Filter Models and Easy Terms ⋆

F.Alessi[1], M. Dezani-Ciancaglini[2], F. Honsell[1]

[1] Dipartimento di Matematica ed Informatica, Università di Udine,
Via delle Scienze 208, 33100 Udine, Italy
{alessi, honsell}@dimi.uniud.it
[2] Dipartimento di Informatica, Università di Torino,
Corso Svizzera 185, 10149 Torino, Italy
dezani@di.unito.it

Abstract. We illustrate the use of intersection types as a tool for synthesizing λ-models which exhibit special purpose features. We focus on semantical proofs of *easiness*. This allows us to prove that the class of λ-theories induced by *graph models* is strictly included in the class of λ-theories induced by *non-extensional filter models.*

Introduction

Intersection types were introduced in the late 70's by Dezani and Coppo [10, 12, 6], to overcome the limitations of Curry's type discipline. They are a very expressive type language which allows to describe and capture various properties of λ-terms. For instance, they have been used in [29] to give the first type theoretic characterization of *strongly normalizable* terms and in [13] to capture *persistently normalizing terms* and *normalizing terms*. See [15] for a more complete account of this line of research.

Intersection types have a very significant realizability semantics with respect to applicative structures. This is a generalization of Scott's natural semantics [31] of simple types. According to this interpretation types denote subsets of the applicative structure, an arrow type $A \to B$ denotes the sets of points which map all points belonging to the interpretation of A to points belonging to the interpretation of B, and an intersection type $A \cap B$ denotes the intersections of the interpretation of A and the interpretation of B. Building on this, intersection types have been used in [6] to give a proof of the completeness of the natural semantics of Curry's simple type assignment system in applicative structures, introduced in [31]. See [14] for a more complete treatment of completeness of intersection type assignment systems.

But intersection types have also an alternative semantics based on *duality* which is related to Abramsky's *Domain Theory in Logical Form* [1]. Actually it amounts to the application of that paradigm to the special case of ω-algebraic complete lattice models of pure lambda calculus, [11]. Namely, types correspond

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to *compact elements*: the type Ω denoting the least element, intersections denoting *joins* of compact elements, and arrow types denoting *step functions* of compact elements. A typing judgment then can be interpreted as saying that a given term belongs to a pointed compact open set in a ω-algebraic complete lattice model of λ-calculus. By duality, type theories give rise to *filter λ-models*. Intersection type assignment systems can then be viewed as *finitary logical* definitions of the interpretation of λ-terms in such models, where the meaning of a λ-term is the set of types which are deducible for it.

This duality lies at the heart of the success of intersection types as a powerful tool for the analysis of λ-models, see *e.g.* [2, 6, 11, 13, 3, 17, 21, 16, 28, 19, 30].

In this paper we illustrate the use of intersection types as a tool for synthesizing λ-models which exhibit special purpose features. For building our models we will introduce a strengthened version of intersection type theories, namely the *easy* ones. We focus on semantical proofs of *easiness* [23], [5] (Definition 15.3.8) (we recall that a closed term P is *easy* if, for any other closed term M, the theory $\lambda\beta + \{M = P\}$ is consistent). More specifically we will consider the terms $\omega_2\omega_2$ and $\omega_3\omega_3\mathsf{I}$ where $\omega_2 \equiv \lambda x.xx$, $\omega_3 \equiv \lambda x.xxx$ and I is the identity combinator. Let P be $\omega_2\omega_2$ or $\omega_3\omega_3\mathsf{I}$. For any closed term M we will build a non-trivial filter model (that is a non-trivial λ-model built on intersection type theories) where the interpretations of M and P coincide. From this fact it follows that the theory $\lambda\beta + \{P = M\}$ is consistent, hence P is easy. A feature of the present model construction is that with very small changes we can show the consistency of $\lambda\beta\eta + \{P = M\}$ for $P \equiv \omega_2\omega_2$ and $P \equiv \omega_3\omega_3\mathsf{I}$.

