntax:*

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}.$$

Much of the expressive power of intersection type languages comes from the fact that they are endowed with a *preorder relation*, which induces, on the set of types, the structure of a meet semi-lattice with respect to intersection. We consider here a class of preorder relations we call natural, for the general definition see [ADCH03].

Definition 6 (Natural intersection type preorder).

1. *A* natural intersection type preorder (nitp) *Σ is a pair $(\mathbb{C}^{\Sigma}, \leq_{\Sigma})$ where $\mathbb{C}^{\Sigma}$ is a set of type constants and $\leq_{\Sigma}$ is a binary relation over $\mathbb{T}^{\Sigma} = \mathbb{T}(\mathbb{C}^{\Sigma})$ satisfying the following set ∇^0 ("nabla-zero") of axioms and rules:*

$$(refl)\quad A \leq_{\Sigma} A \qquad\qquad (idem)\quad A \leq_{\Sigma} A \cap A$$

$$(incl_L)\quad A \cap B \leq_{\Sigma} A \qquad\qquad (incl_R)\quad A \cap B \leq_{\Sigma} B$$

$$(mon)\quad \frac{A \leq_{\Sigma} A' \quad B \leq_{\Sigma} B'}{A \cap B \leq_{\Sigma} A' \cap B'} \qquad\qquad (trans)\quad \frac{A \leq_{\Sigma} B \quad B \leq_{\Sigma} C}{A \leq_{\Sigma} C}$$

$$(\Omega)\quad A \leq_{\Sigma} \Omega \qquad\qquad (\Omega\text{-}\eta)\quad \Omega \leq_{\Sigma} \Omega \to \Omega$$

$$(\to\text{-}\cap)\quad (A \to B) \cap (A \to C) \leq_{\Sigma} A \to B \cap C \qquad\qquad (\eta)\quad \frac{A' \leq_{\Sigma} A \quad B \leq_{\Sigma} B'}{A \to B \leq_{\Sigma} A' \to B'}$$

2. *A recursive set ∇ of axioms and rules of the shape $A \leq_{\nabla} B$ over $\mathbb{T}^{\nabla} = \mathbb{T}(\mathbb{C}^{\nabla})$ is said to generate the nitp $\Sigma^{\nabla} = (\mathbb{C}^{\nabla}, \leq_{\nabla})$ if $A \leq_{\nabla} B$ holds iff it can be derived from the axioms and rules of $\nabla \cup \nabla^0$.*

Axiom (Ω) states that each nitp has a maximal element.

The meaning of the last three axioms and rules can be grasped if we consider types to denote subsets of a domain of discourse and we look at $\to$ as the function space constructor in the light of Curry-Scott semantics, see [Sco75]. Thus the type $A \to B$ denotes the set of *total* functions which map each element of A into an element of B. Axiom $(\Omega\text{-}\eta)$ expresses the fact that all the objects in our domain of discourse are total functions, i.e. that Ω is equal to $\Omega \to \Omega$ [BCDC83]. This is so since $\Omega \to \Omega$ is the set of functions which applied to an arbitrary element return again an arbitrary element.

The intended interpretation of arrow types motivates axiom $(\to\text{-}\cap)$, which implies that if a function maps A into B, and the same function maps also A into C, then, actually, it maps the whole A into the intersection between B and C (i.e. into $B \cap C$), see [BCDC83].

Rule (η) is also very natural in view of the set-theoretic interpretation. It implies that the arrow constructor is contravariant in the first argument and covariant in the second one. It is clear that if a function maps A into B, and we take a subset A' of A and a superset B' of B, then this function will map also A' into B', see [BCDC83].

Notation.
- $A \sim_\Sigma B$ and $A \sim_\nabla B$ will be short for $A \leq_\Sigma B \leq_\Sigma A$ and $A \leq_\nabla B \leq_\nabla A$, respectively.
- Since $\cap$ is commutative and associative (modulo $\sim_\Sigma$), we shall write $\bigcap_{i \leq n} A_i$ for $A_1 \cap \ldots \cap A_n$. Similarly we shall write $\cap_{i \in I} A_i$, where I denotes always a finite set. Moreover we make the convention that $\cap_{i \in \emptyset} A_i$ is Ω.

Before going on, we give a simple lemma, whose proof is obtained combining $(\rightarrow$-$\cap)$ and (η).

Lemma 1. *Let Σ be a nitp. Then, for any I, $A_i, B_i \in \mathbb{T}^\Sigma$ $(i \in I)$, we have:*

$$\bigcap_{i \in I}(A_i \rightarrow B_i) \leq_\Sigma \bigcap_{i \in I} A_i \rightarrow \bigcap_{i \in I} B_i$$

We can devise semantic domains out of intersection types by means of an appropriate notion of filter over a type preorder. This is a particular case of filter over a generic meet semi-lattice (see [Joh86]).

Definition 7 (Σ-filters). *A Σ-filter (or a filter over $\mathbb{T}^\Sigma$) is a set $X \subseteq \mathbb{T}^\Sigma$ such that*

1. *$\Omega \in X$;*
2. *if $A \leq_\Sigma B$ and $A \in X$, then $B \in X$;*
3. *if $A, B \in X$, then $A \cap B \in X$.*

$\mathcal{F}^\Sigma$ denotes the set of Σ-filters.

Given $X \subseteq \mathbb{T}^\Sigma$, $\uparrow X$ denotes the Σ-filter generated by X. For $A \in \mathbb{T}^\Sigma$, we write $\uparrow A$ instead of $\uparrow \{A\}$.

Proposition 4. *The set of Σ-filters $\mathcal{F}^\Sigma$, ordered by subset inclusion, is an ω-algebraic complete lattice, where $\uparrow \Omega$ is the bottom, and $\mathbb{T}^\Sigma$ is the top. Moreover if $X, Y \in \mathcal{F}^\Sigma$:*

$$X \sqcup Y = \uparrow (X \cup Y);$$
$$X \sqcap Y = X \cap Y.$$

If $\chi \subseteq \mathcal{F}^\Sigma$ is a directed set, then $\bigsqcup \chi = \bigcup \chi$.
The finite elements are exactly the principal filters.

It is possible to turn the space of filters into a natural λ-structure.

Definition 8 (Filter structures).

1. *Application* $\_\cdot\_ : \mathcal{F}^\Sigma \times \mathcal{F}^\Sigma \to \mathcal{F}^\Sigma$ *is defined as*

$$X \cdot Y = \uparrow\{B \mid \exists A \in Y. A \to B \in X\}.$$

2. *The maps* $F^\Sigma : \mathcal{F}^\Sigma \to [\mathcal{F}^\Sigma \to \mathcal{F}^\Sigma]$ *and* $G^\Sigma : [\mathcal{F}^\Sigma \to \mathcal{F}^\Sigma] \to \mathcal{F}^\Sigma$ *are defined by:*

$$\begin{array}{ll} F^\Sigma(X) = \lambda Y \in \mathcal{F}^\Sigma . X \cdot Y; \\ G^\Sigma(f) \ = \uparrow \{A \to B \mid B \in f(\uparrow A)\} \end{array}$$

The triple $\langle \mathcal{F}^\Sigma, F^\Sigma, G^\Sigma \rangle$ *is called the* filter structure *induced by* Σ.

We now give a simple proposition whose results will be useful later on.

