An irregular filter model

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Abstract

In this paper we introduce a new filter model, which is of a kind that escaped investigation up to now: it is induced by an intersection type theory generated in a non-standard way, by a preorder which puts into relation an atom with an arrow type, *without equating* them. We study the domain-theoretic implications of this choice, that are not trivial: in order to describe this filter model a new category is introduced and a special purpose functor defined. The filter model is then characterized as the initial algebra of the functor.

1 Introduction

Intersection type theories (ITT's) were introduced in the late 70's by Dezani and Coppo [14, 15, 12] to overcome the limitations of Curry's type discipline. They were used as a powerful tool for describing existing λ-models and synthesising new ones. The main features of ITT's are two:
- they give the possibility of describing in a finitary way the interpretation of λ-terms, through the Type Assignment Systems, see e.g. [12, 17, 21, 25, 20];
- they can be calibrated exactly for inducing λ-models which exhibit certain desired properties, see e.g. [2, 26, 7, 8, 18].

As concerns this second aspect, ITT's exploit the axiomatic presentation through which they are defined. Suppose we want to define an ITT $\mathcal{T}$ with certain properties: essentially the desired properties are kept into account since the very definition of $\mathcal{T}$, introducing "ad hoc" axioms and rules, modelled on the properties. The space of the filters over $\mathcal{T}$ allows to recover a domain with the desired properties, built as a colimit in **ALG** (the category of ω-algebraic lattices and Scott continuous functions) of the ω-chain

$$D_0 \xrightarrow{f_0} F(D_0) \xrightarrow{F(f_0)} F^2(D_0) \dots$$

generated by a suitable functor F. Therefore ITT's provide a rather simple (set-theoretic) way for building interesting domains, which can be otherwise recovered by using the complex category theoretic apparatus.

The other side of the coin is that the balance which guarantees the categorical description of ITT's is rather delicate, depending on a somewhat rigid scheme used for defining ITT's: among other steps, it requires to choose specific axioms of the shape $\alpha \sim B$ where α is an atom and B a suitable intersection of arrow types; the emphasis has to be put on the symmetry of "$\sim$", since $\alpha \sim B$ corresponds to the pair of axioms $\alpha \leq B$ and $\alpha \geq B$. This restriction allows to put at work the Stone duality paradigm of [1], first preconized in [33], and ITT's can be used, for instance, for giving presentation to λ-models built as reflexive objects in the category of ω-algebraic lattices, such as the Scott D_∞ [32] and the Park model [28].

The questions that we consider are the following. What happens if we break the symmetry of the specific axioms, by considering ITT's where atoms are just related, but not equated to any intersection of arrow types (that is, we have $\alpha \leq B$, or $B \leq \alpha$, but not $\alpha \sim B$)? Which is the categorical construction associated to these "irregular" ITT's?

Before going on with this discussion, an historical remark: irregular intersection type theories appeared just once in the literature (in [16]), in order to give an instance of λ-model where not all the continuous functions were representable. As far as the author knows, there are no other instances in the literature.

Coming back to the questions above, we have no general answer. What is shown in the paper is that breaking the simmetry, even in a seemingly simple case, brings heavy consequences on the domain-theoretic side, so confirming that the balance between ITT 's and their categorical description is difficult to keep.

In the paper we consider an irregular version of the Park model. As already mentioned in [16], the Park model is isomorphic to the filter model PA induced by the intersection type theory over the set of atoms $\{\Omega, \phi\}$, generated by the axiom

$$(\mathrm{pa}) \quad \phi \sim \phi \to \phi.$$

What we do is to take just one half of (pa), namely (1/2-pa) $\phi \leq \phi \to \phi$. Consequently, we call "1/2 Park filter model" (1/2-PA for short) the irregular λ-model induced by the intersection type theory generated by (1/2-pa).

The break of the simmetry has immediate consequence on the categorical side: as far as the author could check, the well-known property of PA of being a colimit in **ALG** for a ω-chain generated by the functor $F(X) = [X \to X]$, does not hold anymore for 1/2-PA, nor the author succeeded in finding an alternative satisfactory characterization in **ALG**.

In the paper we show how it is possible to recover a good categorical characterization of 1/2-PA.

The main part of the paper comes when introducing the "pointed" version of **ALG**, named $\mathbf{ALG}_*$, whose objects are pairs $\langle D, \delta \rangle$, where δ is a compact element in D, and tailor the special functor $\mathcal{G}_*$ which captures the domain-theoretic aspects of 1/2-PA: in particular we can describe 1/2-PA as an initial algebra, hence as a colimit, of a suitable functor over $\mathbf{ALG}_*$, giving the complete categorical characterization of 1/2-PA. Note that the definition of $\mathcal{G}_*$ would be impossible in **ALG**. The paper is organized as follows.

Section 2 contains a list of preliminary results on lattices.

In Section 3 we recall basic notions and facts about intersection type theories and filter models and introduce the irregular ITT $\mathcal{T}_*$. Even though it is not relevant for what concerns the paper, it is proved that $\mathcal{T}_*$ induces a λ-model.

In Section 4 we see the difficulties in characterizing 1/2-PA as a colimit of a functor working inside **ALG**.

In Section 5 we introduce the new category $\mathbf{ALG}_*$ and prove that it is cartesian closed.

In Section 6 we introduce the functor $\mathcal{A}_*$ over $\mathbf{ALG}_*$: this functor enriches a pointed lattice $\langle D, \delta\rangle$ with a new point, put just above δ. Thanks to $\mathcal{A}_*$ we can understand the axiom (1/2-pa) from a categorical point of view.

Section 7 is devoted to the technique for solving domain equations in $\mathbf{ALG}_*$: we basically use the classical technique of [34] for **O**-categories. At the end of the section we introduce the fundamental functor $\mathcal{G}_*$, built on the exponential functor and the above mentioned functor $\mathcal{A}_*$, and prove that the equation $X \simeq \mathcal{G}_*(X)$ has a solution $\mathbb{A}$ which is an initial $\mathcal{G}_*$-algebra in the subcategory $\mathbf{ALG}_*^{E\bot}$ of $\mathbf{ALG}_*$.

Section 8 contains the characterization result. After some technical lemma on $\mathcal{T}_*$ it is proved that 1/2 Park filter model is an initial $\mathcal{G}_*$-algebra in $\mathbf{ALG}_*^{E\bot}$, so proving that it is isomorphic to $\mathbb{A}$. This gives the complete semantic characterization of $\mathcal{T}_*$.

The paper is self-contained as much as possible, but for what concerns category theory.

2 Domain-theoretic preliminaries

This auxiliary section is devoted to fix notation and recall domain-theoretic definitions and results which concern lattices. The first part of the section concerns mainly ω-algebraic lattices, and the material presented will be used throughout the paper. The last part of the section present some notions and results on prime algebraic lattices, and will be used in the last section. A complete reference on lattice theory is [23].

- Let $P = \langle P, \sqsubseteq\rangle$ be a poset. We say that P is a *complete lattice* if any subset $Z \subseteq P$ has a least upper bound $\bigsqcup Z$.

From now on we denote complete lattices with D, E, possibly with indexes, while posets are denoted by P, Q.

- We say that a subset $A \subseteq P$ is *lower-closed* [*upper-closed*] if $a \in A$ and $a \sqsupseteq a'$ [resp. $a \sqsubseteq a'$] implies $a' \in A$. Given $B \subseteq P$, we define $\downarrow B = \{p \in P \mid \exists b \in B.p \sqsubseteq b\}$, $\uparrow B = \{p \in P \mid \exists b \in B.p \sqsupseteq b\}$; we write $\downarrow p$ for $\downarrow \{p\}$, and $\uparrow p$ for $\uparrow \{p\}$ $(p \in P)$.

- A subset $Z \subseteq D$ is *directed* if it is non-empty and for any $z, z' \in Z$ there exists $z'' \in Z$ such that $z, z' \sqsubseteq z''$.

- A monotone function $f : D \to E$ is *continuous* if for any directed $Z \subseteq D$

$$f(\bigsqcup Z) \quad = \quad \bigsqcup f(Z)$$

The space of continuous functions from D to E, ordered with the pointwise ordering, is denoted $[D \to E]$.

- A function $f : D \to E$ is *additive* if for any $Z \subseteq D$

$$f(\bigsqcup Z) \quad = \quad \bigsqcup f(Z)$$

- An element $d \in D$ is *compact* if for any directed $Z \subseteq D$, $d \sqsubseteq \bigsqcup Z$ implies that there exists $z \in Z$ such that $d \sqsubseteq z$. $K(D)$ denotes the subset of the compact elements of D.

- D is an ω-*algebraic* lattice (alg) if $K(D)$ is countable and moreover, for any $x \in D$,

$$x \quad = \quad \bigsqcup\{d \in K(D) \mid p \sqsubseteq x\}$$

ALG is the category of ω-algebraic lattices and continuous functions.

- Given two ω-algebraic lattices D and E, and two compact elements $d \in K(D)$, $e \in K(E)$, we define the step function

$$(d \Rightarrow e)(x) \quad = \quad \begin{cases} e & \text{if } d \sqsubseteq x \\ \perp & \text{otherwise} \end{cases}$$

$(d \Rightarrow e) \sqsubseteq f$ if and only if $e \sqsubseteq f(d)$ (hence $(d \Rightarrow e) \sqsubseteq (d' \Rightarrow e')$ if and only if $[d' \sqsubseteq d$ and $e \sqsubseteq e']$).

Finite sups of step functions are the compact elements in $[D \to E]$.

Lemma 2.1 *If D and E are alg's, then*

1. $K([D \to E]) = \{\bigsqcup_{i \in I}(d_i \Rightarrow e_i) \mid (\forall i \in I. d_i, e_i$ *compact) and I finite*$\}$
2. $[D \to E]$ *is ω-algebraic.*

Morever **ALG** *is a CCC (cartesian closed category) with "enough points".*

- Let $i : D \to E$, $j : E \to D$ be continuous functions. We say that $\iota = \langle i, j\rangle : D \to E$ is an *embedding-projection pair* (ep for short) if

$$\begin{array}{rcl} j \circ i & = & Id_D \\ i \circ j & \sqsubseteq & Id_E \end{array}$$

If $\langle i, j\rangle : D \to E$ and $\langle h, k\rangle : E \to E'$, then $\langle i, j\rangle \circ \langle h, k\rangle = \langle h \circ i, j \circ k\rangle$.
$\mathbf{ALG}^E$ is the category of ω-algebraic lattices and ep's.

Next lemma on ep's is very useful. Its proof can be recovered by using the results of Section 0-3 of [23] on basic properties of Galois connections.

Lemma 2.2 *Let D, E be alg's and $\iota = \langle i, j\rangle : D \to E$ be an ep.*

1. $\forall x \in D, y \in E.\ i(x) \sqsubseteq y \ \Leftrightarrow x \sqsubseteq j(y)$.
   j is the right adjoint of i and it is often denoted by i^R.

2. *ι is completely determined by the the embedding i, since j is forced to satisfy the following equality*

$$j(y) = \bigsqcup\{x \mid i(x) \sqsubseteq y\}$$

3. *i is additive, injective and preserves compact elements.*

- Let D be a lattice. An element $p \in D$ is *prime* if for any $Z \subseteq D$, $d \sqsubseteq \bigsqcup Z$ implies that there exists $z \in Z$ such that $d \sqsubseteq z$. $Pr(D)$ denotes the subset of prime elements of D. As obvious from definition, prime implies compact.
- D is a *prime algebraic* lattice (prime alg for short) if $Pr(D)$ is countable and moreover for any $x \in D$,

$$x \quad = \quad \bigsqcup\{p \in Pr(D) \mid p \sqsubseteq x\}$$

- In a prime algebraic lattice, compact elements are finite joins of prime elements. If D is prime algebraic, then it is ω-algebraic. A prime alg is always a distributive lattice.

Lemma 2.3 *If D and E are prime alg's, then*

1. $Pr([D \to E]) = \{d \Rightarrow e \mid d \in K(D), e \in Pr(E)\}$.
2. $[D \to E]$ *is prime algebraic.*

Lemma 2.4 *Let D be a prime alg, E be an alg, and $i^- : Pr(D) \to K(E)$ satisfy*

$$\text{(p-refl)} \quad \forall p, p' \in Pr(D).\ p \leq p' \ \Leftrightarrow \ i^-(p) \leq i^-(p')$$

Then i^- can be extended to an ep $i : D \to E$ by defining, for any $x \in D$,

$$i(x) \quad = \quad \bigsqcup\{i^-(p) \mid p \in Pr(D)\ \&\ p \sqsubseteq x\}$$

From now on we identify an ep $\iota = \langle i, i^R \rangle$ with its embedding i.