The easiness of both $\omega_2\omega_2$ (see [23], [26]) and $\omega_3\omega_3\mathsf{I}$ (see [24], [9]) have been shown by syntactic arguments. These and other easiness results have been mainly obtained either using the "Jacopini technique" [23], [27] (Section 4.4.4), [26], or using Church-Rosser relations which extend $\lambda\beta$ [22], [8], [9]. Actually a semantic proof of the easiness of $\omega_2\omega_2$ has already appeared in the literature [4] with a proof based on non-standard $\mathcal{P}(\omega)$ models. Moreover [20] builds extensional filter models equating $\omega_2\omega_2$ to arbitrary closed terms. See also [33].

One of the by-product of this paper is that it provides a negative answer to an interesting question, namely whether the class of λ-theories induced by *graph models*, (e.g. Scott, Park and Engeler models are instances of graph models), coincide with the class of λ-theories induced by *non-extensional filter models* as defined in [16]. In [25] it was shown that the equation of $\omega_3\omega_3\mathsf{I} = \mathsf{I}$ cannot be proved in any graph model, whilst our paper shows that this is possible with non-extensional filter models. Hence the two mentioned classes differ (actually [16] shows that graph model λ-theories are included in non-extensional filter model λ-theories).

It is an open question of this paper to single out which classes of λ-terms can be proved easy using filter models as semantical tools.

Also a "philosophic" question arises: a part from classical semantics tools, such as Fixed Point Induction, what do we gain when we have an ordered cpo model for a λ-theory? At present, we have not a clear cut answer to such question.

The present paper is organized as follows. In Section 1 we present easy intersection type theories and type assignment systems for them. We prove some meta-theoretic properties including a Generation Theorem. In Section 2 we introduce λ-models based on spaces of filters in easy intersection type theories. Sections 3 and 4 exhibit the mentioned models which allow to prove easiness of $\omega_2\omega_2$ and $\omega_3\omega_3\mathsf{I}$. Section 5 discusses the extensional versions of the above models.

1 Intersection Type Assignment Systems

Intersection types are syntactical objects built inductively by closing a given set $\mathbb{C}$ of *type atoms* (constants) under the *function type* constructor $\to$ and the *intersection* type constructor $\cap$. In this paper we only need to consider intersection types which contain the universal type Ω and possibly the "isolated" type ι in their set of atoms.

Definition 1 (Intersection Type Language).
Let $\mathbb{C}$ be a countable set of constants such that $\Omega \in \mathbb{C}$. The Ω-intersection type language *over $\mathbb{C}$, denoted by $\mathbb{T} = \mathbb{T}(\mathbb{C})$ is defined by the following abstract syntax:*

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}. \ \square$$

Notice that the most general form of an intersection type is a finite intersection of arrow types and type constants.

Notation. Upper case Roman letters, i.e. $A, B, \ldots$, will denote arbitrary types. Greek letters will denote constants in $\mathbb{C}$. When writing intersection types we shall use the following convention: the constructor $\cap$ takes precedence over the constructor $\to$ and it associates to the right.

Much of the expressive power of intersection type disciplines comes from the fact that types can be endowed with a *preorder relation* $\leq$, which induces the structure of a meet semi-lattice with respect to $\cap$, the top element being Ω.

The notion we introduce of *easy* intersection type theory is new in the literature. It is tailored in order both to include the type theories of Sections 3, 4, 5 and to have easier proofs of the following Theorems 1, 2, 3. We refer the interested reader to [7] for the general definition of intersection type theory.

$$\begin{array}{llll}
(\text{refl}) & A \leq A & (\text{idem}) & A \leq A \cap A \\
(\text{incl}_L) & A \cap B \leq A & (\text{incl}_R) & A \cap B \leq B \\
(\text{mon}) & \dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'} & (\text{trans}) & \dfrac{A \leq B \quad B \leq C}{A \leq C} \\
(\Omega) & A \leq \Omega & (\Omega\text{-}\eta) & \Omega \leq \Omega \to \Omega \\
(\to\text{-}\cap) & (A \to B) \cap (A \to C) \leq A \to B \cap C & (\eta) & \dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}
\end{array}$$