Proposition 5. *1. Each* $f \in [\mathcal{F}^\Sigma \to \mathcal{F}^\Sigma]$ *satisfies*

$$B \in f(\uparrow A) \iff \uparrow A \Rightarrow \uparrow B \sqsubseteq f$$

and

$$f = \bigsqcup\{\uparrow A \Rightarrow \uparrow B \mid B \in f(\uparrow A)\}.$$

2. For all $A, B \in \mathbb{T}^\Sigma$,

$$B \in X \cdot \uparrow A \textit{ iff } A \to B \in X.$$

Proof. (1) Immediate by Proposition 1(2), taking into account that $\uparrow A \Rightarrow \uparrow B$ are all and only the step functions in $[\mathcal{F}^\Sigma \to \mathcal{F}^\Sigma]$.
(2) ($\Rightarrow$) If $B \sim_\Sigma \Omega$ then $\Omega \to \Omega \leq_\Sigma A \to B$ by rule (η). So $A \to B \in X$ by definition of Σ-filter (Definition 7). Otherwise by definition of application (Definition 8(1)) $B \in X \cdot \uparrow A$ iff $B \in \uparrow \{D \mid \exists C \in \uparrow A.\ C \to D \in X\}$. Then there is I and types C_i, D_i such that $A \leq_\Sigma \bigcap_{i\in I} C_i$, $\bigcap_{i \in I} D_i \leq_\Sigma B$ and $C_i \to D_i \in X$ for all $i \in I$ by definition of Σ-filter (Definition 7). So we get $A \to B \in X$ by axiom ($\to$-$\cap$) and rule (η).
($\Leftarrow$) Trivial.

Arrow types allow to describe the functional behaviour of filters, as shown in the next proposition which relates them with step functions, F^Σ and G^Σ.

Proposition 6.

1. *For all* $X \in \mathcal{F}^\Sigma$ *we get* $F^\Sigma(X) = \bigsqcup\{\uparrow A \Rightarrow \uparrow B \mid A \to B \in X\}$.
2. *For all* $A, B \in \mathbb{T}^\Sigma$ *we get* $G^\Sigma(\uparrow A \Rightarrow \uparrow B) = \uparrow (A \to B)$.

Proof. (1) Let $\Xi = \bigsqcup\{\uparrow A \Rightarrow \uparrow B \mid A \to B \in X\}$. It suffices to show

$$D \in \Xi(\uparrow C) \ \Leftrightarrow \ D \in F^\Sigma(X)(\uparrow C).$$

We first prove ($\supseteq$). If $D \in F^\Sigma(X)(\uparrow C)$, then, by Proposition 5(2), it follows $C \to D \in X$. From this fact and $D \in (\uparrow C \Rightarrow \uparrow D)(\uparrow C)$, a fortiori we get immediately $D \in \Xi(\uparrow C)$.

$(\subseteq)$. If $D \in \Xi(\uparrow C)$, then, by definition of step function, we get $\uparrow C \Rightarrow \uparrow D \sqsubseteq \Xi$. By compactness $\uparrow C \Rightarrow \uparrow D$ and Proposition 1(1), there exist I finite set and $A_i, B_i \in \mathbb{T}^{\Sigma}$, such that $\forall i \in I$, $A_i \to B_i \in X$, $\bigsqcup_{i\in I} \uparrow A_i \subseteq \uparrow C$, and $\uparrow D \subseteq \bigsqcup_{i\in I} \uparrow B_i$. We rewrite the previous three statements using the fact that X is a filter and Proposition 4 as follows:
(a) $\bigcap_{i\in I}(A_i \to B_i) \in X$;
(b) $C \leq_{\Sigma} \bigcap_{i\in I} A_i$;
(c) $\bigcap_{i\in I} B_i \leq_{\Sigma} D$.
Using (η) and (b), (c) above, we get $\bigcap_{i\in I} A_i \to \bigcap_{i\in I} B_i \leq_{\Sigma} C \to D$. This last judgment, along with $(trans)$ and Lemma 1, implies $\bigcap_{i\in I}(A_i \to B_i) \leq_{\Sigma} C \to D$. By (a) above and the fact that X is a filter, we get $C \to D \in X$, hence $D \in F^{\Sigma}(X)(\uparrow C)$ by Proposition 5(2).
(2)

$$\begin{aligned} G^{\Sigma}(\uparrow A \Rightarrow \uparrow B) &= \uparrow \{C \to D \mid D \in (\uparrow A \Rightarrow \uparrow B)(\uparrow C)\} \text{ by definition of } G^{\Sigma} \\ &\supseteq \uparrow (A \to B). \end{aligned}$$

$$\begin{aligned} G^{\Sigma}(\uparrow A \Rightarrow \uparrow B) &= \uparrow \{C \to D \mid D \in (\uparrow A \Rightarrow \uparrow B)(\uparrow C)\} \text{ by definition of } G^{\Sigma} \\ &= \uparrow \{C \to D \mid C \leq_{\Sigma} A \text{ and } B \leq_{\Sigma} D\} \\ &\subseteq \uparrow \{C \to D \mid A \to B \leq_{\Sigma} C \to D\} \qquad \text{by rule } (\eta) \\ &= \uparrow (A \to B). \end{aligned}$$

3.1 Interpreting λ-terms in filter structures

Any filter structure $\mathcal{F}^{\Sigma}$, being endowed with the two mappings F^{Σ} and G^{Σ}, can be turned into a domain where to interpret λ-calculus by using the interpretation function $[\![\ ]\!]^{\mathcal{F}^{\Sigma}}$ as defined in Definition 3. In this subsection we will see how this interpretation can be built by means of a suitable *type assignemnt system*. The advantage of using type assignment systems consists in the possibility of calculating term interpretation in a finitary way, as filters of types that can be assigned to terms.

Definition 9 (Type assignment system). *The* intersection type assignment system *relative to the nitp Σ, notation $\lambda\cap^{\Sigma}$, is a formal system for deriving judgements of the form $\Gamma \vdash^{\Sigma} M : A$, where the* subject *$M$ is an untyped λ-term, the* predicate *A is in $\mathbb{T}^{\Sigma}$, and Γ is a Σ-basis. Its axioms and rules are the following:*

$$(Ax)\ \frac{(x{:}A) \in \Gamma}{\Gamma \vdash^{\Sigma} x{:}A} \qquad\qquad (Ax\text{-}\Omega)\ \Gamma \vdash^{\Sigma} M : \Omega$$

$$(\to \mathrm{I})\ \frac{\Gamma, x{:}A \vdash^{\Sigma} M : B}{\Gamma \vdash^{\Sigma} \lambda x.M : A \to B} \qquad\qquad (\to \mathrm{E})\ \frac{\Gamma \vdash^{\Sigma} M : A \to B \quad \Gamma \vdash^{\Sigma} N : A}{\Gamma \vdash^{\Sigma} MN : B}$$

$$(\cap \mathrm{I})\ \frac{\Gamma \vdash^{\Sigma} M : A \quad \Gamma \vdash^{\Sigma} M : B}{\Gamma \vdash^{\Sigma} M : A \cap B} \qquad\qquad (\leq)\ \frac{\Gamma \vdash^{\Sigma} M : A \quad A \leq_{\Sigma} B}{\Gamma \vdash^{\Sigma} M : B}$$

It is easy to verify that the following rules are admissible [2] :

$$(\leq \mathrm{L})\ \frac{\Gamma, x : A \vdash M : B \quad A' \leq_\Sigma A}{\Gamma, x : A' \vdash M : B}$$

$$(\mathrm{W})\ \frac{\Gamma \vdash M : B \quad x \notin \Gamma}{\Gamma, x : A \vdash M : B} \qquad (\mathrm{S})\ \frac{\Gamma, x : A \vdash M : B \quad x \notin FV(M)}{\Gamma \vdash M : B}$$

We continue with a standard Generation Theorem, which is necessary for proving the main result of this subsection.

Theorem 1 (Generation Theorem).