3 Intersection type theories and filter models

In this section we recall the basic notions and results on intersection types theories and filter models.

Through the years, many definitions of intersection type theories have been proposed in the literature, following a trend tending to strenghten the generality of definition (compare for instance the definition in [12] with [19]). Since we focus on a specific intersection type theory, which is a variation of very classical ones, we give a definition which is similar to the elder ones in the literature. Essentially our presentation follows [16] (in that paper intersection type theories were called "extended abstract type structures", see Definition 1.1).

Definition 3.1 *1. Let* $\mathbb{C}$ *a countable set of atoms containing* Ω*. An* intersection type languages *over* $\mathbb{C}$*, denoted by* $\mathbb{T} = \mathbb{T}(\mathbb{C})$ *is defined by the following abstract syntax:*

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}$$

Capital letters $A, B, \ldots$ *range over* $\mathbb{T}$*.*

2. An intersection type theory *(ITT) over* $\mathbb{C}$*, denoted by* $\mathcal{T}(\mathbb{C})$ *or simply* $\mathcal{T}$*, is a set of statements including the following axioms and closed under the following rules:*

(refl)	$A \leq A$	*(Ωη)*	$\Omega \leq \Omega \to \Omega$
(idem)	$A \leq A \cap A$	*(mon)*	$\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$
(incl$_L$)	$A \cap B \leq A$	*(trans)*	$\dfrac{A \leq B \quad B \leq C}{A \leq C}$
(incl$_R$)	$A \cap B \leq B$	*(η)*	$\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$
(Ω)	$A \leq \Omega$	*(→-∩)*	$(A \to B) \cap (A \to C) \leq A \to B \cap C$

Moreover $\leq$ *is required to be antisymmetric over* $\mathbb{C}$*.*

As mentioned, several different definitions of ITT's have been proposed. In order to avoid confusion when referring to the literature, we will call an ITT *standard* or *not-standard* according to whether it is captured by Definition 3.1. Standard ITT's are defined for instance in [7], [8], [12], [16], [17], [18], [20] for investigating a wide variety of topics related to the semantics of λ-calculus. Non-standard ITT's are defined e.g. in the seminal papers [14] and [15], in [2] and [3] for the study of lazy λ-calculus, in [21] and [25] in order to study properties of call-by-value.

We introduce some notation.
- Atoms will be ranged over by greek letters.
- $\mathbb{B}$ range over intersections of arrow types $\bigcap_{i \in I}(A_i \to B_i)$, which are not equivalent to Ω.
- Syntactical equality between types is denoted by $\equiv$.
- Ω is called the *top* atom.
- We agree that $\cap_{i \in \emptyset} A_i$ is Ω.
- We will write $A \sim B$ when both $A \leq B$ and $B \leq A$ (so for instance $A \to (B \cap C) \sim (A \to B) \cap (A \to C)$).
- When $A \sim B$ we say that A and B are *equivalent*.
- From now on I, J, K will denote finite set of indexes.

Associativity and commutativity of $\cap$ follows from above axioms and rules.
By applying (idem), (mon) and (trans), the following rule is derivable

$$\text{(meet)} \quad \frac{A \leq B \quad A \leq C}{A \leq B \cap C}$$

(meet), along with the two (incl) rules, proves that intersection is the meet operation with respect to $\leq$.

Syntactical equality up to commutativity and associativity of $\cap$ is denoted by $=$. So $A \cap B = B \cap A$, $A \to (B \cap C) \sim (A \to B) \cap (A \to C)$, but $A \to (B \cap C) \neq (A \to B) \cap (A \to C)$.

We give a useful technical lemma.

Lemma 3.2 *Let $\mathcal{T}(\mathbb{C})$ be an ITT. For any I, J, $A_i, B_i, A'_j, B'_j \in \mathbb{T}(\mathbb{C})$ ($i \in I$, $j \in J$), it holds*

1. $\bigcap_{i\in I}(A_i \to B_i) \leq (\bigcap_{i\in I} A_i) \to (\bigcap_{i\in I} B_i)$
2. $\forall j \in J. \exists I' \subseteq I.\ [A'_j \leq \bigcap_{i\in I'} A_i \ \&\ \bigcap_{i\in I'} B_i \leq B'_j] \quad \Rightarrow$ $\bigcap_{i\in I}(A_i \to B_i) \leq \bigcap_{j\in J}(A'_j \to B'_j)$.

Definition 3.3 *Let $\mathcal{T}(\mathbb{C})$ be an ITT.*

1. *A* filter *is a set $x \subseteq \mathbb{T}(\mathbb{C})$ such that:*
   (a) $\Omega \in x$
   (b) $A \leq B$ *and* $A \in x$ *imply* $B \in x$
   (c) $A \in x$ *and* $B \in x$ *imply* $A \cap B \in x$
2. *$Flt^{\mathcal{T}}$ is the set of filters over $\mathcal{T}$, ordered by set-theoretic inclusion, and is called the* filter structure *over $\mathcal{T}$.*
3. *If $y \subseteq \mathbb{T}(\mathbb{C})$, $\Uparrow y$ denotes the filter generated by y. If $y = \{A\}$, we write $\Uparrow A$ instead of $\Uparrow \{A\}$. $\Uparrow A$ is called a* principal *filter. Actually it coincides with $\uparrow A$, the upper closure of A.*
4. $\cdot : Flt^{\mathcal{T}} \times Flt^{\mathcal{T}} \to Flt^{\mathcal{T}}$ *is defined by*
$$x \cdot y \quad = \quad \{B \mid \exists A \in y. A \to B \in x\}$$
5. $F^{\mathcal{T}} : Flt^{\mathcal{T}} \to [Flt^{\mathcal{T}} \to Flt^{\mathcal{T}}]$ *and* $G^{\mathcal{T}} : [Flt^{\mathcal{T}} \to Flt^{\mathcal{T}}] \to Flt^{\mathcal{T}}$ *are introduced:*
$$\begin{array}{rcl} F^{\mathcal{T}}(x) & = & (y \mapsto x \cdot y) \\ G^{\mathcal{T}}(f) & = & \Uparrow \{A \to B \mid B \in f(\uparrow A)\} \end{array}$$

It is well-known (see [16]) that $Flt^{\mathcal{T}}$ is an ω-algebraic lattice: given $X \subseteq Flt^{\mathcal{T}}$

(fil-sup) $\quad \bigsqcup X = \uparrow \{\bigcap_{1\leq j\leq n} A_j \mid n \in \mathbb{N}, \forall 1 \leq j \leq n.\ A_j \in x_j \in X\}$,

Moreover $x \sqcap y = x \cap y$, the bottom filter is $\Uparrow \Omega$, the top filter $\mathbb{T}(\mathbb{C})$. Compact elements in $Flt^{\mathcal{T}}$ are the principal filters: they inherit the order $\leq^{op}$:

$$\uparrow A \subseteq \uparrow B \quad \Leftrightarrow \quad B \leq A$$

(we will use this equivalence quite often later on).

$\cdot$, $F^{\mathcal{T}}$ and $G^{\mathcal{T}}$ are continuous functions.

Given an ITT $\mathcal{T}$, a notion of interpretation of λ-terms is introduced (see [16])[1]:

$$\begin{array}{lcl} [\![x]\!]^{\mathcal{T}}_{\rho} & = & \rho(x) \\ [\![MN]\!]^{\mathcal{T}}_{\rho} & = & (F^{\mathcal{T}}([\![M]\!]^{\mathcal{T}}_{\rho}))([\![N]\!]^{\mathcal{T}}_{\rho}) \\ [\![\lambda x.M]\!]^{\mathcal{T}}_{\rho} & = & G^{\mathcal{T}}(d \mapsto [\![M]\!]^{\mathcal{T}}_{\rho[x/d]}) \quad (d \in \mathit{Flt}^{\mathcal{T}}) \end{array}$$

Not any filter structure $\mathit{Flt}^{\mathcal{T}}$ gives rise to a lambda model $\langle \mathit{Flt}^{\mathcal{T}}, F^{\mathcal{T}}, G^{\mathcal{T}}, [\![\_]\!]\rangle$. According to the definition of λ-model of Hindley-Longo [24], a sufficient condition for obtaining a λ-model is that $F^{\mathcal{T}} \circ G^{\mathcal{T}} = Id_{\mathit{Flt}^{\mathcal{T}}}$.

This is guaranteed by the *legality* condition of [30].

Definition 3.4 *We say that $\mathcal{T}(\mathbb{C})$ is* legal, *if for any I and $A_i, B_i, C, D \in \mathbb{T}(\mathbb{C})$ $(i \in I)$, with $D \not\sim \Omega$, we have:*

$$\textstyle\bigcap_{i\in I}(A_i \to B_i) \leq C \to D \quad \Rightarrow \quad \bigcap_{i\in J} B_i \leq D \quad \textit{where } J = \{i \in I \mid C \leq A_i\}.$$

Next lemma is Theorem 2.13(iii) of [16], or a special case of Theorem 10.1.11 of [30].

Lemma 3.5 *If $\mathcal{T}$ is legal, then $F^{\mathcal{T}} \circ G^{\mathcal{T}} = Id_{\mathit{Flt}^{\mathcal{T}}}$, hence $\langle \mathit{Flt}^{\mathcal{T}}, F^{\mathcal{T}}, G^{\mathcal{T}}, [\![\_]\!]\rangle$ is a λ-model.*

Intersection type theories are a powerful tool for synthesizing new λ-models or describing existing ones. Their definition relies mostly on using special purpose axioms that put into relation atoms with (intersection of) arrow types. In order to reason in a clean way about it we introduce the following definition.

Definition 3.6 *Let $\mathcal{T}(\mathbb{C})$ be an ITT. Let $\mathbb{C}^{-}$ denote the atoms different from Ω.*

1. *An* atom-arrow judgment for $\alpha \in \mathbb{C}^{-}$ *is any judgment of the shape $\alpha \leq \mathbb{B}$ or $\mathbb{B} \leq \alpha$ (where $\mathbb{B}$ is an intersection of arrow types not equivalent to Ω).*
2. *$\alpha \in \mathbb{C}^{-}$ is called* arrow-related *if an atom-arrow judgment for α is derivable in $\mathcal{T}$.*
3. *$\alpha \in \mathbb{C}^{-}$ is called* arrow-equated *if for some $\mathbb{B}$, both $\alpha \leq \mathbb{B}$ and $\mathbb{B} \leq \alpha$ are derivable in $\mathcal{T}$ (that is, $\alpha \sim \mathbb{B}$).*

Note that, for $\alpha \neq \beta$, it is not possible to have $\alpha \sim \mathbb{B} \sim \beta$, since this would contrast the antisymmetry of the relation $\leq$ on atoms.

Now we come to the "regularity" condition (reg) generally respected in the literature, both for standard and non-standard intersection type theories. In defining an ITT, special purpose axioms are always introduced in such a way to satisfy the following restriction:

[1] It coincides with the interpretation defined through the *type assignment system*, see [16].

$$\text{(reg)} \quad \begin{array}{l}\text{either no atom in } \mathbb{C}^- \text{ is arrow-related or} \\ \text{each atom in } \mathbb{C}^- \text{ is arrow-equated.}\end{array}$$

Definition 3.7 *We call* regular *any ITT $\mathcal{T}$ defined respecting (reg).*

As far as the author could check, all the intersection type theories defined in the literature are regular, with just the mentioned exception of the intersection type theory of Definition 4.9 in [16].

As relevant examples of ITT's defined according to (reg), and in view of the definition of our irregular ITT, we consider the following two intersection type theories. Consider $\mathbb{C} = \{\Omega, \phi\}$ and the following axioms:

$$\begin{array}{ll}\text{(sc)} & \phi \sim \Omega \to \phi \\ \text{(pa)} & \phi \sim \phi \to \phi\end{array}$$

Let $\mathcal{T}_{sc}$ and $\mathcal{T}_{pa}$ be the intersection type theories generated by (sc) and (pa) respectively. Then $Flt^{\mathcal{T}_{sc}}$ is isomorphic to Scott D_∞ λ-model of [32], built starting from $\{\bot, \top\}$, while $Flt^{\mathcal{T}_{pa}}$ is isomorphic to Park λ-model of [28].