Fig. 1. The set ∇_0 of axioms and rules

Definition 2 (Easy intersection type theories).
Let $\mathbb{T} = \mathbb{T}(\mathbb{C})$ be an Ω-intersection type language. The easy intersection type theory *(*eitt *for short) $\Sigma(\mathbb{C}, \bigtriangledown)$ over $\mathbb{T}$ is the set of all judgments $A \leq B$ derivable from $\bigtriangledown$, where $\bigtriangledown$ is a collection of axioms and rules such that (we write $A \sim B$ for $A \leq B$ & $B \leq A$):*

1. *$\bigtriangledown$ contains the set $\bigtriangledown_0$ of axioms and rules of Figure 1;*
2. *further axioms can be of the following three shapes only:*

$$\psi \leq \psi', \psi \sim (\phi \to A), \psi \sim (\phi \to \zeta) \cap (\xi \to A),$$

   where $\psi, \psi', \phi, \zeta, \xi \in \mathbb{C} \setminus \{\iota\}$, $A \in \mathbb{T}$, and $\psi, \psi' \not\equiv \Omega$;
3. *$\bigtriangledown$ does not contain further rules;*
4. *for each $\psi \not\equiv \Omega, \iota$ there is exactly one axiom in $\bigtriangledown$ of the shape $\psi \sim A$;*
5. *let $\bigtriangledown$ contain $\psi \sim A$ and $\psi' \sim A'$ where either $A \equiv \phi \to \zeta$ or $A \equiv (\phi \to \zeta) \cap (\xi \to B)$. Then $\bigtriangledown$ contains also $\psi \leq \psi'$ iff $A' \equiv \phi' \to \zeta'$ and $\bigtriangledown$ contains both $\phi' \leq \phi$ and $\zeta \leq \zeta'$.*

Notice that:
(a) since $\Omega \sim \Omega \to \Omega \in \Sigma(\mathbb{C}, \bigtriangledown)$ by axioms (Ω) and (Ω-η), it follows that all atoms in $\mathbb{C}$ different from ι are equivalent to suitable (intersections of) arrow types;
(b) $\bigtriangledown$ cannot contain axioms of the shape $\iota \leq A$ or $A \leq \iota$: this justifies the label "isolated" for ι;
(c) associativity and commutativity of $\cap$ (modulo $\sim$) follow easily from the axioms and rules of $\bigtriangledown_0$ as defined in Definition 2.

A consequence of (a) is that an eitt induces an extensional λ-model iff its set of constants does not contain ι: this will be proved in Theorem 3.

Notation.
When we consider an eitt $\Sigma(\mathbb{C}, \bigtriangledown)$, we will write $\mathbb{C}^{\bigtriangledown}$ for $\mathbb{C}$, $\mathbb{T}^{\bigtriangledown}$ for $\mathbb{T}(\mathbb{C})$ and $\Sigma^{\bigtriangledown}$ for $\Sigma(\mathbb{C}, \bigtriangledown)$. Moreover $A \leq_{\bigtriangledown} B$ will be short for $(A \leq B) \in \Sigma^{\bigtriangledown}$ and $A \sim_{\bigtriangledown} B$ for $A \leq_{\bigtriangledown} B \leq_{\bigtriangledown} A$. We will consider syntactic equivalence "$\equiv$" of types up to associativity and commutativity of $\cap$. We will write $\bigcap_{i \leq n} A_i$ for $A_1 \cap \ldots \cap A_n$. Similarly we will write $\bigcap_{i \in I} A_i$, where I denotes always a finite non-empty set. $\Box$

Theorem 1 gives useful properties of eitt's.