1. *Assume $A \not\sim_\Sigma \Omega$. Then $\Gamma \vdash^\Sigma x : A$ iff $(x : B) \in \Gamma$ and $B \leq_\Sigma A$ for some $B \in \mathbb{T}^\Sigma$.*
2. *$\Gamma \vdash^\Sigma MN : A$ iff $\Gamma \vdash^\Sigma M : B \to A$, and $\Gamma \vdash^\Sigma N : B$ for some $B \in \mathbb{T}^\Sigma$.*
3. *$\Gamma \vdash^\Sigma \lambda x.M : A$ iff $\Gamma, x{:}B_i \vdash^\Sigma M : C_i$ and $\bigcap_{i \in I}(B_i \to C_i) \leq_\Sigma A$, for some I and $B_i, C_i \in \mathbb{T}^\Sigma$.*

Proof. The proof of each $(\Leftarrow)$ is easy. So we only treat $(\Rightarrow)$.

(1) Easy by induction on derivations, since only the axioms (Ax), (Ax-Ω), and the rules $(\cap\mathrm{I})$, $(\leq)$ can be applied. Notice that the condition $A \not\sim_\Sigma \Omega$ implies that $\Gamma \vdash^\Sigma x : A$ cannot be obtained just using axiom (Ax-Ω).

(2) If $A \sim_\Sigma \Omega$ we can choose $B \sim_\Sigma \Omega$. Otherwise, the proof is by induction on derivations. The only interesting case is when $A \equiv A_1 \cap A_2$ and the last applied rule is $(\cap\mathrm{I})$:

$$(\cap\mathrm{I})\ \frac{\Gamma \vdash^\Sigma MN : A_1 \quad \Gamma \vdash^\Sigma MN : A_2}{\Gamma \vdash^\Sigma MN : A_1 \cap A_2}.$$

The condition $A \not\sim_\Sigma \Omega$ implies that we cannot have $A_1 \sim_\Sigma A_2 \sim_\Sigma \Omega$. We give the proof for $A_1 \not\sim_\Sigma \Omega$ and $A_2 \not\sim_\Sigma \Omega$, the other cases can be treated similarly. By induction there are B_1, B_2 such that

$$\begin{array}{l} \Gamma \vdash^\Sigma M : B_1 \to A_1,\ \Gamma \vdash^\Sigma N : B_1, \\ \Gamma \vdash^\Sigma M : B_2 \to A_2,\ \Gamma \vdash^\Sigma N : B_2. \end{array}$$

Then $\Gamma \vdash^\Sigma M : (B_1 \to A_1) \cap (B_2 \to A_2)$ and by rules (η), $(\to$-$\cap)$:

$$(B_1 \to A_1) \cap (B_2 \to A_2) \leq_\Sigma B_1 \cap B_2 \to A_1 \cap A_2 \leq_\Sigma B_1 \cap B_2 \to A.$$

We are done, since $\Gamma \vdash^\Sigma N : B_1 \cap B_2$ by rule $(\cap\mathrm{I})$.

(3) The proof is very similar to the proof of Point (2). It is again by induction on derivations and again the only interesting case is when the last applied rule is $(\cap\mathrm{I})$:

[2] Recall that a rule is *admissible* in a system if, for each instance of the rule, if its premises are derivable in the system then so is its conclusion.

$$(\cap\mathrm{I})\ \frac{\Gamma \vdash^{\Sigma} \lambda x.M : A_1 \quad \Gamma \vdash^{\Sigma} \lambda x.M : A_2}{\Gamma \vdash^{\Sigma} \lambda x.M : A_1 \cap A_2}.$$

By induction there are I, B_i, C_i, J, D_j, G_j such that

$$\begin{array}{l}\forall i \in I.\ \Gamma, x{:}B_i \vdash^{\Sigma} M : C_i, \forall j \in J.\ \Gamma, x{:}D_j \vdash^{\Sigma} M : G_j,\\ \bigcap_{i\in I}(B_i \to C_i) \leq_{\Sigma} A_1 \quad \& \quad \bigcap_{j\in J}(D_j \to G_j) \leq_{\Sigma} A_2.\end{array}$$

So we are done since $(\bigcap_{i\in I}(B_i \to C_i)) \cap (\bigcap_{j\in J}(D_j \to G_j)) \leq_{\Sigma} A$.

We are now in position for proving the main result of this subsection: in filter structures the interpretation of a term coincides with the set of types which are deducible for it.

Theorem 2. *Let $\langle \mathcal{F}^{\Sigma}, F^{\Sigma}, G^{\Sigma}\rangle$ be a filter structure. Then, for any λ-term M and environment $\rho : Var \to \mathcal{F}^{\Sigma}$,*

$$[\![M]\!]^{\Sigma}_{\rho} = \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho.\ \Gamma \vdash^{\Sigma} M : A\},$$

where $[\![\ ]\!]^{\Sigma}$ is the interpretation function $[\![\ ]\!]^{\mathcal{F}^{\Sigma}}$ and $\Gamma \models \rho$ iff $\rho(x : B) \in \Gamma$ implies $B \in \rho(x)$.

Proof. First notice that $\Gamma \models \rho$ and $\Gamma' \models \rho$ imply (by definitions of $\models$ and of filter) $\Gamma \uplus \Gamma' \models \rho$, where we use $\uplus$ to denote the union between bases defined by:

$$\begin{array}{rl}\Gamma_1 \uplus \Gamma_2 = & \{(x{:}\tau) \mid (x{:}\tau) \in \Gamma_1\ \&\ x \notin \Gamma_2\} \cup \\ & \{(x{:}\tau) \mid (x{:}\tau) \in \Gamma_2\ \&\ x \notin \Gamma_1\} \cup \\ & \{(x{:}\tau_1 \cap \tau_2) \mid (x{:}\tau_1) \in \Gamma_1\ \&\ (x{:}\tau_2) \in \Gamma_2\}.\end{array}$$

Moreover notice that by rules (W) and ($\leq$ L) if $\Gamma \vdash^{\Sigma} M : A$ then $\Gamma \uplus \Gamma' \vdash^{\Sigma} M : A$ for all Γ'. We can conclude that:

$(\heartsuit)\quad \Gamma \models \rho, \Gamma' \models \rho$, and $\Gamma \vdash^{\Sigma} M : A$ imply $\Gamma \uplus \Gamma' \models \rho$ and $\Gamma \uplus \Gamma' \vdash^{\Sigma} M : A$.

We prove now the thesis by induction on M.
If $M \equiv x$, then

$$\begin{array}{rl}[\![x]\!]^{\Sigma}_{\rho} & = \rho(x) \\ & = \{A \in \mathbb{T}^{\Sigma} \mid \exists B \in \rho(x).\ B \leq_{\Sigma} A\} \\ & = \{A \in \mathbb{T}^{\Sigma} \mid \exists B \in \rho(x).\ x : B \vdash^{\Sigma} x : A\} \text{ by Theorem 1(1)} \\ & = \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho.\ \Gamma \vdash^{\Sigma} x : A\}.\end{array}$$

If $M \equiv NL$, then

$$
\begin{aligned}
[\![NL]\!]^{\Sigma}_{\rho} &= [\![N]\!]^{\Sigma}_{\rho} \cdot [\![L]\!]^{\Sigma}_{\rho} \\
&= \uparrow \{C \in \mathbb{T}^{\Sigma} \mid \exists B \in [\![L]\!]^{\Sigma}_{\rho}.B \to C \in [\![N]\!]^{\Sigma}_{\rho}\} \\
&\quad\ \text{by definition of application} \\
&= \{A \in \mathbb{T}^{\Sigma} \mid \exists I, B_i, C_i.\ B_i \to C_i \in [\![N]\!]^{\Sigma}_{\rho}, B_i \in [\![L]\!]^{\Sigma}_{\rho}, \\
&\quad \textstyle\bigcap_{i \in I} C_i \leq_{\Sigma} A\} \\
&\quad\ \text{by definition of filter} \\
&= \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho, I, B_i, C_i.\ \Gamma \vdash^{\Sigma} N : B_i \to C_i, \\
&\quad \Gamma \vdash^{\Sigma} L : B_i, \textstyle\bigcap_{i \in I} C_i \leq_{\Sigma} A\} \\
&\quad \text{by induction and } (\heartsuit) \\
&= \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho.\ \Gamma \vdash^{\Sigma} NL : A\} \\
&\quad\ \text{by Theorem 1(2) and rule } (\leq).
\end{aligned}
$$