The application of (reg) guarantees a good categorical characterization of ITT's as colimits in **ALG** (we will see it in Section 4), but at the price of cutting off some intermediate options on the choice of axioms, which do not respect (reg).

Next definition is an exemplary case of ITT's defined without respecting (reg).

Definition 3.8 *An ITT $\mathcal{T}$ is called* irregular *if there is an atom $\alpha \neq \Omega$ such that α is arrow-related without being arrow-equated. $Flt^{\mathcal{T}}$ is a* irregular *filter structure if $\mathcal{T}$ is irregular.*

In the rest of the paper we consider an irregular version of the Park intersection type theory, generated by the axiom

$$\text{(1/2-pa)} \quad \phi \leq \phi \to \phi$$

Definition 3.9 *We call $\mathcal{T}_*$ the intersection type theory over $\mathbb{C}_* = \{\Omega, \phi\}$ generated by (1/2-pa). We call $Flt^{\mathcal{T}_*}$* 1/2 Park filter model *(1/2-PA for short).*

From now on we will write $\mathbb{T}_*$ for $\mathbb{T}(\mathbb{C}_*)$.

Note that $\mathcal{T}_*$ differs from the ITT associated to the Park model for a small syntactic change: in defining $\mathcal{T}_*$, we have not considered the axiom $\phi \to \phi \leq \phi$ as the Park ITT does. Nevertheless the semantic consequences of this change are relevant.

In particular, we will see the difficulties of capturing in **ALG** the categorical characterization of (1/2-PA), and how to get the required characterization working in a suitable new category.

At present, we do not know that $\mathcal{T}_*$ is irregular. This will be shown in Lemma 3.12. First we need some preliminary results.

We characterize the types that are equivalent to the top type Ω.

Lemma 3.10 *Let $\mathcal{E}_\Omega \subseteq \mathbb{T}_*$ be generated by the following abstract syntax:*

$$\mathcal{E}_\Omega = \Omega \mid \mathcal{E}_\Omega \cap \mathcal{E}_\Omega \mid A \to \mathcal{E}_\Omega$$

Then

1. $\uparrow \Omega = \mathcal{E}_\Omega$
2. *for any $A \in \mathbb{T}_*$ and $E \in \mathcal{E}_\Omega$, $A \sim A \cap E$.*

Proof: (1) By induction on the structure of types in $\mathcal{E}_\Omega$, it follows $\Omega \leq E$ for any $E \in \mathcal{E}_\Omega$, hence $\mathcal{E}_\Omega \subseteq\uparrow \Omega$. By induction on derivation it follows that if $E \in \mathcal{E}_\Omega$ and $E \leq A$, then $A \in \mathcal{E}_\Omega$. Therefore, since $\Omega \in \mathcal{E}_\Omega$, it follows $\mathcal{E}_\Omega \supseteq\Uparrow \Omega$.
(2) Immediate from (1), applying (incl) and (meet).□

From now on, when we write a type of the shape $\bigcap_{i\in I}(A_i \to B_i) \cap E$, with $E \in \mathcal{E}_\Omega$, it is understood that for any $i \in I$, $B_i \notin \mathcal{E}_\Omega$. This is not restrictive. In fact, if $B_{i'} \in \mathcal{E}_\Omega$, then the corresponding arrow type $A_{i'} \to B_{i'}$ is in $\mathcal{E}_\Omega$ too, so we can read the type as $\bigcap_{i\in I\setminus\{i'\}}(A_i \to B_i) \cap (E \cap (A_{i'} \to B_{i'}))$.

After Lemma 3.10(2), from now on we reason up to the equivalence $A \sim A \cap E$.

Lemma 3.11 *If $A \leq \phi \cap B \in \mathcal{T}_*$, then $A = \phi \cap A'$, for some A'.*

Proof: Immediate by induction on derivations.□

Note that for any type $A \in \mathbb{T}_*$, either $A = B\cap\phi$, or $A = \bigcap_{i\in I}(B_i \to C_i)\cap E$, with $E \in \mathcal{E}_\Omega$.

We can now prove that $\mathcal{T}_*$ is irregular.

Lemma 3.12 *$\mathcal{T}_*$ is an irregular intersection type theory.*

Proof: $\phi \neq \Omega$ is obviously arrow-related by axiom (1/2-pa). On the other hand, it is not possible to have $\mathbb{B} \sim \phi$ for some intersection of arrows $\mathbb{B}$, since this would imply $\mathbb{B} \leq \phi$. By Lemma 3.11 this is not possble. Therefore $\mathbb{B} \leq \phi$ is not derivable.□

Although the topic is not central in the paper, we now investigate some properties of $\mathcal{T}_*$ in relation to λ-calculus. Next two lemmata are useful for proving that $\mathcal{T}_*$ is legal.

Lemma 3.13 *Let $\bigcap_{i\in I}(A_i \to B_i) \cap E \leq A' \in \mathcal{T}_*$, with $E \in \mathcal{E}_\Omega$. Then there exists J, C_j, D_j and $E' \in \mathcal{E}_\Omega$ such that $A' = \bigcap_{j\in J}(C_j \to D_j) \cap E'$.*

Proof: By induction on derivations.□

Lemma 3.14 *Let $E_1, E_2 \in \mathcal{E}_\Omega$. Then*

$$\begin{array}{l}\bigcap_{i\in I}(A_i \to B_i) \cap E_1 \leq \bigcap_{j\in J}(C_i \to D_j) \cap E_2 \in \mathcal{T}_* \;\Leftrightarrow \\ \forall j \in J.\exists I' \subseteq I.C_j \leq \bigcap_{i\in I'} A_i \;\&\; \bigcap_{i\in I'} B_i \leq D_j\end{array}$$

Proof: ($\Leftarrow$) easy, by applying Lemma 3.2(1) and Lemma 3.10.
($\Rightarrow$) by induction on derivations. The non-trivial step is the case of (trans). Suppose that the conclusion depends on an application of (trans) to the two judgments

$$\begin{array}{ll} \text{(i)} & \bigcap_{i\in I}(A_i \to B_i) \cap E_1 \leq A'' \\ \text{(ii)} & A'' \leq \bigcap_{j\in J}(C_j \to D_j) \cap E_2 \end{array}$$

By Lemma 3.13, we have $A'' = \bigcap_{k\in K}(A'_k \to B'_k) \cap E'_1$. Then we can apply the inductive hypothesis to (ii), and derive that

$$\forall j \in J.\exists K' \subseteq K.\ [C_j \leq \bigcap_{k\in K'} A'_k \ \&\ \bigcap_{k\in K'} B'_k \leq D_j]$$

Then, applying induction to (i),

$$\forall k \in K'.\ \exists I_k \subseteq I.\ [A'_k \leq \bigcap_{i\in I_k} A_i \ \&\ \bigcap_{i\in I_k} B_i \leq B'_k]$$

Defining $I' = \bigcup_{k\in K'} I_k$, we get

$$C_j \leq \bigcap_{i\in I'} A_i \ \&\ \bigcap_{i\in I'} B_i \leq D_j$$

which proves the thesis.$\square$

Theorem 3.15 *1. $\mathcal{T}_*$ is legal.*

2. $Flt^{\mathcal{T}_*}$ is a λ-model.

Proof: (1) Let $A = \bigcap_{i\in I}(A_i \to B_i) \leq C \to D$. If $D \in \mathcal{E}_\Omega$, it is sufficient to choose the set J of Definition 3.4 as $\emptyset$. Otherwise, we have $A = \bigcap_{i\in I'}(A_i \to B_i) \cap E$, where $I' = \{i \in I \mid B_i \notin \mathcal{E}_\Omega\}$, and $E \in \mathcal{E}_\Omega$. As a consequence of $A \leq C \to D$ it follows $\bigcap_{i\in I'}(A_i \to B_i) \cap E \leq (C \to D) \cap \Omega$. Applying Lemma 3.14 to this last judgment, there exists $J \subseteq I'$ such that $C \leq \bigcap_{i\in J} A_i$ and $\bigcap_{i\in J} B_i \leq D$, proving that $\mathcal{T}_*$ is legal.

(2) By (1) and Lemma 3.5, we get that $Flt^{\mathcal{T}_*}$ is a λ-model.$\square$

Lemma 3.16 *Let $\phi \leq A \leq \phi \to \phi$. Then $A \leq \bigcap_{i\in I}(B_i \to C_i)$ if and only if for any $i \in I$, $B_i \leq \phi$ and $\phi \leq C_i$.*

Proof By induction on derivations. The unique non-trivial step is when rule (trans) is applied. Suppose that the derivation ends with an application of (trans) whose premises are (a) $A \leq A'$ and (b) $A' \leq \bigcap_{i\in I}(B_i \to C_i)$. There are two cases. If A' has the shape $D \cap \phi$, then $A' \sim \phi$, and the thesis follows by induction applied to (b). Otherwise, $A' = \bigcap_{j\in J}(B'_j \to C'_j) \cap E$, with $E \in \mathcal{E}_\Omega$. By Lemma 3.14, it follows that for any $i \in I$, there exists $J_i \subseteq J$ such that

$$\text{(c)} \quad B_i \leq \bigcap_{j\in J_i} B'_j \ \&\ \bigcap_{j\in J_i} C'_j \leq C_i$$

By induction on (a), we get $B'_j \leq \phi$ and $\phi \leq C'_j$, for any $j \in J$. This, along with (c) and (trans), implies $B_i \leq \phi$ and $\phi \leq C_i$, for any $i \in I$.$\square$

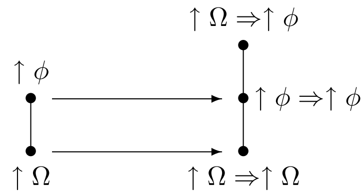

Figure 1: $i_0 : D_0 \to [D_0 \to D_0]$ induced by $\mathcal{T}_{pa}$

4 Why $\mathcal{T}_*$ does not fit to ALG

This section contains a digression for explaining how the correspondence between ITT's and colimits in **ALG** works in the regular case, and which are the difficulties in finding the correspondence in the irregular case of $\mathcal{T}_*$.

First we introduce some notation. Let ρ be a triple $\langle F, D_0, i_0 \rangle$, where $F : \mathbf{ALG} \to \mathbf{ALG}$ is a locally continuous endofunctor, D_0 an initial domain, $i_0 : D_0 \to F(D_0)$ an initial embedding. We define $\Delta(\rho)$ as the ω-chain

$$D_0 \stackrel{i_0}{\to} F(D_0) \stackrel{F(i_0)}{\to} F^2(D_0) \stackrel{F^2(i_0)}{\to} F^3(D_0) \ldots$$

$colim(\rho)$ denotes the colimit of $\Delta(\rho)$.

Given a regular ITT $\mathcal{T}$, quite often it is easy to recover, by the Stone duality paradigm of [1], a colimit characterization of $Flt^{\mathcal{T}}$ of the shape

$$Flt^{\mathcal{T}} \simeq colim(\rho_{\mathcal{T}})$$

for a suitable triple $\rho_{\mathcal{T}}$. The way for determine $\rho_{\mathcal{T}}$ amounts to define D_0 as $\{\uparrow \bigcap_{i \in I} \alpha_i \mid \alpha_i \in \mathbb{C}\}$, ordered by set-theoretic inclusion; the functor F is built using the exponential functor; the embedding i_0 depends on the judgments that arrow-equate atoms, according to the rule

$$i_0(\uparrow \alpha) = \bigsqcup_{i \in I} (\uparrow \beta_i \Rightarrow \uparrow \gamma_i) \quad \Leftrightarrow \quad \alpha \sim \bigcap_{i \in I} (\beta_i \to \gamma_i)$$

As an example, consider the Park ITT $\mathcal{T}_{pa}$ over $\{\Omega, \phi\}$, already introduced in Section 3. Then the triple ρ_{pa} consists of: the two-point lattice, the functor $F(X) = [X \to X]$, the embedding $i_0(d) = d \Rightarrow d$, as pictured in Figure 1. This corresponds exactly to the starting point of the ω-chain whose colimit is the Park model of [28].