Theorem 1.
For all I, and $A_i, B_i, C, D \in \mathbb{T}^{\bigtriangledown}$,

$$\begin{array}{l} \bigcap_{i \in I}(A_i \to B_i) \leq_{\bigtriangledown} C \to D \;\&\; D \not\sim_{\bigtriangledown} \Omega \Rightarrow \\ \exists J \subseteq I.\ C \leq_{\bigtriangledown} \bigcap_{i \in J} A_i \;\&\; \bigcap_{i \in J} B_i \leq_{\bigtriangledown} D. \end{array}$$

Proof. In this proof we assume that ψ, ϕ, ξ denote constants different from ι. Let $A(\cap\iota) \leq_{\bigtriangledown} B(\cap\iota)$ be short for $A \leq_{\bigtriangledown} B$ or $A \cap \iota \leq_{\bigtriangledown} B$ or $A \cap \iota \leq_{\bigtriangledown} B \cap \iota$, where $A, B \not\leq_{\bigtriangledown} \iota$. Notice that we cannot have $A \leq_{\bigtriangledown} B \cap \iota$ and $A \not\leq_{\bigtriangledown} \iota$.

By assumption for each constant ψ there exists in $\bigtriangledown$ exactly one judgment of the shape $\psi \sim \bigcap_{l \in L(\psi)}(\xi_l^{(\psi)} \to E_l^{(\psi)})$. We can prove by simultaneous induction on the definition of $\leq_{\bigtriangledown}$ two statements, the first of which implies the thesis.

1. if $(\bigcap_{i\in I}(A_i \to B_i))\cap(\bigcap_{h\in H}\psi_h)(\cap\iota) \leq_{\bigtriangledown} (\bigcap_{j\in J}(C_j \to D_j))\cap(\bigcap_{k\in K}\phi_k)(\cap\iota)$, and $D_j \not\sim_{\bigtriangledown} \Omega$, then for each $j \in J$ there exist $I' \subseteq I$, $H' \subseteq H$ and, for all $h \in H'$, $L^{(\psi_h)'} \subseteq L^{(\psi_h)}$ such that $C_j \leq_{\bigtriangledown} (\bigcap_{i\in I'} A_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\psi_h)'}} \xi_l^{(\psi_h)}))$ and $(\bigcap_{i\in I'} B_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\psi_h)'}} E_l^{(\psi_h)})) \leq_{\bigtriangledown} D_j$;
2. if $(\bigcap_{i\in I}(A_i \to B_i))\cap(\bigcap_{h\in H}\psi_h)(\cap\iota) \leq_{\bigtriangledown} (\bigcap_{j\in J}(C_j \to D_j))\cap(\bigcap_{k\in K}\phi_k)(\cap\iota)$, and $\phi_k \not\sim_{\bigtriangledown} \Omega$, then for each $m \in L^{(\phi_k)}$ there exist $I' \subseteq I$, $H' \subseteq H$ and, for all $h \in H'$, $L^{(\psi_h)'} \subseteq L^{(\psi_h)}$ such that $\xi_m^{(\phi_k)} \leq_{\bigtriangledown} (\bigcap_{i\in I'} A_i)\cap(\bigcap_{h\in H'}(\bigcap_{l\in L^{(\psi_h)'}} \xi_l^{(\psi_h)}))$ and $(\bigcap_{i\in I'} B_i) \cap (\bigcap_{h\in H'}(\bigcap_{l\in L^{(\psi_h)'}} E_l^{(\psi_h)})) \leq_{\bigtriangledown} E_m^{(\phi_k)}$. □

Inside the set of types we single out those which are not ι and not intersections containing ι. Moreover we associate to each type the maximum number of nested arrows in the leftmost path.

Definition 3. *1. A type $A \in \mathbb{T}^{\bigtriangledown}$ is* functional *iff $A \not\equiv \iota$ and there is no $B \in \mathbb{T}^{\bigtriangledown}$ such that $A \equiv \iota \cap B$;*
2. The mapping $\# : \mathbb{T}^{\bigtriangledown} \to \mathbb{N}$ is defined inductively on types as follows:

$$\begin{array}{lll} \#(A) & = 0 & \text{if } A \in \mathbb{C}^{\bigtriangledown}; \\ \#(A \to B) & = \#(A) + 1; & \\ \#(A \cap B) & = \max\{\#(A), \#(B)\}. \ \Box & \end{array}$$

Trivially all types in $\mathbb{T}^{\bigtriangledown}$ are functional when $\iota \notin \mathbb{C}$.

Some properties of functional types are crucial.