If $M \equiv \lambda x.N$, then

$$
\begin{aligned}
[\![\lambda x.N]\!]^{\Sigma}_{\rho} &= G^{\Sigma}(\boldsymbol{\lambda} X \in \mathcal{F}^{\Sigma}.[\![N]\!]^{\Sigma}_{\rho[x:=X]}) \\
&= \uparrow \{B \to C \in \mathbb{T}^{\Sigma} \mid C \in [\![N]\!]^{\Sigma}_{\rho[x:=\uparrow B]}\} \\
&\quad\ \text{by definition of } G^{\Sigma} \\
&= \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho, I, B_i, C_i.\ \Gamma, x : B_i \vdash^{\Sigma} N : C_i, \\
&\quad \textstyle\bigcap_{i \in I} (B_i \to C_i) \leq_{\Sigma} A\} \\
&\quad \text{by induction and } (\heartsuit) \\
&= \{A \in \mathbb{T}^{\Sigma} \mid \exists \Gamma \models \rho.\ \Gamma \vdash^{\Sigma} \lambda x.N : A\} \\
&\quad\ \text{by Theorem 1(3) and rule } (\leq).
\end{aligned}
$$

4 Isomorphism results

In this section we will see that nitps are closely related to natural λ-structures. On one hand, any nitp induces a filter structure which is a natural λ-structure. On the other hand, for any natural λ-structure $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$, it is possible to find a presentation of it by means of a nitp Σ, i.e. $\langle \mathcal{F}^{\Sigma}, F^{\Sigma}, G^{\Sigma} \rangle$ and $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ are isomorphic as natural λ-structures. This kind of presentation is not always given by means of a recursive set of axioms and rules, but it will be so in the case of D_{∞} models as shown in the final section of the paper.

The correspondence between nitps and natural λ-structures can be refined in a categorical setting, showing that both natural λ-structures and nitps are objects of suitable categories, which turn out to be equivalent. In the present paper we give instead a direct proof.

We begin the present section by showing the first (easy) isomorphism result.

Theorem 3 (Isomorphism I). *Each $\langle \mathcal{F}^{\Sigma}, F^{\Sigma}, G^{\Sigma} \rangle$ is a natural λ-structure.*

Proof. We have just to prove that F^{Σ} and G^{Σ} set up a Galois connection, that is

$$
\begin{aligned}
F^{\Sigma} \circ G^{\Sigma} &\sqsupseteq Id_{[\mathcal{F}^{\Sigma} \to \mathcal{F}^{\Sigma}]} \\
G^{\Sigma} \circ F^{\Sigma} &\sqsubseteq Id_{\mathcal{F}^{\Sigma}}.
\end{aligned}
$$

The first inequality is given by:

$$\begin{aligned}
F^{\Sigma}(G^{\Sigma}(f)) &= \bigsqcup\{\uparrow A \Rightarrow \uparrow B \mid A \to B \in G^{\Sigma}(f)\} && \text{by Proposition 6(1)}\\
&\sqsupseteq \bigsqcup\{\uparrow A \Rightarrow \uparrow B \mid B \in f(\uparrow A)\} && \text{by definition of } G^{\Sigma}\\
&= f && \text{by Proposition 1(2).}
\end{aligned}$$

For the second inequality we get

$$\begin{aligned}
G^{\Sigma}(F^{\Sigma}(X)) &= \uparrow\{A \to B \mid B \in F^{\Sigma}(X)(\uparrow A)\} && \text{by definition of } G^{\Sigma}\\
&= \uparrow\{A \to B \mid A \to B \in X\} && \text{by Proposition 5(2)}\\
&\subseteq X.
\end{aligned}$$

In the remaining of the present subsection we will prove the vice versa, i.e. that each natural λ-structure can be generated by a suitable nitp.

To each λ-structure $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}}\rangle$ we associate a nitp $\Sigma^{\mathbb{D}}$. The preorder relation on types takes into account both the partial order between compact elements of $\mathcal{D}$ and the mapping $G_{\mathcal{D}}$.

Definition 10. *Let $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}}\rangle$ be a λ-structure. We define:*

1. $\mathbb{C}^{\mathbb{D}} = \{\psi_c \mid c \in \mathcal{K}(\mathcal{D})\}$*, where $\psi_\perp$ is Ω and ψ_c is a fresh constant for each other $c \in \mathcal{K}(\mathcal{D})$;*
2. $\leq_{\mathbb{D}} \subseteq \mathbb{T}^{\mathbb{D}} \times \mathbb{T}^{\mathbb{D}}$ *as the preorder relation generated by adding to ∇^0:*

$$\begin{aligned}
\mathbb{D} = {} &\{\psi_c \leq_{\mathbb{D}} \psi_d \mid d \sqsubseteq c\} \cup \{\psi_c \cap \psi_d \leq_{\mathbb{D}} \psi_e \mid e = c \sqcup d\}\\
&\cup \{\psi_c \to \psi_d \sim_{\mathbb{D}} \psi_e \mid e = G_{\mathcal{D}}(c \Rightarrow d)\}
\end{aligned}$$

   where $\psi_c, \psi_d, \psi_e \in \mathbb{C}^{\mathbb{D}}$;
3. $\Sigma^{\mathbb{D}} = \langle \mathbb{C}^{\mathbb{D}}, \leq_{\mathbb{D}}\rangle$.

The nitp $\Sigma^{\mathbb{D}}$ enjoys some useful properties.

Proposition 7. *1. For all $A \in \mathbb{T}^{\mathbb{D}}$ there is $c \in \mathcal{K}(\mathcal{D})$ such that $A \sim_{\mathbb{D}} \psi_c$.*
2. For all $\psi_c, \psi_d \in \mathbb{C}^{\mathbb{D}}$: $\psi_c \leq_{\mathbb{D}} \psi_d$ iff $d \sqsubseteq c$;
3. For all $\psi_c, \psi_d, \psi_e \in \mathbb{C}^{\mathbb{D}}$: $\psi_e \leq_{\mathbb{D}} \psi_c \to \psi_d$ iff $G_{\mathcal{D}}(c \Rightarrow d) \sqsubseteq e$.

Proof. (1) By induction on A. Let $B \sim_{\mathbb{D}} \psi_b$ and $C \sim_{\mathbb{D}} \psi_c$.
If $A \equiv B \cap C$ then $A \sim_{\mathbb{D}} \psi_{b \sqcup c}$.
If $A \equiv B \to C$ then $A \sim_{\mathbb{D}} \psi_d$ where $d = G_{\mathcal{D}}(b \Rightarrow c)$.
For (2) define $\mathsf{pp} : \mathbb{T}^{\mathbb{D}} \to \mathcal{K}(\mathcal{D})$ by:

$$\begin{aligned}
\mathsf{pp}(\psi_c) &= c;\\
\mathsf{pp}(A \cap B) &= \mathsf{pp}(A) \sqcup \mathsf{pp}(B);\\
\mathsf{pp}(A \to B) &= G_{\mathcal{D}}(\mathsf{pp}(A) \Rightarrow \mathsf{pp}(B)).
\end{aligned}$$

It is easy to verify by induction on $\leq_{\mathbb{D}}$ that $A \leq_{\mathbb{D}} B$ implies $\mathsf{pp}(B) \sqsubseteq \mathsf{pp}(A)$. This yields $\psi_c \leq_{\mathbb{D}} \psi_d \Rightarrow d \sqsubseteq c$. The other implication is immediate by definition of $\mathbb{D}$.

(3) follows from (2) since $\psi_c \to \psi_d \sim_{\mathbb{D}} \psi_{G_{\mathcal{D}}(c \Rightarrow d)}$.