Consider now $\mathcal{T}_*$. We would like to characterize $\mathcal{T}_*$ as above, by finding a suitable $\rho_{\mathcal{T}_*}$ such that

$$(*\text{-ch}) \quad Flt^{\mathcal{T}_*} \simeq colim(\rho_{\mathcal{T}_*})$$

We can translate easily the definition of $\mathcal{T}_*$ into an embedding, as done in Figure 2. Axioms (Ω) and $(\Omega\eta)$ force $i_0(\uparrow \Omega)$ to be the constant bottom function

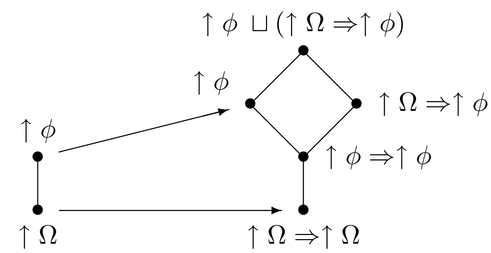

Figure 2: $i_0 : D_0 \to D_?$ induced by $\mathcal{T}_*$

$\uparrow \Omega \Rightarrow \uparrow \Omega$. But as a consequence of the irregularity of $\mathcal{T}_*$, $\uparrow \phi$ is not identified via the embedding to any function. As a consequence, what is missing is the categorical interpretation of the domain on the right in Figure 2 (called at the moment $D_?$): the axiom $\phi \leq \phi \to \phi$ leads to the lifting of the point $\uparrow \phi$ above the step function $\uparrow \phi \Rightarrow \uparrow \phi$. It is not known to the author whether there is a functor over **ALG** which can perform the action required for building $D_?$ out of D_0, and moreover it allows to recover the colimit characterization of $Flt^{\mathcal{T}_*}$. The solution proposed in the next sections amounts to find an alternative category, where a functor that captures the construction of $D_?$ does exist.

5 The category $\mathbf{ALG}_*$

In this section we introduce the category $\mathbf{ALG}_*$ that we will use in order to solve the problem related to the categorical characterization of $\mathcal{T}_*$. $\mathbf{ALG}_*$ is defined as a variation of the category **ALG**: objects are chosen as ω-algebraic lattices $\langle D, \delta\rangle$, where the special point δ is in $K(D)$. The notion of morphism is relaxed with respect to the preservation of the special points. We will not require, given a morphism $f : \langle D, \delta\rangle \to \langle E, \eta\rangle$, (ht) $f(\delta) = \eta$, since this condition would forbid the existence of the exponential functor (needed for defining the fundamental domain equation in the paper). So we will just require on morphisms the weaker condition $\eta \sqsubseteq f(\delta)$ (however (ht) will hold for embedding-projection pairs).

Definition 5.1 *1. A* pointed *alg (p-alg) is a pair $\langle D, \delta\rangle$ where D is an object in* **ALG** *and $\delta \in K(D)$. δ is called the* special *element. Quite often we will write simply D instead of $\langle D, \delta\rangle$.*

2. *Let $\langle D, \delta\rangle$ and $\langle E, \eta\rangle$ be two p-alg's. We say that a monotone function $f : D \to E$ is* tidy *if for any $x \in D$, $\eta \sqsubseteq f(\delta)$. We say that f is* hypertidy *if the stronger condition $f(\delta) = \eta$ holds.*

3. *$\mathbf{ALG}_*$ consists of p-alg's and tidy continuous functions;*

4. *$\mathbf{ALG}_*^E$ consists of p-alg's and ep's $\langle i, j\rangle$ such that both i and j are tidy.*

Lemma 5.2 *Given $\langle i,j\rangle : \langle D,\delta\rangle \to \langle E,\eta\rangle$ in $\mathbf{ALG}_*^E$, then i and j are hypertidy.*

Proof: From tidiness of j we have $\delta \sqsubseteq j(\eta)$. This implies $i(\delta) \sqsubseteq i(j(\eta)) \sqsubseteq \eta$. Moreover $\eta \sqsubseteq i(\delta)$, since i is tidy, so it must hold $i(\delta) = \eta$, proving that i is hypertidy. As to j, by above $j(\eta) = j(i(\delta))$, and the thesis follows from $j \circ i = Id_D$.$\square$

In view of the future characterization of $Flt^{\mathcal{T}_*}$, we introduce $\mathbf{ALG}_*^{E\perp}$, the full subcategory of $\mathbf{ALG}_*^E$ whose objects are p-alg's $\langle D,\delta\rangle$ with $\delta \neq \perp$. The interest of $\mathbf{ALG}_*^{E\perp}$ is based on the fact that, differently from $\mathbf{ALG}_*^E$, it has an initial object.

Corollary 5.3 $\mathbf{ALG}_*^{E\perp}$ *has as initial object $\Xi_0 = \langle\{\perp,\xi_0\},\xi_0\rangle$.*

Proof: let $i : \Xi_0 \to \langle E,\eta\rangle$. $i(\perp) = \perp$, since i is additive by Lemma 2.2(3). Moreover by previous lemma it holds $i(\xi_0) = \eta$. So i is uniquely determined. Its right adjoint is $i^R = (\eta \Rightarrow \xi_0)$.$\square$

Next lemma shows a relevant property of $\mathbf{ALG}_*$.

Lemma 5.4 $\mathbf{ALG}_*$ *is a concrete CCC.*

Proof: $\mathbf{ALG}_*$ is concrete (see [4], Definition 5.1) since the forgetful functor $\mathcal{U} : \mathbf{ALG}_* \to \mathbf{Set}$ is trivially faithful. The terminal object is the one-point pointed lattice $\mathbf{1} = \langle\{\star\},\star\rangle$. Let $\langle D,\delta\rangle$, $\langle E,\eta\rangle$ and $\langle C,\gamma\rangle$ be p-alg's. The cartesian product $\langle D,\delta\rangle \times \langle E,\eta\rangle$ is $\langle D\times E,(\delta,\eta)\rangle$, with the usual projections (details are omitted). As to the exponential E^D, we take $\langle [D\to E],\delta \Rightarrow \eta\rangle$ (we recall that $[D \to E]$ contains all the continuous functions, also the non-tidy ones).

The proof of continuity of the functions below is as in Section 2 of [29], so we just prove tidiness.

Define $apply_{D,C} : \langle D\times[D\to C],(\delta,\delta\Rightarrow\gamma)\rangle \to \langle C,\gamma\rangle$ by (we omit subscripts)

$$apply(d,f) = f(d)$$

apply is hypertidy, since

$$\begin{aligned} apply(\delta,\delta\Rightarrow\gamma) &= (\delta\Rightarrow\gamma)(\delta) \\ &= \gamma \end{aligned}$$

Given a tidy continuous function $f : \langle D\times E,(\delta,\eta)\rangle \to \langle C,\gamma\rangle$, define $curry(f) : \langle E,\eta\rangle \to \langle [D\to C],\delta\Rightarrow\gamma\rangle$ by

$$\forall y \in E.\ curry(f)(y) \quad = \quad (x\in D \mapsto f(x,y))$$

$curry(f)$ is tidy. In fact, for any $x \in D$, such that $x \sqsupseteq \delta$, we have

$$\begin{aligned} (curry(f)(\eta))(x) &= f(x,\eta) \\ &\sqsupseteq \gamma && \text{since } f \text{ is tidy} \\ &= (\delta\Rightarrow\gamma)(x) \end{aligned}$$

Therefore $curry(f)(\eta) \sqsupseteq \delta \Rightarrow \gamma$, proving that *curry* is tidy. Finally, $curry(f)$ is the unique continuous function such that $apply \circ (Id \times curry(f)) = f$.□

The exponential functor $\rightarrow_* : \mathbf{ALG}_*^{op} \times \mathbf{ALG}_* \rightarrow \mathbf{ALG}_*$ is explicitely defined as follows (as usual we use the infix notation for it):
- on objects: given p-alg's $D = \langle D, \delta \rangle$ and $E = \langle E, \eta \rangle$,

$$(D \rightarrow_* E) \quad = \quad \langle [D \rightarrow E], \delta \Rightarrow \eta \rangle$$

- on morphisms: let $f \in \mathbf{ALG}_*(D', D)$, $g \in \mathbf{ALG}_*(E, E')$. $(f \rightarrow_* g) : [D \rightarrow E] \rightarrow [D' \rightarrow E']$ is defined by

$$\forall u \in [D \rightarrow E] \quad (f \rightarrow_* g)(u) = g \circ u \circ f$$

$(f \rightarrow_* g)$ is a morphisms in $\mathbf{ALG}_*$. It is continuous (see e.g. Section 2 of [29]), and tidy, as checked directly in the following lemma.

Lemma 5.5 *Let $f : \langle D', \delta' \rangle \rightarrow \langle D, \delta \rangle$ and $g : \langle E, \eta \rangle \rightarrow \langle E', \eta' \rangle$ be tidy continuous functions. Then $(f \rightarrow_* g)$ is tidy.*

Proof We have to prove that

$$(\delta' \Rightarrow \eta') \sqsubseteq g \circ (\delta \Rightarrow \eta) \circ f$$

Since f is tidy, we have $\delta \sqsubseteq f(\delta')$, so $\eta = (\delta \Rightarrow \eta)(f(\delta'))$. Therefore

$$\begin{array}{lll} \text{true} & \Leftrightarrow \quad \eta' \sqsubseteq g(\eta) & \text{since } g \text{ is tidy} \\ & \Leftrightarrow \quad \eta' \sqsubseteq g((\delta \Rightarrow \eta)(f(\delta'))) & \text{by above} \\ & \Leftrightarrow \quad (\delta' \Rightarrow \eta') \sqsubseteq g \circ (\delta \Rightarrow \eta) \circ f \square & \end{array}$$

Now a short investigation on the relationship between **ALG** and $\mathbf{ALG}_*$.

Lemma 5.6 *Let $\mathcal{U} : \mathbf{ALG}_* \rightarrow \mathbf{ALG}$ be the forgetful functor. Then $\mathcal{U}$ has as right adjoint the full and faithful functor $\mathcal{I} : \mathbf{ALG} \rightarrow \mathbf{ALG}_*$ whose action, for any D object and f morphism in* **ALG**, *is:*

$$\begin{array}{rcl} \mathcal{I}(D) & = & \langle D, \bot \rangle \\ \mathcal{I}(f) & = & f. \end{array}$$

Proof: Let $\mathcal{D} = \langle D, \delta \rangle$. Note that $hom_{\mathbf{ALG}_*}(\mathcal{D}, \langle E, \bot \rangle) = hom_{\mathbf{ALG}}(D, E)$, since the tidiness condition trivially holds for any continuous function from D to E, when the special point in E is $\bot$. As a consequence $\mathcal{I}$ is well-defined on morphisms, full and faithful, and the identity

$$hom_{\mathbf{ALG}_*}(\mathcal{D}, \mathcal{I}(E)) = hom_{\mathbf{ALG}}(\mathcal{U}(\mathcal{D}), E)$$

is the natural isomorphism between $hom_{\mathbf{ALG}_*}(\_, \mathcal{I}(\_))$ and $hom_{\mathbf{ALG}}(\mathcal{U}(\_), \_)$.□

For sake of completeness, we end this section by investigating the relation between $\mathbf{ALG}_*$ and λ-models.

An object X in a CCC is *reflexive* if there exists a pair of morphisms $\Phi : X \to X^X$ and $\Psi : X^X \to X$ such that

$$\Phi \circ \Psi = Id_{X^X}$$

(we use the standard notation $[X \to X] \trianglelefteq X$ when X is reflexive).

Note that, although it is a CCC, $\mathbf{ALG}_*$ has not "enough points", since $hom_{\mathbf{ALG}_*}(\mathbf{1}, \langle D, \delta\rangle)$ contains just those functions which send the unique point $\star \in \mathbf{1}$ to some point above δ, so covering just $\uparrow \delta$, which in general is a proper subset of D (unless $\delta = \bot$). For instance, consider Ξ_0 and the two endofunctions $const_{\xi_0}, Id_{\Xi_0} : \Xi_0 \to \Xi_0$; they are different, since $const_{\xi_0}(\bot) = \xi_0 \neq \bot = Id_{\Xi_0}(\bot)$. But the unique morphism $m : \mathbf{1} \to D$ is $m(\star) = \xi_0$, which does not allow to discriminate $const_{\xi_0}$ from Id_{Ξ_0}, since $const_{\xi_0} \circ m = Id_{\Xi_0} \circ m$.