Theorem 2. *1. If $A \leq_{\bigtriangledown} B$ and A is functional, then B is functional.*
2. Let $A \in \mathbb{T}^{\bigtriangledown}$ be a functional type. Then, if $\#(A) \geq 1$, there is $B \in \mathbb{T}^{\bigtriangledown}$ such that $A \sim_{\bigtriangledown} B$, $B \equiv \bigcap_{i\in I}(C_i \to D_i)$, and $\#(B) = \#(A)$.

Proof. (1) Easy by induction on $\leq_{\bigtriangledown}$.
(2) Let $A \equiv (\bigcap_{j\in J}(C'_j \to D'_j)) \cap (\bigcap_{h\in H}\psi_h)$, where $C'_j, D'_j \in \mathbb{T}^{\bigtriangledown}$, $\psi_h \in \mathbb{C}^{\bigtriangledown}$. Being A functional, $\forall h \in H.\ \psi_h \not\equiv \iota$, hence for each $h \in H$ there are I_h, $\xi_{i,h} \in \mathbb{C}^{\bigtriangledown}$, $A_{i,h} \in \mathbb{T}^{\bigtriangledown}$, such that $\psi_h \sim_{\bigtriangledown} \bigcap_{i\in I_h}(\xi_{i,h} \to A_{i,h})$. We can choose $B \equiv (\bigcap_{j\in J}(C'_j \to D'j)) \cap (\bigcap_{h\in H}(\bigcap_{i\in I_h}(\xi_{i,h} \to A_{i,h})))$. □

Before giving the key notion of *intersection-type assignment system*, we introduce bases and some related definitions.

Definition 4 (Bases).

1. A $\bigtriangledown$-basis *is a (possibly infinite) set of statements of the shape $x:B$, where $B \in \mathbb{T}^{\bigtriangledown}$, with all variables distinct.*
2. $x \in \Gamma$ is short for $\exists A \in \mathbb{T}^{\bigtriangledown}.\ (x:A) \in \Gamma$ and $\Gamma, x:A$ is short for $\Gamma \cup \{x:A\}$ when $x \notin \Gamma$.
3. Let Γ and Γ' be $\bigtriangledown$-bases. The $\bigtriangledown$-basis $\Gamma \uplus \Gamma'$ is defined as follows:

$$\begin{aligned} \Gamma \uplus \Gamma' = &\ \{x\colon A \cap B \mid x\colon A \in \Gamma \text{ and } x\colon B \in \Gamma'\} \\ &\cup \{x\colon A \mid x\colon A \in \Gamma \text{ and } x \notin \Gamma'\} \\ &\cup \{x\colon B \mid x\colon B \in \Gamma' \text{ and } x \notin \Gamma\}. \end{aligned}$$

Accordingly we define:

$$\Gamma \Subset \Gamma' \ \Leftrightarrow\ \exists \Gamma''.\, \Gamma \uplus \Gamma'' = \Gamma'. \ \square$$

Definition 5 (The type assignment system).
The intersection type assignment system *relative to the eitt Σ^∇, notation $\lambda\cap^\nabla$, is a formal system for deriving judgments of the form $\Gamma \vdash^\nabla M : A$, where the* subject *$M$ is an untyped λ-term, the* predicate *A is in $\mathbb{T}^\nabla$, and Γ is a ∇-basis. Its axioms and rules are the following:*

$$(Ax)\ \frac{(x{:}A) \in \Gamma}{\Gamma \vdash^\nabla x{:}A} \qquad\qquad (Ax\text{-}\Omega)\ \Gamma \vdash^\nabla M : \Omega$$

$$(\to \mathrm{I})\ \frac{\Gamma, x{:}A \vdash^\nabla M : B}{\Gamma \vdash^\nabla \lambda x.M : A \to B} \qquad (\to \mathrm{E})\ \frac{\Gamma \vdash^\nabla M : A \to B \quad \Gamma \vdash^\nabla N : A}{\Gamma \vdash^\nabla MN : B}$$

$$(\cap\mathrm{I})\ \frac{\Gamma \vdash^\nabla M : A \quad \Gamma \vdash^\nabla M : B}{\Gamma \vdash^\nabla M : A \cap B} \qquad (\leq_\nabla)\ \frac{\Gamma \vdash^\nabla M : A \quad A \leq_\nabla B}{\Gamma \vdash^\nabla M : B} \quad \square$$

As usual we consider λ-terms modulo α-conversion.