Notice that the first two points of the above proposition imply that for each type A in $\mathbb{T}^{\mathbb{D}}$ there is exactly one compact element c in $\mathcal{D}$ such that $A \sim_{\mathbb{D}} \psi_c$.

We define now a lattice isomorphism between the set $\mathcal{F}^{\mathbb{D}}$ of DD-filters over $\mathbb{T}^{\mathbb{D}}$ and $\mathcal{D}$.

Definition 11. *The mapping* $\mathsf{m} : \mathcal{F}^{\mathbb{D}} \to \mathcal{D}$ *is defined by*

$$\mathsf{m}(X) = \bigsqcup_{\psi_c \in X} c.$$

It is not difficult to verify that $\mathsf{m}(\uparrow \psi_c) = c$ and that m is a lattice isomorphism between $\mathcal{F}^{\mathbb{D}}$ and $\mathcal{D}$.

We show that m commutes with application.

Lemma 2. $\mathsf{m}(X \cdot Y) = \mathsf{m}(X) \cdot \mathsf{m}(Y)$.

Proof. By the continuity of m and of application we need to consider only finite elements in $\mathcal{F}^{\mathbb{D}}$, i.e. using also Proposition 7(1) we only need to show:

$$\mathsf{m}(\uparrow \psi_c \cdot \uparrow \psi_d) = \mathsf{m}(\uparrow \psi_c) \cdot \mathsf{m}(\uparrow \psi_d).$$

First notice that

$$\begin{array}{lll} \psi_c \leq_{\mathbb{D}} \psi_d \to \psi_b & \Leftrightarrow G_{\mathcal{D}}(d \Rightarrow b) \sqsubseteq c & \text{by Proposition 7(3)} \\ & \Leftrightarrow d \Rightarrow b \sqsubseteq F_{\mathcal{D}}(c) & \text{by condition (1) of Definition 2} \\ & \Leftrightarrow b \sqsubseteq F_{\mathcal{D}}(c)(d) & \text{by definition of step function} \\ & \Leftrightarrow b \sqsubseteq c \cdot d & \text{by definition of application.} \end{array}$$

We get (using three times rule (η))

$$\begin{array}{ll} \mathsf{m}(\uparrow \psi_c \cdot \uparrow \psi_d) & = \mathsf{m}(\uparrow \{A \mid \psi_c \leq_{\mathbb{D}} \psi_d \to A\}) \\ & \quad \text{by definition of application} \\ & = \mathsf{m}(\uparrow \{\psi_b \mid b \in \mathcal{K}(\mathcal{D}) \text{ and } \psi_c \leq_{\mathbb{D}} \psi_d \to \psi_b\}) \\ & \quad \text{by Proposition 7(1)} \\ & = \bigsqcup\{b \in \mathcal{K}(\mathcal{D}) \mid \psi_c \leq_{\mathbb{D}} \psi_d \to \psi_b\} \\ & \quad \text{by definition of } \mathsf{m} \\ & = \bigsqcup\{b \in \mathcal{K}(\mathcal{D}) \mid b \sqsubseteq c \cdot d\} \\ & \quad \text{by above} \\ & = c \cdot d \\ & = \mathsf{m}(\uparrow \psi_c) \cdot \mathsf{m}(\uparrow \psi_d). \end{array}$$

Finally we can give the second isomorphism result, whose proof follows immediately from the previous lemma and Proposition 3.

Theorem 4 (Isomorphism II). *Let* $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ *be a natural* λ*-structure, then the nitp* $\Sigma^{\mathbb{D}}$ *of Definition 10 is such that* $\langle \mathcal{D}, F_{\mathcal{D}}, G_{\mathcal{D}} \rangle$ *and* $\langle \mathcal{F}^{\mathbb{D}}, F^{\mathbb{D}}, G^{\mathbb{D}} \rangle$ *are isomorphic.*

5 D_∞-models and filter models

Since all ω-algebraic complete lattices which are extensional λ-models are clearly natural λ-structures, Theorem 4 implies that any such λ-model is isomorphic to a filter λ-model. However the finitary logical description provided by the proof of Theorem 4 is rather opaque. In this section we show that in the special case of D_∞ inverse limit models, one can obtain far more concise type theoretic descriptions. Remarkably the nitp which induces a filter model isomorphic to D_∞, starting from D_0, is exactly the nitp *freely generated* by a nitp which induces D_0 together with the equalities which arise from encoding the initial projections.

First of all we fix some notations and recall the standard D_∞ construction.

Definition 12. *1. Let D_0 be an ω-algebraic complete lattice and*

$$\langle \mathsf{i}_0, \mathsf{j}_0 \rangle : D_0 \to [D_0 \to D_0]$$

be an embedding-projection *pair, i.e.* $\mathsf{i}_0 : D_0 \to [D_0 \to D_0]$ *and* $\mathsf{j}_0 : [D_0 \to D_0] \to D_0$ *satisfy* $\mathsf{i}_0 \circ \mathsf{j}_0 \sqsubseteq Id_{[D_0 \to D_0]}$ *and* $\mathsf{j}_0 \circ \mathsf{i}_0 = Id_{D_0}$.

2. Define a tower $\langle \mathsf{i}_n, \mathsf{j}_n \rangle : D_n \to D_{n+1}$ *in the following way:*

- $D_{n+1} = [D_n \to D_n]$;
- $\mathsf{i}_n(f) = \mathsf{i}_{n-1} \circ f \circ \mathsf{j}_{n-1}$ *for any* $f \in D_n$;
- $\mathsf{j}_n(g) = \mathsf{j}_{n-1} \circ g \circ \mathsf{i}_{n-1}$ *for any* $g \in D_{n+1}$.

3. The set D_∞ is defined by

$$D_\infty = \{\langle d_n \rangle \mid \forall n.\ d_n \in D_n \text{ and } \mathsf{j}_n(d_{n+1}) = d_n\},$$

where $\langle d_n \rangle$ is short for $\langle d_n \rangle_{n \in N}$.

4. The ordering on D_∞ is given by

$$\langle d_n \rangle \sqsubseteq \langle e_n \rangle \Leftrightarrow \forall k.\ d_k \sqsubseteq e_k.$$

5. Let $\langle \Phi_{m\infty}, \Phi_{\infty m} \rangle$ denotes the standard embedding-projection pair from D_m to D_∞: for any $d \in D_m$, $\langle d_n \rangle \in D_\infty$,
$\Phi_{m\infty}(d) = \langle \ldots \mathsf{j}_{m-2}(\mathsf{j}_{m-1}(d)), \mathsf{j}_{m-1}(d), d, \mathsf{i}_m(d), \mathsf{i}_{m+1}(\mathsf{i}_m(d)) \ldots \rangle$,
$\Phi_{\infty m}(\langle d_n \rangle) = d_m$.

6. Let $\Phi_{mn} : D_m \to D_n$ be $\Phi_{\infty n} \circ \Phi_{m\infty}$.

7. Let $F_\infty : D_\infty \to [D_\infty \to D_\infty]$ be defined by

$$F_\infty(\langle d_n \rangle)(\langle e_n \rangle) = \textstyle\bigsqcup_{n \in N} \Phi_{n\infty}(d_{n+1}(e_n)),$$

and $G_\infty : [D_\infty \to D_\infty] \to D_\infty$ by

$$G_\infty(f) = \textstyle\bigsqcup_{n \in N} \Phi_{(n+1)\infty}(\Phi_{\infty n} \circ f \circ \Phi_{n\infty}).$$

Remark 1. From previous definition it follows easily that, if $n \leq p \leq k$ and $d \in D_n$, $e \in D_p$, then $\Phi_{np}(d) \sqsubseteq e$ iff $\Phi_{nk}(d) \sqsubseteq \Phi_{pk}(e)$ iff $\Phi_{n\infty}(d) \sqsubseteq \Phi_{p\infty}(e)$.