As a consequence, the classical result (see e.g. [11], Chapter 5, Proposition 5.5.7) which connects CCC to λ-models, namely

Any reflexive object in a CCC, which has "enough points", is a λ-model

does not apply to the case of $\mathbf{ALG}_*$. Anyway we can prove easily in another way that reflexive objects in $\mathbf{ALG}_*$ are λ-models.

Theorem 5.7 *Any reflexive object $\mathcal{D} = \langle D, \delta\rangle$ in $\mathbf{ALG}_*$ is a λ-model.*

Proof: If $\mathcal{D}$ is reflexive, there are morphisms in $\mathbf{ALG}_*$, $\Phi : \mathcal{D} \to \mathcal{D}^{\mathcal{D}}$ and $\Psi : \mathcal{D}^{\mathcal{D}} \to \mathcal{D}$, such that $\Phi \circ \Psi = Id_{\mathcal{D}^{\mathcal{D}}}$. By Lemma 5.4, $\mathcal{D}^{\mathcal{D}}$ is $\langle [D \to D], \delta \Rightarrow \delta\rangle$; therefore, by definition of morphism in $\mathbf{ALG}_*$, we have that $\Phi \in hom_{\mathbf{ALG}}(D, [D \to D])$, $\Psi \in hom_{\mathbf{ALG}}([D \to D], D)$, with $\Phi \circ \Psi = Id_{D^D}$, hence D is a reflexive object in $\mathbf{ALG}$. Since this last category is a CCC with enough points, we get that D, hence $\mathcal{D}$, is a λ-model.$\Box$

6 The endofunctor $\mathcal{A}_*$ over $\mathbf{ALG}_*$

Given a category $\mathcal{C}$ we say that T is a functor *over* $\mathcal{C}$ if $T : (\mathcal{C}^{op})^m \times \mathcal{C}^n \to \mathcal{C}$, for some $m, n \in I\!N$. In this section we introduce the endofunctor $\mathcal{A}_*$ over $\mathbf{ALG}_*$, which adds to a lattice a new compact point just above the special element. This functor will be crucial for the construction of the domain $D_?$ and for the characterization ($*$-ch).

Let $\langle D, \delta\rangle$ be a p-alg. We define

$$\mathcal{A}(D) = D \cup \{(0, x) \mid x \sqsupseteq \delta\}$$

The element $(0, \delta)$ is shortly written ξ_D. $\mathcal{A}(D)$ is ordered as follows:

$$v \sqsubseteq z \quad \Leftrightarrow \quad \begin{cases} v \sqsubseteq_D z & \text{if } v, z \in D \\ v \sqsubseteq_D y & \text{if } v \in D,\ z = (0, y) \\ x \sqsubseteq_D y & \text{if } v = (0, x),\ z = (0, y) \end{cases}$$

In order to characterize sups in $\mathcal{A}(D)$, we introduce the following notation. Given $Z \subseteq \mathcal{A}(D)$, let

$$\begin{array}{rcl} Z^{(1)} & = & Z \cap D; \\ Z^{(2)} & = & \{x \in \uparrow \delta \mid (0,x) \in Z\} \end{array}$$

It easy to check that

$$(*\text{-}\bigsqcup) \qquad \bigsqcup Z \quad = \quad \left\{ \begin{array}{ll} \bigsqcup^D Z & \text{if } Z \subseteq D \\ (0, \bigsqcup^D (Z^{(1)} \cup Z^{(2)}) & \text{otherwise} \end{array} \right.$$

where the sup $\bigsqcup$ is taken in $\mathcal{A}(D)$, while $\bigsqcup^D$ is taken in D. Applying $(*\text{-}\bigsqcup)$ at the binary case, we have:

$$(*\text{-}\bigsqcup\text{-bin}) \qquad \forall x \in D. x \sqsupseteq \delta \ \Rightarrow \ \xi_D \sqcup x = (0,x)$$

Finally note that

$$(*\text{-}\bigsqcup\text{-dir}) \qquad \begin{array}{l} Z \subseteq D \text{ directed and } Z^{(2)} \text{ non-empty imply} \\ Z^{(2)} \text{ directed and } \bigsqcup Z = \bigsqcup\{(0,y) \mid y \in Z^{(2)}\} = (0, \bigsqcup^D Z^{(2)}). \end{array}$$

The proof of next lemma is omitted. It follows easily from $(*\text{-}\bigsqcup)$.

Lemma 6.1 *1. For any p-alg $\langle D, \delta \rangle$, $\mathcal{A}(D)$ is an ω-algebraic lattice, whose compact elements are*

$$K(\mathcal{A}(D)) = K(D) \cup \{(0,d) \mid d \in K(D)\}$$

2. The set-theoretic inclusion $D \subseteq \mathcal{A}(D)$ is an embedding.

3. If moreover D is prime algebraic, then so it $\mathcal{A}(D)$. In such a case we have

$$Pr(\mathcal{A}(D)) = Pr(D) \cup \{\xi_D\}.$$

Definition 6.2 *Given a p-alg $D = \langle D, \delta \rangle$, $\mathcal{A}_*(D)$ is defined as $\langle \mathcal{A}(D), \xi_D \rangle$. By Lemma 6.1 $\mathcal{A}_*(D)$ in an object in* $\mathbf{ALG}_*$.

We now extend the action of $\mathcal{A}_*$ on morphisms.

Lemma 6.3 *Let $D = \langle D, \delta \rangle$ and $E = \langle E, \eta \rangle$ be p-alg's, and $f : \langle D, \delta \rangle \to \langle E, \eta \rangle$ be a tidy continuous function. For any $z \in \mathcal{A}_*(D)$, define*

$$(\star\star) \quad \mathcal{A}_*(f)(z) \quad = \quad \left\{ \begin{array}{ll} f(z) & \textit{if } z \in D \\ (0, f(y)) & \textit{if } z = (0,y) \end{array} \right.$$

Then $\mathcal{A}_*(f) : \mathcal{A}_*(D) \to \mathcal{A}_*(E)$ is a tidy continuous function.

Proof: First of all note that if $z = (0,y)$, then $f(z) = (0, f(y))$ is an element of $\mathcal{A}_*(E)$. In fact, in such a case $y \sqsupseteq \delta$, since $z \in \mathcal{A}_*(D)$. Then it follows $f(y) \sqsupseteq \eta$, since f is tidy, hence $f(z) \in \mathcal{A}_*(E)$.

Note moreover that by definition, $\mathcal{A}_*(f)$ satisfies the following commutativity properties with respect to the operation $(\_)^{(1)}$ and $(\_)^{(2)}$: for any $Z \subseteq \mathcal{A}_*(D)$

$$
\begin{array}{ll}
\text{(a)} & (\mathcal{A}_*(f)(Z))^{(1)} = f(Z^{(1)}) \\
\text{(b)} & (\mathcal{A}_*(f)(Z))^{(2)} = \{f(y) \mid y \in Z^{(2)}\}
\end{array}
$$

$\mathcal{A}_*(f)$ is tidy: by definition of $\mathcal{A}_*$, it follows immediately $\xi_E \sqsubseteq \mathcal{A}_*(f)(\xi_D)$. $\mathcal{A}_*(f)$ is monotone. Let $v, z \in \mathcal{A}_*(D)$, with $v \sqsubseteq z$. There are three possible cases:

- $v \sqsubseteq_D z$. In such a case $\mathcal{A}_*(f)(v) = f(v) \sqsubseteq f(z) = \mathcal{A}_*(f)(z)$.
- $v \in D$, $z = (0, y)$ (with $y \sqsupseteq \delta$), and $v \sqsubseteq_D y$. Then

$$
\begin{array}{rcll}
\mathcal{A}_*(f)(v) & = & f(v) & \\
& \sqsubseteq & f(y) & \text{since } f \text{ is monotone and } \sqsubseteq_D \subseteq \sqsubseteq \\
& \sqsubseteq & (0, f(y)) & \\
& = & \mathcal{A}_*(f)(z) &
\end{array}
$$

- $v = (0, x)$, $z = (0, y)$, and $x \sqsubseteq_D y$. In this case we have

$$
\begin{array}{rcl}
\mathcal{A}_*(f)(v) & = & (0, f(x)) \\
& \sqsubseteq & (0, f(y)) \\
& = & \mathcal{A}_*(f)(z)
\end{array}
$$

$\mathcal{A}_*(f)$ is continuous. In fact, let Z any directed set in $\mathcal{A}_*(D)$. If $Z^{(2)}$ is empty, then

$$
\begin{array}{rcll}
\mathcal{A}_*(f)(\bigsqcup Z) & = & \mathcal{A}_*(f)(\bigsqcup Z^{(1)}) & \\
& = & f(\bigsqcup^D Z^{(1)}) & \\
& = & \bigsqcup^D \{f(z) \mid z \in Z^{(1)}\} & \text{since } f \text{ is continuous} \\
& = & \bigsqcup \{\mathcal{A}_*(f)(z) \mid z \in Z^{(1)}\} & \text{since } Z^{(1)} \subseteq D \\
& = & \bigsqcup \{\mathcal{A}_*(f)(z) \mid z \in Z\} &
\end{array}
$$

If $Z^{(2)}$ is non-empty, then

$$
\begin{array}{rcll}
\mathcal{A}_*(f)(\bigsqcup Z) & = & \mathcal{A}_*(f)(0, \bigsqcup^D Z^{(2)}) & \text{by } (*\text{-}\bigsqcup\text{-dir}) \\
& = & (0, f(\bigsqcup^D Z^{(2)})) & \text{by definition of } \mathcal{A}_*(f) \\
& = & (0, \bigsqcup^D \{f(y) \mid y \in Z^{(2)}\} & \text{since } f \text{ is continuous} \\
& = & \bigsqcup \{(0, f(y)) \mid y \in Z^{(2)}\} & \text{by } (*\text{-}\bigsqcup) \\
& = & \bigsqcup \{0\} \times (\mathcal{A}_*(f)(Z))^{(2)} & \text{by (b)} \\
& = & (0, \bigsqcup^D (\mathcal{A}_*(f)(Z)^{(2)}) & \\
& = & \bigsqcup (\mathcal{A}_*(f)(Z)) & \text{by } (*\text{-}\bigsqcup\text{-dir})
\end{array}
$$

The proof is complete.□

From previous lemma and the routine check that $\mathcal{A}_*$ commutes with compositions and preserves identities, it follows

Proposition 6.4 $\mathcal{A}_* : \mathbf{ALG}_* \to \mathbf{ALG}_*$ *is a functor.*

7 Domain Equations in $\mathbf{ALG}_*$

In this section we study the existence of solutions of domain equation $X \simeq F(X)$, where F is an endofunctor over $\mathbf{ALG}_*$.

Once we have shown how to solve domain equations, we concentrate on defining the right equation for describing 1/2-PA.

For all the categorical notions involved in this section we refer to [34]. We just recall the definition of $T^E : (\mathbf{ALG}_*^E)^{m+n} \to \mathbf{ALG}_*^E$ starting from $T : (\mathbf{ALG}_*^{op})^m \times \mathbf{ALG}_*^n \to \mathbf{ALG}_*$. Let $D_1 \dots D_m$, $E_1 \dots E_n$, be p-alg's, and $i_s : D_s \to D'_s$, $h_r : E_r \to E'_r$ be ep's, for any $1 \leq s \leq m$, $1 \leq r \leq n$. Then

$$
\begin{aligned}
T^E(D_1, \dots, D_m, E_1, \dots, E_n) \; &= \; T(D_1, \dots, D_m, E_1, \dots, E_n); \\
T^E(i_1, \dots, i_m, h_1, \dots, h_n) \; &= \\
= \; &\langle T(i_1^R, \dots, i_m^R, h_1, \dots, h_n), T(i_1, \dots, i_m, h_1^R, \dots, h_n^R) \rangle
\end{aligned}
$$

For instance, in the case of $\to_*$, $(i \to_*^E h) = \langle (i^R \to_* h), (i \to_* h^R) \rangle$.

Lemma 7.1 *1. $\mathbf{ALG}_*$ is an $\mathbf{O}$-category.*

2. In $\mathbf{ALG}_*$ every ω-chain of embeddings has an $\mathbf{O}$-colimit, i.e. a universal cocone $\langle A, \mu_n \rangle_n$ such that

$$\textit{(O-col)} \quad \bigsqcup_n \mu_n \circ \mu_n^R = Id_A$$

Proof: (1) From e.g. Section 4 of [29], we know that:
- every ascending chain of (tidy) continuous functions is continuous;
- composition of continuous functions is continuous.