Notice that intersection elimination rules

$$(\cap E)\ \ \frac{\Gamma \vdash^\nabla M : A \cap B}{\Gamma \vdash^\nabla M : A} \qquad \frac{\Gamma \vdash^\nabla M : A \cap B}{\Gamma \vdash^\nabla M : B}$$

can be immediately proved to be derivable in all $\lambda\cap^\nabla$.

A first simple proposition, which can be proved straightforwardly by induction on the structure of derivations is the following.

Proposition 1.

1. *If $x \notin FV(M)$ and $\Gamma, x{:}B \vdash^\nabla M : A$, then $\Gamma \vdash^\nabla M : A$;*
2. *If $\Gamma \vdash^\nabla M : A$ and $\Gamma \Subset \Gamma'$, then $\Gamma' \vdash^\nabla M : A$.*

We end this section by stating a Generation, or Inversion, Theorem for the type assignment systems $\lambda\cap^\nabla$.

Theorem 3 (Generation Theorem).

1. *Assume $A \not\sim_\nabla \Omega$. $\Gamma \vdash^\nabla x : A$ iff $(x{:}B) \in \Gamma$ and $B \leq_\nabla A$ for some $B \in \mathbb{T}^\nabla$.*
2. *$\Gamma \vdash^\nabla MN : A$ iff $\Gamma \vdash^\nabla M : B \to A$, and $\Gamma \vdash^\nabla N : B$ for some $B \in \mathbb{T}^\nabla$.*
3. *$\Gamma \vdash^\nabla \lambda x.M : A$ iff $\Gamma, x{:}B_i \vdash^\nabla M : C_i$ and $\bigcap_{i\in I}(B_i \to C_i) \leq_\nabla A$, for some I and $B_i, C_i \in \mathbb{T}^\nabla$.*
4. *If $\Gamma \vdash^\nabla \lambda x.M : A$ then A is a functional type.*
5. *$\Gamma \vdash^\nabla \lambda x.M : B \to C$ iff $\Gamma, x{:}B \vdash^\nabla M : C$.*

Proof. The proof of each $(\Leftarrow)$ is easy. So we only treat $(\Rightarrow)$.

(1) Easy by induction on derivations, since only the axioms (Ax), (Ax-Ω), and the rules $(\cap\mathrm{I})$, $(\leq_\nabla)$ can be applied. Notice that the condition $A \not\sim_\nabla \Omega$ implies that $\Gamma \vdash^\nabla x : A$ cannot be obtained just using axioms (Ax-Ω).

(2) If $A \sim_\nabla \Omega$ we can choose $B \sim_\nabla \Omega$. Otherwise the proof is by induction on derivations. The only interesting case is when $A \equiv A_1 \cap A_2$ and the last applied rule is $(\cap\mathrm{I})$:

$$(\cap\mathrm{I})\ \frac{\Gamma \vdash^\nabla MN : A_1 \quad \Gamma \vdash^\nabla MN : A_2}{\Gamma \vdash^\nabla MN : A_1 \cap A_2}.$$

The condition $A \not\sim_\nabla \Omega$ implies that we cannot have $A_1 \sim_\nabla A_2 \sim_\nabla \Omega$. We do the proof for $A_1 \not\sim_\nabla \Omega$ and $A_2 \not\sim_\nabla \Omega$, the other cases can be treated similarly. By induction there are B, C, D, E such that

$$\begin{array}{cc} \Gamma \vdash^\nabla M : B \to C, & \Gamma \vdash^\nabla N : B, \\ \Gamma \vdash^\nabla M : D \to E, & \Gamma \vdash^\nabla N : D, \\ C \leq_\nabla A_1 & \&\quad E \leq_\nabla A_2. \end{array}$$

So we are done being $(B \to C) \cap (D \to E) \leq_\nabla B \cap D \to C \cap E \leq B \cap D \to A$ by rules $(\to\text{-}\cap)$ and (η) since $C \cap E \leq_\nabla A$.