Theorem 5. *([Sco72])* $\langle D_\infty, F_\infty, G_\infty\rangle$ *is a* λ*-model.*

Next definition exhibits nitps which induce filter λ-models isomorphic to D_∞ models. Notice the similarities with Definition 10. In particular, the equivalences between arrow types and constants are built in both cases by considering the action of the compact element preserving map ($G_{\mathcal{D}}$ in the case of Definition 10, i_0 here). A difference with respect to Definition 10 is that we are forced to define such equivalences by means of intersections and sups. The reason for this is that we do not have a constant for each compact function, which could lead to an apparently smoother definition such as in the case of Definition 10 (which actually yields a lot of redundant types), but rather we represent a compact function as a sup of suitable step functions. Dually, in the nitp, the compact function will be represented by the intersection of the arrow types which correspond to the involved step functions.

Definition 13. *Define:*

1. $\mathbb{C}^\infty = \{\psi_c \mid c \in \mathcal{K}(D_0)\}$, *where* $\psi_\perp$ *is* Ω *and* ψ_c *is a fresh constant for each other* $c \in \mathcal{K}(D_0)$;
2. $\leq_\infty$ *as the preorder relation generated by adding to* ∇^0:

$$\begin{aligned}\infty = \{\psi_c \leq_\infty \psi_d \mid d \sqsubseteq c\} \cup \{\psi_c \cap \psi_d \sim_\infty \psi_e \mid e = c \sqcup d\} \\ \cup \{\textstyle\bigcap_{j\in J}(\psi_{c_j} \to \psi_{d_j}) \sim_\infty \psi_d \mid \mathsf{i}_0(d) = \bigsqcup_{j\in J}(c_j \Rightarrow d_j)\}\end{aligned}$$

   where $\psi_c, \psi_d, \psi_e, c_j, d_j \in \mathbb{C}^\infty$;
3. $\Sigma^\infty = \langle \mathbb{C}^\infty, \leq_\infty\rangle$.

The nitp Σ^∞ enjoys some useful properties.

Lemma 3. *1.* $\bigcap_{i\in I}\psi_{c_i} \sim_\infty \psi_{\bigsqcup_{i\in I} c_i}$.

2. $\bigcap_{i\in I}(C_i \to D_i) \leq_\infty A \to B$ *implies* $\bigcap_{i\in J} D_i \leq_\infty B$ *where* $J = \{i \in I \mid A \leq_\infty C_i\}$.
3. $\uparrow\bigcap_{i\in I}(C_i \to D_i)\cdot\uparrow A = \uparrow\bigcap_{i\in J} D_i$ *where* $J = \{i \in I \mid A \leq_\infty C_i\}$.

Proof. (1) follows easily from Definition 13.

For (2) notice that by definition for each constant $\alpha \in \mathbb{C}^\infty$ there is exactly one judgement of the shape $\alpha \sim_\infty \bigcap_{l\in L^{(\psi_d)}}(\gamma_l^{(\alpha)} \to \delta_l^{(\alpha)})$, where $\gamma_l^{(\alpha)}, \delta_l^{(\alpha)} \in \mathbb{C}^\infty$.

We can prove by simultaneous induction on the definition of $\leq_\infty$ two statements, the first of which implies the thesis.

– if $(\bigcap_{i\in I}(A_i \to B_i)) \cap (\bigcap_{h\in H}\alpha_h) \leq_\infty (\bigcap_{j\in J}(C_j \to D_j)) \cap (\bigcap_{k\in K}\beta_k)$, then for each $j \in J$: $(\bigcap_{i\in I'} B_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\alpha_h)'}}\delta_l^{(\alpha_h)})) \leq_\infty D_j$ where $I' = \{i \in I \mid C_j \leq_\infty A_i\}$, $H' = \{h \in H \mid \exists l \in L^{(\alpha_h)} C_j \leq_\infty \gamma_l^{(\alpha_h)}\}$, and $L^{(\alpha_h)'} = \{l \in L^{(\alpha_h)} \mid C_j \leq_\infty \gamma_l^{(\alpha_h)}\}$;

– if $(\bigcap_{i\in I}(A_i \to B_i)) \cap (\bigcap_{h\in H} \alpha_h) \leq_\infty (\bigcap_{j\in J}(C_j \to D_j)) \cap (\bigcap_{k\in K} \beta_k)$, then for each $k \in K, m \in L^{(\beta_k)}$ $(\bigcap_{i\in I'} B_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\alpha_h)'}} \delta_l^{(\alpha_h)})) \leq_\infty \delta_m^{(\beta_k)}$ where $I' = \{i \in I \mid \gamma_m^{(\beta_k)} \leq_\infty A_i\}$, $H' = \{h \in H \mid \exists l \in L^{(\alpha_h)} \gamma_m^{(\beta_k)} \leq_\infty \gamma_l^{(\alpha_h)}\}$, and $L^{(\alpha_h)'} = \{l \in L^{(\alpha_h)} \mid \gamma_m^{(\beta_k)} \leq_\infty \gamma_l^{(\alpha_h)}\}$.

For (3) the inclusion $\subseteq$ follows immediately from the definition of filter application. We show the reverse inclusion.

$$
\begin{aligned}
B \in \uparrow \textstyle\bigcap_{i\in I}(C_i \to D_i)\cdot \uparrow A &\Rightarrow A \to B \in \uparrow \textstyle\bigcap_{i\in I}(C_i \to D_i) \\
&\quad \text{by Proposition 5(2)} \\
&\Rightarrow \textstyle\bigcap_{i\in I}(C_i \to D_i \leq_\infty A \to B \\
&\quad \text{by definition of filter} \\
&\Rightarrow \textstyle\bigcap_{i\in J} D_i \leq_\infty B \text{ where } J = \{i \in I \mid A \leq_\infty C_i\} \\
&\quad \text{by (2).}
\end{aligned}
$$

The proof of the isomorphism will be postponed because several preliminary results are needed. These are the subjects of Lemmata 4, 5 and 6.

First we classify the types in $\mathbb{T}^\infty$ according to the maximal number of nested arrow occurrences they may contain.

Definition 14. *1. We define the map* rank $rk : \mathbb{T}^\infty \to N$ *by:*

$$
\begin{aligned}
rk(\psi_c) &= 0; \\
rk(A \to B) &= max\{rk(A), rk(B)\} + 1; \\
rk(A \cap B) &= max\{rk(A), rk(B)\}.
\end{aligned}
$$

2. Let $\mathbb{T}_n^\infty = \{A \in \mathbb{T}^\infty \mid rk(A) \leq n\}$.

We can associate to each type in $\mathbb{T}_n^\infty$ an element in D_n: this will be crucial for defining the mapping which gives the desidered isomorphism (see Definition 16).

Definition 15. *We define, for each* $n \in N$, *a map* $w_n : \mathbb{T}_n^\infty \to D_n$ *by a double induction on* n *and on the construction of types in* $\mathbb{T}^\infty$*:*

$$
\begin{aligned}
w_n(\psi_c) &= \Phi_{0n}(c); \\
w_n(A \cap B) &= w_n(A) \sqcup w_n(B); \\
w_n(A \to B) &= w_{n-1}(A) \Rightarrow w_{n-1}(B).
\end{aligned}
$$

The following property of w_n shows that no information is lost if we map a type into any D_n with n greater than the rank of the type.

Lemma 4. *For all* $A \in \mathbb{T}_n^\infty$ *and for all* $m, p \geq n$ *we have* $\Phi_{m\infty}(w_m(A)) = \Phi_{p\infty}(w_p(A))$.