In order to conclude that $\mathbf{ALG}_*$ is an $\mathbf{O}$-category it remains to show that an ascending chain of tidy continuous functions is tidy, which is trivial.

(2) Consider an ω-chain of embeddings, $\Gamma = \langle \langle D_n, \delta_n \rangle, i_n \rangle_n$, where $i_n : D_n \to D_{n+1}$. Following the Scott construction of direct limit (see e.g. Lemma 4, Section 4, of [29]), we define $D_\Gamma \subseteq \Pi_n D_n$ by

$$D_\Gamma = \{ \langle x_n \rangle_n \mid \forall n. x_n \in D_n \;\&\; i_n^R(x_{n+1}) = x_n \}$$

D_Γ is a alg with

$$K(D_\Gamma) = \{ \langle i_{nm}(p) \rangle_m \mid p \in K(D_n) \}$$

where $i_{nm} : D_n \to D_m$ is defined as $i_{m-1} \circ \dots i_n$ if $n < m$, as $i_{n-1}^R \circ \dots \circ i_m^R$ if $n > m$, and as Id_{D_n} if $n = m$.

We now have to transform D_Γ into a p-alg, by choosing the special element: we select $\delta_\Gamma = \langle \delta_n \rangle_n$. δ_Γ is in D_Γ, since i_n and i_n^R are hypertidy by Lemma 5.2, so $i_n^R(\delta_{n+1}) = \delta_n$. We make D_Γ a cocone for Γ in the usual way, by defining the tidy ep's $\alpha_n : D_n \to D_\Gamma$:

$$
\begin{aligned}
&\forall x \in D_n. \; \alpha_n(x) = \langle i_{nm}(x_n) \rangle_m \\
&\forall \langle x_m \rangle_m \in D_\Gamma. \; \alpha_n^R(\langle x_m \rangle_m) = x_n
\end{aligned}
$$

(O-col) holds for D_Γ, exaclty with the same proof of Lemma 4, Section 4 of [29]).□

From point 2 of previous lemma, and Theorem 2 of [34], it follows that $\mathbf{ALG}_*$ *locally determines* colimits of ω-chain of embeddings $\langle i_n : D_n \to D_{n+1}\rangle_n$ (see Definition 8 of [34]). So we have

Theorem 7.2 *Let T be a* locally continuous *functor over* $\mathbf{ALG}_*$. *Then*

1. *T^E is ω-continuous.*
2. *The domain equation $X \simeq T(X)$ has solution in* $\mathbf{ALG}_*$.

Proof: (1) Immediate from Theorem 3 of [34].
(2) X can be chosen as the colimit of $\Delta(\langle T, D_0, i_0\rangle)$ (for the proof, see e.g. Section 2 of [34]).□.

Now we concentrate on the two functors $\mathcal{A}_*$ and $\to_*$, and will prove that they are locally continuous.

Proposition 7.3 1. *$\to_*$ is locally continuous.*

2. *$\mathcal{A}_*$ is locally continuous.*

Proof: (1) $\to_*$ is locally continuous by exactly the same proof presented after Definition 4, Section 4, of [29].
(2) $\mathcal{A}_*$ is locally monotonic. If $f, g \in \mathbf{ALG}_*(D, E)$ with $f \sqsubseteq g$, then $\mathcal{A}_*(f) \sqsubseteq \mathcal{A}_*(g)$ follows immediately from $(\star\star)$ of Lemma 6.3. Let $\langle f_n\rangle_n$ be an ascending chain of morphisms $f_n : D \to E$. Let $z \in \mathcal{A}_*(D)$. There are two cases. If $z \in D$, then

$$\begin{array}{rcl} \mathcal{A}_*(\bigsqcup_n (f_n))(z) & = & (\bigsqcup_n f_n)(z) \\ & = & \bigsqcup_n^D f_n(z) \\ & = & \bigsqcup_n^D \mathcal{A}_*(f_n)(z) \\ & = & (\bigsqcup_n \mathcal{A}_*(f_n))(z) \end{array}$$

If $z = (0, y)$, we have

$$\begin{array}{rcll} \mathcal{A}_*(\bigsqcup_n f_n)(z) & = & \mathcal{A}_*(\bigsqcup_n f_n)(0, y) & \\ & = & (0, (\bigsqcup_n f_n)(y)) & \text{by } (\star\star) \\ & = & (0, \bigsqcup_n^D f_n(y)) & \\ & = & \bigsqcup_n (0, f_n(y)) & \text{by } (*\text{-}\bigsqcup) \\ & = & \bigsqcup_n \mathcal{A}_*(f_n)(0, y) & \\ & = & (\bigsqcup_n \mathcal{A}_*(f_n))(0, y) & \end{array}$$

The proof is so complete.□

By Theorem 7.3 we get immediately

Theorem 7.4 *$\mathcal{A}_*^E$ and $\to_*^E$ are ω-continuous functors.*

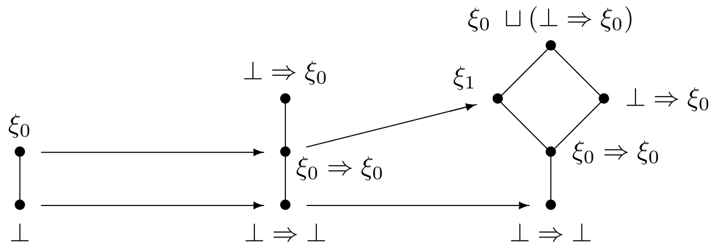

Figure 3: The unique embedding from Ξ_0 to $\mathcal{G}_*(\Xi_0)$.

Consider the functor $\tilde{\mathcal{F}}_* = \rightarrow_* \circ \mathsf{D} : \mathbf{ALG}_* \rightarrow \mathbf{ALG}_*$, where $\mathsf{D} : \mathbf{ALG}_* \rightarrow \mathbf{ALG}_* \times \mathbf{ALG}_*$ is the diagonal functor, namely

$$\begin{array}{rcl} \tilde{\mathcal{F}}_*(\langle D, \delta \rangle) & = & \langle [D \rightarrow D], \delta \Rightarrow \delta \rangle \\ \tilde{\mathcal{F}}_*(\langle i, i^R \rangle) & = & \langle i^R \rightarrow i, i \rightarrow i^R \rangle \end{array}$$

$\tilde{\mathcal{F}}_*$ is locally continuous, since it is obtained as composition of locally continuous functors.

Theorem 7.5 *Let* $\mathcal{G}_* = \mathcal{A}_* \circ \tilde{\mathcal{F}}_* : \mathbf{ALG}_* \rightarrow \mathbf{ALG}_*$. *Then the equation*

$$(\diamondsuit) \qquad X \simeq \mathcal{G}_*(X)$$

has solution in $\mathbf{ALG}_*$.

Proof: $\mathcal{G}_*$ is locally continuous since it is composition of locally continuous functors. The conclusion follows from Theorem 7.2(2).□

Note the action of $\mathcal{G}_*$ on Ξ_0. By applying first $\tilde{\mathcal{F}}_*$ we obtain the domain pictured in the middle of Figure 3; then by applying $\mathcal{A}_*$ we obtain the domain $\mathcal{G}_*(\Xi_0)$ shown in the right of the figure. The unique embedding possible from Ξ_0 to $\mathcal{G}_*$ in $\mathbf{ALG}_*^E$ is obtained by composing the two pairs of arrows in the figure.

One can compare Figure 2 with Figure 3 in order to see that $\mathcal{G}_*$ performs exactly the action required for building $D_?$ out of Ξ_0.

Relying on $\mathcal{G}_*$, we can introduce the colimit which, as we will prove, satisfies ($*$-ch), so providing the required characterization of 1/2-PA.

Let ρ^* be the triple $\langle \mathcal{G}_*, \Xi_0, i_0 \rangle$ of Figure 3, that is $i_0 : \Xi_0 \rightarrow \mathcal{G}_*(\Xi_0)$ is the unique morphism given by

$$\begin{array}{rcl} i_0(\perp) & = & \perp \\ i_0(\xi_0) & = & \xi_1 \end{array}$$

Definition 7.6 *Let* $\mathbb{A} = colim(\rho^*)$ *be the solution of equation* $(\diamondsuit)$. *We call* τ *the mediating isomorphism from* $\mathcal{G}_*(\mathbb{A})$ *to* $\mathbb{A}$.

Theorem 7.7 *$\langle \mathbb{A}, \tau\rangle$ is the initial $\mathcal{G}_*$-algebra in $\mathbf{ALG}_*^{E\perp}$.*

Proof: $\mathcal{G}_*$ is ω-continuous over $\mathbf{ALG}_*^{E\perp}$, by the same proof by which it is ω-continuous over $\mathbf{ALG}_*^{E}$. Moreover, by Lemma 5.3, Ξ_0 is the initial object of $\mathbf{ALG}_*^{E\perp}$. The conclusion follows from Lemma 2 of [34].$\square$

Note that by Lemma 6.1(2), $[\mathbb{A} \to \mathbb{A}]$ embeds into $\mathcal{A}_*(\mathbb{A} \to_* \mathbb{A})$. Since this last domain is exactly $\mathcal{G}_*(\mathbb{A})$ which is isomorphic to $\mathbb{A}$, we conclude, using Theorem 5.7

Theorem 7.8 *$\mathbb{A}$ is a reflexive object, hence a λ-model.*

In view of Theorem 8.10, Theorem 7.8 can be viewed as the semantic counterpart of Theorem 3.15(2).

8 Characterizion of $Flt^{\mathcal{T}_*}$

This section contains the main result of the paper, namely the domain-theoretic characterization of $Flt^{\mathcal{T}_*}$.

Before that, we have to pave the way with some technical results on $\mathcal{T}_*$.

Definition 8.1 *The subset $\mathbb{P} \subseteq \mathbb{T}_*$ of* prime *types*[2] *is defined by the following abstract syntax:*

$$\mathbb{P} = \phi \mid (\mathbb{P} \cap \ldots \cap \mathbb{P}) \to \mathbb{P}$$

(intersection may be empty).

Prime types are ranged over by P, Q, R.

We define two functions: the *prime decomposition* of types $dec : \mathbb{T}_* \to \mathcal{P}_{fin}(\mathbb{P})$ and the *canonical forms* of types $(\_)^\sharp : \mathbb{T}_* \to \mathbb{T}_*$. They are defined by mutual induction on the construction of types.

$$\begin{array}{lcl}
A^\sharp & = & \bigcap\{P \mid P \in dec(A)\} \\
dec(\Omega) & = & \emptyset \\
dec(\phi) & = & \{\phi\} \\
dec(A \cap B) & = & dec(A) \cup dec(B) \\
dec(A \to B) & = & \{A^\sharp \to P \mid P \in dec(B)\}
\end{array}$$

By induction on the construction of A, it follows that $dec(A)$ is a finite set.

Lemma 8.2 *For any $A \in \mathbb{T}_*$ we have:*

1. *$dec(A) \subseteq \mathbb{P}$*

2. *$A \sim A^\sharp$.*

[2] Prime types are more commonly known with the name of *strict* types, see e.g. [10].

Proof: (1) By induction on definition of *dec*. We give the proof for arrow types $A \to B$: $A^\sharp$ is by induction an intersection of prime types, so $A^\sharp \to P$, for $P \in dec(B)$, is a prime type.

(2) By induction on the construction of the type. For Ω and ϕ the thesis is trivial. For intersections $A \cap B$, it follows from definition of $(\_)^\sharp$ that $(A \cap B)^\sharp \leq A^\sharp$, and $(A \cap B)^\sharp \leq B^\sharp$. By induction $A^\sharp \sim A$ and $B^\sharp \sim B$, hence $(A \cap B)^\sharp \leq A \cap B$. With a similar argument, we get $A \cap B \leq (A \cap B)^\sharp$. Finally, for arrow types:

$$
\begin{array}{rcll}
(A \to B)^\sharp & \sim & \bigcap\{A^\sharp \to P \mid P \in dec(B)\} & \\
& \sim & \bigcap\{A \to P \mid P \in dec(B)\} & \text{by induction and } (\eta) \\
& \sim & A \to (\bigcap\{P \mid P \in dec(B)\}) & \text{by Lemma 3.2(1)} \\
& \sim & A \to B^\sharp & \text{by definition of } \sharp \\
& \sim & A \to B & \text{by induction } \square
\end{array}
$$

Lemma 8.3 *Let $A, B \in \mathbb{T}_*$. $A \leq B$ if and only if for any $B' \in dec(B)$ there exists $A' \in dec(A)$ such that $A' \leq B'$.*

Proof: By induction on the derivation of the judgment.$\square$

Lemmas 8.2 and 8.3 allow to show that $Flt^{\mathcal{T}_*}$ is a prime alg.

Proposition 8.4 *$Flt^{\mathcal{T}_*}$ is a prime alg, with*

$$Pr(Flt^{\mathcal{T}_*}) \quad = \quad \{\uparrow P \mid P \in \mathbb{P}\}$$

Proof: First of all, each $\uparrow P$, with $P \in \mathbb{P}$ is prime element. In fact, let $\uparrow P \subseteq \bigsqcup_{i \in \mathcal{I}} x_i$. Then, by (fil-sup), there exists a finite subset $I \subseteq \mathcal{I}$, and, for any $i \in I$, types $A_i \in x_i$, such that $\bigcap_{i \in I} A_i \leq P$. Let $B = \bigcap_{i \in I} A_i$. By Lemma 8.3, there exists $A' \in dec(B)$ such that $A' \leq P$. Since $dec(B) = \bigcup_{i \in I} dec(A_i)$, it follows that there exists $i' \in I$ such that $A' \in dec(A_{i'})$. Since $A_{i'} \leq A'$, we get $A_{i'} \leq P$ by (trans), hence $P \in x_{i'}$, proving that $\uparrow P$ is prime.
On the other hand, for any $\uparrow A$, we have

$$
\begin{array}{rcll}
\uparrow A & = & \uparrow A^\sharp & \text{by Lemma 8.2(2)} \\
& = & \uparrow \bigcap\{P \mid P \in dec(A)\} & \text{by definition of } (\_)^\sharp \\
& = & \bigsqcup\{\uparrow P \mid P \in dec(A)\} & \text{by (fil-sup)}
\end{array}
$$

If $\uparrow A$ is prime then, by above, there exists $P \in dec(A)$ such that $\uparrow A = \uparrow P$. By Lemma 8.2(1) we have $P \in \mathbb{P}$. So we have proved that every prime element in $Flt^{\mathcal{T}_*}$ have the shape $\uparrow P$, for some $P \in \mathbb{P}$.
Finally we prove that for any filter x, $x = \bigsqcup\{\uparrow P \mid P \in x\}$. ($\supseteq$) is trivial. As concerns ($\subseteq$), let $A \in x$. By Lemma 8.2 (use both points of it), it follows that $A \sim \bigcap_{i \in I} P_i$ for suitable prime types P_i, which implies both $A \in \bigsqcup_{i \in I} \uparrow P_i$ (by (fil-sup)) and $P_i \in x$ (since x is upward closed). Therefore it follows $A \in \bigsqcup\{\uparrow P \mid P \in x\}$.$\square$

The fact that $Flt^{\mathcal{T}_*}$ is a prime alg will allow us to use Lemma 2.4 for defining ep's.

We now turn $Flt^{\mathcal{T}_*}$ into a prime p-alg.

Definition 8.5 *As an object of* $\mathbf{ALG}_*$, *we define* $Flt^{\mathcal{T}_*} = \langle Flt^{\mathcal{T}_*}, \uparrow \phi\rangle$.

We are now in position for turning $Flt^{\mathcal{T}_*}$ into a $\mathcal{G}_*$-algebra in $\mathbf{ALG}_*^E$, by defining a suitable morphism $\theta : \mathcal{G}_*(Flt^{\mathcal{T}_*}) \to Flt^{\mathcal{T}_*}$.

In what follows, we write ξ instead of $\xi_{[Flt^{\mathcal{T}_*} \to Flt^{\mathcal{T}_*}]}$. By Lemma 2.3 and Lemma 6.1(3), we have

$$Pr(\mathcal{G}_*(Flt^{\mathcal{T}_*})) \quad = \quad \{\xi\} \cup \{\uparrow A \Rightarrow \uparrow P \mid A \in \mathbb{T}(\mathbb{C}), P \in \mathbb{P}\}$$

In order to define θ, first we define $\theta^- : Pr(\mathcal{G}_*(Flt^{\mathcal{T}_*})) \to Pr(Flt^{\mathcal{T}_*})$, and shows that it satisfies the condition (p-refl) of Lemma 2.4. Then we will define its extension θ. We define

$$\begin{aligned} \theta^-(\xi) \quad &= \quad \uparrow \phi \\ \theta^-(\uparrow A \Rightarrow \uparrow P) \quad &= \quad \uparrow (A \to P) \end{aligned}$$

Lemma 8.6 $\theta^- : Pr(\mathcal{G}_*(Flt^{\mathcal{T}_*})) \to Pr(Flt^{\mathcal{T}_*})$ *satisfies (p-refl).*

Proof: We prove that θ^- is monotone. If $\uparrow A \Rightarrow \uparrow P \sqsubseteq \uparrow A' \Rightarrow \uparrow P'$, then $[\uparrow A' \subseteq \uparrow A$ and $\uparrow P \subseteq \uparrow P']$, which is equivalent to $[A \leq A'$ and $P' \leq P]$. By rule (η), we have $A' \to P' \leq A \to P$, which proves monotonicity in this case. If $\uparrow A \Rightarrow \uparrow P \sqsubseteq \xi$, then it is the case $\uparrow A \Rightarrow \uparrow P \sqsubseteq \uparrow \phi \Rightarrow \uparrow \phi$ (which is the special element in $[Flt^{\mathcal{T}_*} \to Flt^{\mathcal{T}_*}]$). This implies $[\uparrow \phi \subseteq \uparrow A$ and $\uparrow P \subseteq \uparrow \phi]$ which is equivalent to $[A \leq \phi$ and $\phi \leq P]$. By applying rule (η) we have $\phi \to \phi \leq A \to P$, and by (1/2-Pa) and (trans) we get $\phi \leq A \to P$, which implies monotonicity of θ^- also in this case.

Finally we prove that θ^- reflects order: if $\theta^-(v) \subseteq \theta^-(z)$, then $v \sqsubseteq z$. Let $v, z \in Pr(Flt^{\mathcal{T}_*})$ and let $\theta^-(v) \subseteq \theta^-(z)$. If $v = \uparrow A \Rightarrow \uparrow P$ and $z = \uparrow A' \Rightarrow \uparrow P'$, then by definition of θ^- it follows $\uparrow (A \to P) \subseteq \uparrow (A' \to P')$, which is equivalent to $A' \to P' \leq A \to P$. Since $\mathcal{T}_*$ is legal, we have $[A \leq A'$ and $P' \leq P]$. This is equivalent to $[\uparrow A' \subseteq \uparrow A$ and $\uparrow P \subseteq \uparrow P']$, which imples $\uparrow A \Rightarrow \uparrow P \sqsubseteq \uparrow A' \Rightarrow \uparrow P'$, that is $v \sqsubseteq z$.
The other case is $v = \uparrow A \Rightarrow \uparrow P$ and $z = \xi$. In such a case $\theta^-(v) \subseteq \theta^-(z)$ corresponds to $\uparrow (A \to P) \subseteq \uparrow \phi$, that is $\phi \leq (A \to P)$. By Lemma 3.16 we have that $[A \leq \phi$ and $\phi \leq P]$, which implies $\uparrow A \Rightarrow \uparrow P \sqsubseteq \uparrow \phi \Rightarrow \uparrow \phi$. Since $\xi \sqsupseteq \uparrow \phi \Rightarrow \uparrow \phi$, we get $\uparrow A \Rightarrow \uparrow P \sqsubseteq \xi$, that is $v \sqsubseteq z$.□

We can now define θ.

Lemma 8.7 *Define* $\theta : \mathcal{G}_*(Flt^{\mathcal{T}_*}) \to Flt^{\mathcal{T}_*}$ *by*

$$\theta(x) \quad = \quad \bigcup \{\theta^-(p) \mid p \in Pr(\mathcal{G}_*(Flt^{\mathcal{T}_*})) \ \& \ p \sqsubseteq x\}$$

θ *induces an ep* $\langle \theta, \theta^R \rangle : \mathcal{G}_*(Flt^{\mathcal{T}_*}) \to Flt^{\mathcal{T}_*}$.

Proof: By Lemma 8.6, θ^- satisfies (p-refl). Therefore we can apply Lemma 2.4 and conclude that the extension θ of θ^- is an ep.□

Proposition 8.8 $\langle Flt^{\mathcal{T}_*}, \theta\rangle$ *is a* $\mathcal{G}_*$*-algebra in* $\mathbf{ALG}_*^E$.

Proof: After Lemma 8.7, in order to prove that θ is a morphism in $\mathbf{ALG}_*^E$, we are left to show that it is hypertidy. This follows from the definition of θ, since $\theta(\xi) = \theta^-(\xi) = \uparrow \phi$.□

Before giving the the semantic characterization of $Flt^{\mathcal{T}_*}$, we recall that a morphism of $\mathcal{G}_*$-algebras $\mu : \mathcal{G}_*(C) \to C$ and $\nu : \mathcal{G}_*(D) \to D$ is a ep $i : C \to D$ such that

$$\nu \circ \mathcal{G}_*(i) \quad = \quad i \circ \mu$$

We will strengthen Proposition 8.8 and show that $Flt^{\mathcal{T}_*}$ is the initial $\mathcal{G}_*$-algebra.

Theorem 8.9 $\langle Flt^{\mathcal{T}_*}, \theta\rangle$ *is an initial* $\mathcal{G}_*$*-algebra in* $\mathbf{ALG}_*^{E\perp}$.

Proof: Let $B = \langle B, \beta\rangle$ be a p-alg with $\beta \neq \perp$, and let $\psi : \mathcal{G}_*(B) \to B$ be a $\mathcal{G}_*$-algebra in $\mathbf{ALG}_*^{E\perp}$ (hence ψ is an ep and satisfies conditions of Lemma 2.2(3)). In order to define $\rho : Flt^{\mathcal{T}_*} \to B$, we first define $\rho^- : Pr(Flt^{\mathcal{T}_*}) \to K(B)$, using the characterization of prime elements given in Proposition 8.4. We proceed by induction on the number of arrows in the type $P \in \mathbb{P}$:

$$\begin{array}{rcl} \rho^-(\uparrow \phi) & = & \beta \\ \rho^-(\uparrow (A \to P)) & = & \psi(\rho^-(\uparrow A) \Rightarrow \rho^-(\uparrow P)) \end{array}$$

where $\rho^-(\uparrow A) = \bigsqcup\{\rho^-(\uparrow Q) \mid Q \in dec(A)\}$.

We now prove that ρ^- is monotone. Let $\uparrow R_1 \subseteq \uparrow R_2$, with $R_1, R_2 \in \mathbb{P}$. We will prove that $\rho^-(\uparrow R_1) \sqsubseteq \rho^-(\uparrow R_2)$ by induction on the number n of arrows in R_1. If $n = 0$, we are in the trivial case $R_1 \equiv R_2 \equiv \phi$. Otherwise we have $R_1 \equiv A \to P$ and two cases are possible.