Proof. We show by induction on the definition of w_n that $w_{n+1}(A) = \mathsf{i}_n(w_n(A))$. Then the desired equality follows from the definition of the function Φ. The only interesting case is when $A \equiv B \to C$. We get

$$\begin{array}{rll} w_{n+1}(B \to C) &= w_n(B) \Rightarrow w_n(C) & \text{by definition} \\ &= \mathsf{i}_{n-1}(w_{n-1}(B)) \Rightarrow \mathsf{i}_{n-1}(w_{n-1}(C)) & \text{by induction} \\ &= \mathsf{i}_n(w_{n-1}(B) \Rightarrow w_{n-1}(C)) & \text{by definition of } \mathsf{i}_n \\ && \text{and of step function} \\ &= \mathsf{i}_n(w_n(B \to C)) & \text{by Definition 15.} \end{array}$$

The maps w_n reverse the order between types, as shown in the following lemma.

Lemma 5. *Let $n \geq rk(A \cap B)$. Then $A \leq_\infty B$ implies $w_n(B) \sqsubseteq w_n(A)$.*

Proof. The proof is by induction on the definition of $\leq_\infty$. We consider just the case of rule (η). Let $A \equiv C \to D$, $B \equiv E \to F$, with $E \leq_\infty C$, $D \leq_\infty F$. Then by induction $w_n(C) \sqsubseteq w_n(E)$ and $w_n(F) \sqsubseteq w_n(D)$, hence $w_n(E) \Rightarrow w_n(F) \sqsubseteq w_n(C) \Rightarrow w_n(D)$. Thus we get, by definition of w_n, $w_{n+1}(B) \sqsubseteq w_{n+1}(A)$, hence, by Lemma 4, $\mathsf{i}_n(w_n(B)) \sqsubseteq \mathsf{i}_n(w_n(A))$. By Remark 1 (since $\mathsf{i}_n = \Phi_{n(n+1)}$) the thesis follows.

Also the reverse implication of Lemma 5 holds.

Lemma 6. *Let $rk(A \cap B) \leq n$. Then $w_n(B) \sqsubseteq w_n(A)$ implies $A \leq_\infty B$.*

Proof. The proof is by induction on $rk(A \cap B)$.
If $rk(A \cap B) = 0$ we have $A \equiv \bigcap_{i \in I} \psi_{c_i}$, $B = \bigcap_{j \in J} \psi_{d_j}$. Then $w_n(B) \sqsubseteq w_n(A)$ implies $\bigsqcup_{j \in J} \Phi_{0n}(d_j) \sqsubseteq \bigsqcup_{i \in I} \Phi_{0n}(c_i)$, that is, by Remark 1, $\bigsqcup_{j \in J} d_j \sqsubseteq \bigsqcup_{i \in I} c_i$. By Definition 13 and Lemma 3(1) it follows $\bigcap_{i \in I} \psi_{c_i} \leq_\infty \bigcap_{j \in J} \psi_{d_j}$, hence $A \leq_\infty B$.
Otherwise, let

$$A \equiv (\bigcap_{i \in I} \psi_{c_i}) \cap (\bigcap_{l \in L} (C_l \to D_l)), B \equiv (\bigcap_{h \in H} \psi_{d_h}) \cap (\bigcap_{m \in M} (E_m \to F_m))$$

where $\psi_{c_i} \sim_\infty \bigcap_{j \in J_i} (\psi_{a_j} \to \psi_{b_j})$, $\psi_{d_h} \sim_\infty \bigcap_{k \in K_h} (\psi_{e_k} \to \psi_{f_k})$. The last two equivalences imply by Lemma 5 that for all $n \geq 1$

$$w_n(\psi_{c_i}) = w_n(\bigsqcup_{j \in J_i} (\psi_{a_j} \Rightarrow \psi_{b_j})), w_n(\psi_{d_h}) = w_n(\bigsqcup_{k \in K_h} (\psi_{e_k} \Rightarrow \psi_{f_k})).$$

So we get

$$\begin{array}{l} \bigsqcup_{h \in H} (\bigsqcup_{k \in K_h} w_n(\psi_{e_k}) \Rightarrow w_n(\psi_{f_k})) \sqcup (\bigsqcup_{m \in M} w_n(E_m) \Rightarrow w_n(F_m)) \sqsubseteq \\ \bigsqcup_{i \in I} (\bigsqcup_{j \in J_i} w_n(\psi_{a_j}) \Rightarrow w_n(\psi_{b_j})) \sqcup (\bigsqcup_{l \in L} w_n(C_l) \Rightarrow w_n(D_l)). \end{array}$$

Now by definition of step function this implies that for each $h \in H$, $k \in K_h$,

$$w_n(\psi_{f_k}) \sqsubseteq \bigsqcup_{i \in I'} (\bigsqcup_{j \in J'_i} w_n(\psi_{b_j})) \sqcup (\bigsqcup_{l \in L'} w_n(D_l))$$

where $I' = \{i \in I \mid \exists j \in J_i \;\&\; w_n(\psi_{a_j}) \sqsubseteq w_n(\psi_{e_k})\}$,
$J'_i = \{j \in J_i \mid w_n(\psi_{a_j}) \sqsubseteq w_n(\psi_{e_k})\}$, $L' = \{l \in L \mid w_n(C_l) \sqsubseteq w_n(\psi_{e_k})\}$.
Since all types involved in the two above judgments have ranks strictly less than $rk(A \cap B)$, by induction and by Lemma 3 we obtain

$$\begin{array}{l} \psi_{e_k} \leq_\infty \bigcap_{i \in I'} (\bigcap_{j \in J'_i} \psi_{a_j}) \cap \bigcap_{l \in L'} C_l, \\ \bigcap_{i \in I'} (\bigcap_{j \in J'_i} \psi_{b_j}) \cap \bigcap_{l \in L'} D_l \leq_\infty \psi_{f_k}. \end{array}$$

Therefore we have $A \leq_\infty \psi_{e_k} \to \psi_{f_k}$ for each $h \in H$, $k \in K_h$. In a similar way we can prove that $A \leq_\infty E_m \to F_m$, for any $m \in M$. Putting together these results we get $A \leq_\infty B$.

We can now prove the isomorphism between $\langle \mathcal{D}_\infty, F_\infty, G_\infty \rangle$ and $\langle \mathcal{F}^\infty, F^\infty, G^\infty \rangle$. First we give the isomorphism map.

Definition 16. *Let* $\hat{\mathsf{m}}$ *be the unique continuous extension of the mapping* $\mathsf{m} : \mathcal{K}(\mathcal{F}^\infty) \to \mathcal{K}(D_\infty)$ *defined by*

$$\mathsf{m}(\uparrow A) = \Phi_{r\infty}(w_r(A)),$$

where $r = rk(A)$.

Notice that by Lemma 4 we have $\mathsf{m}(\uparrow A) = \Phi_{n\infty}(w_n(A))$ for all $n \geq rk(A)$. This will be freely used in the proof of Theorem 6.

We recall that (see [Sco72])

1. $F_\infty \circ G_\infty = Id_{[\mathcal{D}_\infty \to \mathcal{D}_\infty]}$;
2. $G_\infty \circ F_\infty = Id_{\mathcal{D}_\infty}$.

On the other hand, $\langle \mathcal{F}^\infty, F^\infty, G^\infty \rangle$ is a natural λ-structure. So both $\langle \mathcal{D}_\infty, F_\infty, G_\infty \rangle$ and $\langle \mathcal{F}^\infty, F^\infty, G^\infty \rangle$ are natural λ-structures.