- $R_2 = \phi$. In such a case, by Lemma 3.16, we have $A \leq \phi$ and $\phi \leq P$. By Lemma 3.11, it follows that $\phi \in dec(A)$, hence (a) $\rho^-(\uparrow \phi) \sqsubseteq \rho^-(\uparrow A)$. From $\phi \leq P$ we derive by induction (b) $\rho^-(\uparrow P) \sqsubseteq \rho^-(\uparrow \phi)$. From (a) and (b) it follows (c) $\rho^-(\uparrow A) \Rightarrow \rho^-(\uparrow P) \sqsubseteq \rho^-(\uparrow \phi) \Rightarrow \rho^-(\uparrow \phi)$. We have

$$\begin{array}{rcll} \rho^-(\uparrow (A \to P)) & = & \psi(\rho^-(\uparrow A) \Rightarrow \rho^-(\uparrow P)) & \\ & \sqsubseteq & \psi(\rho^-(\uparrow \phi) \Rightarrow \rho^-(\uparrow \phi)) & \text{by (c), since } \psi \text{ is monotone} \\ & = & \psi(\beta \Rightarrow \beta) & \\ & \sqsubseteq & \psi(\xi_{[B \to B]}) & \\ & = & \beta & \text{since } \psi \text{ is hypertidy} \\ & = & \rho^-(\uparrow \phi) & \end{array}$$

- $R_2 = A' \to P'$. In such a case, $\uparrow R_1 \subseteq \uparrow R_2$ implies, by Lemma 3.15(1), $A \leq A'$ and $P' \leq P$. By Lemma 8.3, for any $Q' \in dec(A')$, there exists $Q \in dec(A)$ such that $Q \leq Q'$. By induction we have: for any $Q' \in dec(A')$, there exists $Q \in dec(A)$ such that $\rho^-(\uparrow Q') \sqsubseteq \rho^-(\uparrow Q)$, which implies (a')

$\rho^-(\uparrow A') \sqsubseteq \rho^-(\uparrow A)$. Similarly, by induction we obtain (b') $\rho^-(\uparrow P) \sqsubseteq \rho^-(\uparrow P')$. (a') and (b') imply (c') $\rho^-(\uparrow A) \Rightarrow \rho^-(\uparrow P) \sqsubseteq \rho^-(\uparrow A') \Rightarrow \rho^-(\uparrow P')$. We have

$$\begin{array}{rcll} \rho^-(\uparrow (A \to P)) & = & \psi(\rho^-(\uparrow A) \Rightarrow \rho^-(\uparrow P)) & \\ & \sqsubseteq & \psi(\rho^-(\uparrow A') \Rightarrow \rho^-(\uparrow P')) & \text{by (c'), as } \psi \text{ is monotone} \\ & = & \rho^-(\uparrow (A' \to P')) & \end{array}$$

The proof that ρ^- is monotone is so complete.

The proof that $\rho^-(\uparrow R_1) \sqsubseteq \rho^-(\uparrow R_2)$ implies $\uparrow R_1 \subseteq \uparrow R_2$ descends on the more general result

$$\forall A, B \in \mathbb{T}_*. \rho^-(\uparrow A) \sqsubseteq \rho^-(\uparrow B) \quad \Rightarrow \quad \uparrow A \subseteq \uparrow B$$

which can be proven by induction on types: the technique is quite similar to that used in the previous proof of monotonicity and is omitted; in the basis case of atoms, use the fact that β is different from $\perp$. Therefore we have that ρ^- is well-defined and satisfies (p-refl).

We can now apply Lemma 2.4 and extend ρ^- to an ep

$$\rho : Flt^{\mathcal{T}_*} \to B$$

We prove that ρ is a morphism of $\mathcal{G}_*$-algebras, that is

$$\text{(alg-comm)} \quad \rho \circ \theta = \psi \circ \mathcal{G}_*(\rho)$$

Since all the involved functions are additive and the domain is a prime alg, it is enough to prove that (alg-comm) holds for prime elements z in $Pr(\mathcal{G}_*(Flt^{\mathcal{T}_*}))$. We have two cases to consider. If $z = \xi$, then it is trivial that $\rho(\theta(\xi)) = \psi(\mathcal{G}_*(\rho)(\xi))$, since the special elements are preserved. Otherwise $z = \uparrow A \Rightarrow \uparrow P$, with $A \in \mathbb{T}_*$ and $P \in \mathbb{P}$. We have

$$\begin{array}{rcll} \rho(\theta(\uparrow A \Rightarrow \uparrow P)) & = & \rho(\uparrow (A \to B)) & \text{by definition of } \theta \\ & = & \psi(\rho(\uparrow A) \Rightarrow \rho(\uparrow P)) & \text{by definition of } \rho \\ & = & \psi(\mathcal{G}_*(\rho)(\uparrow A \Rightarrow \uparrow P)) & \text{by definition of } \mathcal{G}_* \end{array}$$

Therefore (alg-comm) holds.

Finally, (alg-comm) forces to define ρ inductively (on the number of arrows in the type) in a unique way. In fact $\uparrow \phi$, as the special element, must be sent to β. Any other prime element $\uparrow (A \to P)$ must be sent to $\psi(\rho(\uparrow A) \Rightarrow \rho(\uparrow P))$ in order to respect (alg-comm). So the action of ρ on prime elements is uniquely determined, and being ρ additive it follows that it is uniquely determined also over $Flt^{\mathcal{T}_*}$.□

Note that as a consequence of the previous theorem and Lemma 1 of [34], θ is an isomorphism.

We can now give the complete characterization of the 1/2 Park model $Flt^{\mathcal{T}_*}$.

Theorem 8.10 *$\langle \mathbb{A}, \tau \rangle$ and $\langle Flt^{\mathcal{T}_*}, \theta \rangle$ are isomorphic.*

Proof: The thesis follows by unicity of initial $\mathcal{G}_*$-algebras, since by Theorem 7.7 and Theorem 8.9 both $\langle \mathbb{A}, \tau \rangle$ and $\langle Flt^{\mathcal{T}_*}, \theta \rangle$ are initial $\mathcal{G}_*$-algebras in $\mathbf{ALG}_*^{E\perp}$.□

9 Conclusions

Although the main result of the paper is rather specific, nevertheless there are some interesting points/questions, so that the paper is without doubt just a starting point.

Here is a list of question related to 1/2 Park model.

- First of all a fundamental question: is it possible to describe properly $Flt^{\mathcal{T}_*}$ working in **ALG**?
- The second question concerns the relation between irregular filter structures and colimits: a categorical characterization as colimits of regular ITT's works inside **ALG**. With irregular ITT's, the problem seems to grow complicate, since, varying the axioms, it is not known in general which is the right category which allows a proper characterization

  $$\text{(char)} \quad Flt^{\mathcal{T}} \simeq colim(\rho)$$

  We found it (namely $\mathbf{ALG}_*$) in the very particular case of (1/2-pa). Is there a general technique which, starting from the axioms of an irregular ITT $\mathcal{T}$, allows to find the right category, along with the right functor, in order to get characterization (char) for $\mathcal{T}$?
- As an instance of the general problem above, the following question arises: which is the right category where to study of (1/2-sc), that is the "1/2-Scott" irregular ITT generated over $\mathbb{C}_*$ by the axiom $\phi \leq \Omega \to \phi$? Are we forced to find a new category different from $\mathbf{ALG}_*$? (It seems to the author that $\mathbf{ALG}_*$ is of no use for characterizing for (1/2-sc)).
- Similarly for the "second half" (pa-1/2) of axiom (pa), namely $\phi \to \phi \leq \phi$: which is the right category and the right functor in order to recover the colimit characterization of the filter structure induced by (pa-1/2)?
- **ALG** can be faithfully and fully embedded into $\mathbf{ALG}_*$ through the functor $\mathcal{I}$. Is there any advantage in developing domain theory in $\mathbf{ALG}_*$? Besides $\mathcal{G}_*$, are there other functors of some interest which exist over $\mathbf{ALG}_*$ but cannot be defined over **ALG**?
- How is the λ-theory induced by $Flt^{\mathcal{T}_*}$?

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References

[1] S. Abramsky. Domain theory in logical form. *Ann. Pure Appl. Logic*, 51(1):1–77, 1991.

[2] S. Abramsky. A domain equation for bisimulation. *Inf. and Comp.*, 92(2):161–218, 1991.

[3] S. Abramsky and L.C. Ong. Full abstraction in the Lazy Lambda Calculus. *Information and Computation*, 105: 159–267, 1993.

[4] J. Adamek, H. Herrlich and G.E. Strecker. *Abstract and Concrete Categories: The Joy of Cats* . Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts, 1990. ftp-available at http://katmat.math.uni-bremen.de/acc/.

[5] F. Alessi. Strutture di tipi, teoria dei domini, e modelli del λ-calcolo. PhD Thesis. University of Turin, 1991.

[6] F. Alessi. Type preorders. In *Trees in algebra and Programming. CAAP '94 (Edinburgh)*, Volume 787 of LNCS, pages 37–51. Springer Verlag, 1994.

[7] F. Alessi, M. Dezani-Ciancaglini and F. Honsell. Filter models and easy terms. In A. Restivo, S. Ronchi Della Rocca, and L. Roversi, editors, *ICTCS '01*, volume 2202 of *LNCS*, pages 17–37. Springer-Verlag, 2001.

[8] F. Alessi, M. Dezani-Ciancaglini and S. Lusin. Intersection types and domain operators. *Theoret. Comput. Sci.*, 316(1-3):25–47, 2004.

[9] A. Asperti Categorical topics in Computer Science. Phd Thesis. University of Pisa, 1990.

[10] S. van Bakel. Cut-elimination in the strict intersection type assignment system is strongly normalising. *Notre Dame Journal of Formal Logic*, 45(1), 2004.

[11] H.P. Barendregt *The Lambda Calculus: Its syntax and semantics*. North-Holland Publishing co., Amsterdam, 1984.

[12] H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *J. Symbolic Logic*, 48(4):931–940, 1983.

[13] C. Berline. From computation to foundations via functions and application: the λ-calculus and its webbed models. *Theoret. Comput. Sci.*, 249:81–161, 2000.

[14] M. Coppo and M. Dezani-Ciancaglini. An extension of the basic functionality theory for the λ-calculus. *Notre Dame J. Formal Logic*, 21(4):685–693, 1980.

[15] M. Coppo, M. Dezani-Ciancaglini and B. Venneri. Principal type schemes and λ-calculus semantics. In *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, pages 535–560. Academic Press, London, 1980.

[16] M. Coppo, M. Dezani-Ciancaglini, F. Honsell, and G. Longo. Extended type structures and filter lambda models. In *Logic Colloquium '82*, pages 241–262. North-Holland, 1984.

[17] M. Coppo, M. Dezani-Ciancaglini, and M. Zacchi. Type theories, normal forms and D_∞ lambda models. *Information and Compuation*, 72(2):85–116, 1987.

[18] M. Dezani-Ciancaglini, S. Ghilezan, and S. Likavec. Behavioural inverse limit models. *Theoret. Comput. Sci.*, 316(1–3):49–74, 2004.

[19] M.Dezani-Ciancaglini, F.Honsell and F.Alessi. A complete characterization of complete intersection-type preorders. *ACM TOCL*, 4(1):120–146, 2003.

[20] M. Dezani-Ciancaglini, F. Honsell, and Y. Motohama. Compositional characterization of λ-terms using intersection types. *Theoret. Comput. Sci.*, 304(3):459–495, 2005.

[21] L. Egidi, F. Honsell, and S. Ronchi della Rocca. Operational, denotational and logical descriptions: a case study. *Fund. Inform.*, 16:149–169, 1992.

[22] E. Engeler. Algebras and combinators. *Algebra Universalis*, 13(3):389–392, 1981.

[23] Gerhard K. Gierz, Karl Heinrich Hofmann, Klaus Keimel, Lawson Jimmie D., Michael W. Mislove, and Dana S. Scott. *Continuous lattices and domains*. Cambridge University Press, 2003.

[24] R. Hindley and G. Longo. Lambda calculus models and extensionality. *Z. Math Logik Grundlag. Math*, 26(4):289–310, 1980.

[25] F. Honsell and M. Lenisa. Semantical analysis of perpetual strategies in λ-calculus. *Theoret. Comput. Sci.*, 212(1-2):183–209, 1999.

[26] F. Honsell and S. Ronchi Della Rocca. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *J. Comput. System Sci.*, 45(1):49–75, 1992.

[27] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 1971.

[28] D. Park. The Y-combinator in Scott's λ-calculus models (revised version). Theory of Computation Report 13, Department of Computer Science, University of Warick, 1976.

[29] G. Plotkin. Pisa Notes (on domain theory). Available at homepages.inf.ed.ac.uk/gdp/publications/Domains.ps University of Edinburgh, 1983.

[30] S. Ronchi della Rocca and L. Paolini. *The Parametric Lambda Calculus. A Metamodel for Computation.* Texts in Theoretical Computer Science. An EACTS Series. Springer-Verlag, 2004.

[31] D. Scott. Data types as lattices. *SIAM J. Comput.*, 5(3):522–587, 1976. Semantics and correctness of programs.

[32] D.S. Scott. Continuous lattices. In *Toposes, algebraic geometry and logic*, volume 274 of *LNM*, pages 97–136. Springer-Verlag, 1972.

[33] D.S. Scott. Domains for Denotational Semantics. in *Automata, languages and programming (Aarhus, 1982)*, pages 577–613. Springer-Verlag, 1982.

[34] Michael B. Smyth and Gordon D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal on Computing*, 11(4):761–783, November 1982.