Theorem 6. *The natural λ-structures $\langle \mathcal{D}_\infty, F_\infty, G_\infty \rangle$ and $\langle \mathcal{F}^\infty, F^\infty, G^\infty \rangle$ are isomorphic.*

Proof. The mapping m is monotone and injective by Lemmas 5 and 6, hence $\hat{\mathsf{m}}$ is so. We prove surjection over D_∞ by induction on n, by showing that each w_n is surjective on D_n. Surjection of w_0 is obvious by definition of w_0 and of the nitp Σ^∞.
Let $f \in D_{n+1}$. By Proposition 1(2) there exist I and $a_i, b_i \in D_n$ such that $f = \bigsqcup_{i \in I}(a_i \Rightarrow b_i)$. By induction there exist types A_i, B_i such that for all $i \in I$, $w_n(A_i) = a_i$ and $w_n(B_i) = b_i$. Therefore

$$\begin{aligned} w_{n+1}(\bigcap_{i \in I}(A_i \to B_i)) &= \bigsqcup_{i \in I}(w_n(A_i) \Rightarrow w_n(B_i)) \\ &= \bigsqcup_{i \in I}(a_i \Rightarrow b_i) \\ &= f \end{aligned}$$

We have so proved that each w_n is surjective. This implies m is surjective onto compact elements of D_∞, hence $\hat{\mathsf{m}} : \mathcal{F}^\infty \to D_\infty$ is surjective.

From Lemma 6 it follows that m^{-1} is monotone, hence $\hat{\mathsf{m}}^{-1}$ is continuous (by definition). We have finally to prove that $\hat{\mathsf{m}}$ commutes with application. Since it is enough to prove the thesis on compact elements, that is on principal filters of $\mathcal{F}^\infty$, we are left to prove that for any $A, B \in \mathbb{T}^\infty$

$$\mathsf{m}(\uparrow A \cdot \uparrow B) = \mathsf{m}(\uparrow A) \cdot \mathsf{m}(\uparrow B).$$

Let $A, B \in \mathbb{T}^\infty$, $A \sim_\infty \bigcap_{i \in I}(C_i \to D_i)$, $J = \{i \in I \mid B \leq_\infty C_i\}$, and n any natural number greater than $rk(A \cap B)$. Then by the definition of $\Phi_{n\infty}$ and Lemmas 5, 6 we get

$$(\flat) J = \{i \in I \mid \Phi_{n\infty}(w_n(C_i)) \sqsubseteq \Phi_{n\infty}(w_n(B))\}.$$

$$\begin{aligned}
\mathsf{m}(\uparrow A \cdot \uparrow B) &= \mathsf{m}(\uparrow \textstyle\bigcap_{i \in J} D_i) \text{ by Lemma 3(3)} \\
&= \Phi_{n\infty}(w_n(\textstyle\bigcap_{i \in J} D_i)) \text{ by definition of } \mathsf{m} \\
&= \textstyle\bigsqcup_{i \in J} \Phi_{n\infty}(w_n(D_i)) \\
&\quad \text{by definition of } w_n \text{ and addittivity of } \Phi_{n\infty} \\
&= \textstyle\bigsqcup_{i \in I}(\Phi_{n\infty}(w_n(C_i)) \Rightarrow \Phi_{n\infty}(w_n(D_i))) \cdot \Phi_{n\infty}(w_n(B)) \\
&\quad \text{by definition of step function and } (\flat) \\
&= \textstyle\bigsqcup_{i \in I} \Phi_{(n+1)\infty}(w_n(C_i) \Rightarrow w_n(D_i)) \cdot \Phi_{n\infty}(w_n(B)) \\
&\quad \text{by definition of } \Phi_{n\infty} \\
&= \Phi_{(n+1)\infty}(w_{n+1}(\textstyle\bigcap_{i \in I}(C_i \to D_i))) \cdot \Phi_{n\infty}(w_n(B)) \\
&\quad \text{by definition of } w_n \\
&= \mathsf{m}(\uparrow A) \cdot \mathsf{m}(\uparrow B) \text{ by definition of } \mathsf{m}.
\end{aligned}$$

This completes the proof that $\langle \mathcal{D}_\infty, F_\infty, G_\infty \rangle$ and $\langle \mathcal{F}^\infty, F^\infty, G^\infty \rangle$ are isomorphic as natural λ-structures, hence as λ-models.

(ω-*Scott*)	$\Omega \to \omega \sim \omega$	(ω-*Park*)	$\omega \to \omega \sim \omega$
$(\omega\varphi)$	$\omega \leq_\Sigma \varphi$	$(\varphi \to \omega)$	$\varphi \to \omega \sim \omega$
$(\omega \to \varphi)$	$\omega \to \varphi \sim \varphi$	(I)	$(\varphi \to \varphi) \cap (\omega \to \omega) \sim \varphi$

Fig. 1. Possible Axioms and Rules concerning $\leq_\Sigma$.

Figure 1 lists axioms and rules used in Figure 2 to define nitps which induce filter models isomorphic to well known inverse limit models. We shall denote such theories as Σ^∇, with various different names ∇ corresponding to the initials of the authors who have first considered the λ-model induced by such a theory. For each such Σ^∇ we specify in Figure 2 the nitp $\Sigma^\nabla = (\mathbb{C}, \leq_\nabla)$ by giving the set of constants $\mathbb{C}^\nabla$ and the set ∇ of extra axioms and rules.

$\mathbb{C}^{\mathcal{S}c}$	$= \{\Omega, \omega\}$	$\mathcal{S}c$	$= \{(\omega\text{-}Scott)\}$
$\mathbb{C}^{\mathcal{P}a}$	$= \{\Omega, \omega\}$	$\mathcal{P}a$	$= \{(\omega\text{-}Park)\}$
$\mathbb{C}^{\mathcal{CDZ}}$	$= \{\Omega, \varphi, \omega\}$	$\mathcal{CDZ}$	$= \{(\omega\varphi), (\varphi \to \omega), (\omega \to \varphi)\}$
$\mathbb{C}^{\mathcal{HR}}$	$= \{\Omega, \varphi, \omega\}$	$\mathcal{HR}$	$= \{(\omega\varphi), (I), (\omega \to \varphi)\}$

Fig. 2. Type Theories: constants, axioms and rules.

As particular cases of Theorem 6 we get that Scott model as defined in [Sco72] is isomorphic to the filter λ-model induced by the nitp $\Sigma^{\mathcal{S}c}$ and Park model as defined in [Par76] is isomorphic to the filter λ-model induced by the nitp $\Sigma^{\mathcal{P}a}$.

The construction of Theorem 6 was first discussed in [CDCHL84]. Other relevant references are [CDCZ87], which presents the filter λ-model induced by the nitp $\Sigma^{\mathcal{CDZ}}$, [HRDR92], where the filter λ-models induced by the nitps $\Sigma^{\mathcal{P}a}$, $\Sigma^{\mathcal{HR}}$ and other models are considered, and [Ale91], [DGH93], [Plo93], where the relation between λ-structures and nitps is studied.

Results similar to Theorem 6 can be given also for other, non-extensional, inverse limit λ-models, obtained as solutions of domain equations involving also other functors. For instance one can consider the *lifted space of functions* $[\ \to\ ]_\perp$, the space of *strict functions* $[\ \to_\perp\ ]$, a *product* $[\ \to\ ] \times A$, or a *sum* $[\ \to\ ] + A$ with a set A of *atoms*, and so on. In all such cases one gets concise type theoretic descriptions of the λ-models obtained as fixed points of such functors corresponding to suitable choices of G [CDL83]. At least the following result is worthwhile mentioning in this respect, see [CDCHL84] for a proof. We define [BCDC83]

$$\mathbb{C}^{\mathcal{BCD}} = \{\Omega\} \cup \mathbb{C}_\infty \qquad \mathcal{BCD} = \{(\Omega\text{-}\eta)\}$$

where $\mathbb{C}_\infty$ is an infinite set of fresh (i.e. different from Ω, ϕ, ω) constants.

Proposition 8. *The filter λ-model induced by $\Sigma^{\mathcal{BCD}}$ is isomorphic to $\langle \mathcal{D}, F, G \rangle$, where $\mathcal{D}$ is the initial solution of the domain equation $[\mathcal{D} \to \mathcal{D}] \times \mathsf{P}(\mathbb{C}_\infty) \equiv \mathcal{D}$, the pair $\langle F, G \rangle$ set up a Galois connection and G is the map which picks always the minimal element in the extensionality classes of all functions.*

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