(3) The proof is very similar to the proof of (2). It is again by induction on derivations and again the only interesting case is when the last applied rule is $(\cap\mathrm{I})$:

$$(\cap\mathrm{I})\ \frac{\Gamma \vdash^\nabla \lambda x.M : A_1 \quad \Gamma \vdash^\nabla \lambda x.M : A_2}{\Gamma \vdash^\nabla \lambda x.M : A_1 \cap A_2}.$$

By induction there are I, B_i, C_i, J, D_j, E_j such that

$$\begin{array}{l} \forall i \in I.\ \Gamma, x{:}B_i \vdash^\nabla M : C_i, \forall j \in J.\ \Gamma, x{:}D_j \vdash^\nabla M : E_j, \\ \bigcap_{i \in I}(B_i \to C_i) \leq_\nabla A_1 \quad \& \quad \bigcap_{j \in J}(D_j \to E_j) \leq_\nabla A_2. \end{array}$$

So we are done since $(\bigcap_{i \in I}(B_i \to C_i)) \cap (\bigcap_{j \in J}(D_j \to E_j)) \leq_\nabla A$.

(4) Immediate from (3) and Theorem 2(1).

(5) The case $C \sim_\nabla \Omega$ is trivial. Otherwise let I, B_i, C_i be as in (3), where $A \equiv B \to C$. Then $\bigcap_{i \in I}(B_i \to C_i) \leq_\nabla B \to C$ implies by Theorem 1 that there exists a $J \subseteq I$ such that $B \leq_\nabla \bigcap_{i \in J} B_i$ and $\bigcap_{i \in J} C_i \leq_\nabla C$. From $\Gamma, x : B_i \vdash^\nabla M : C_i$ we can derive $\Gamma, x{:}B \vdash^\nabla M : C_i$ by Proposition 1(2), so by $(\cap\mathrm{I})$ we have $\Gamma, x{:}B \vdash^\nabla M : \bigcap_{i \in J} C_i$. Finally applying rule $(\leq_\nabla)$ we can conclude $\Gamma, x{:}B \vdash^\nabla M : C$. $\Box$

2 Filter Models

In this section we discuss how to build λ-models out of type theories. We start with the definition of *filter* for eitt's. Then we show how to turn the space of filters into an applicative structure. Finally we will define a notion of interpretation of λ-terms and show that we get λ-models (*filter models*).

Filter models arise naturally in the context of those generalizations of Stone duality that are used in discussing domain theory in logical form (see [1], [11], [32]). This approach provides a conceptually independent semantics to intersection types, the *lattice semantics*. Types are viewed as *compact elements* of domains. The type Ω denotes the least element, intersections denote joins of compact elements, and arrow types allow to internalize the space of continuous endomorphisms. Following the paradigm of Stone duality, type theories give rise to filter models, where the interpretation of λ-terms can be given through a finitary logical description.

Definition 6.

1. *A $\bigtriangledown$-filter (or a filter over $\mathbb{T}^\bigtriangledown$) is a set $X \subseteq \mathbb{T}^\bigtriangledown$ such that:*
   - $\Omega \in X$;
   - *if $A \leq_\bigtriangledown B$ and $A \in X$, then $B \in X$;*
   - *if $A, B \in X$, then $A \cap B \in X$;*
2. *$\mathcal{F}^\bigtriangledown$ denotes the set of $\bigtriangledown$-filters over $\mathbb{T}^\bigtriangledown$;*
3. *if $X \subseteq \mathbb{T}^\bigtriangledown$, $\uparrow X$ denotes the $\bigtriangledown$-filter generated by X;*
4. *a $\bigtriangledown$-filter is* principal *if it is of the shape $\uparrow \{A\}$, for some type A. We shall denote $\uparrow \{A\}$ simply by $\uparrow A$.* $\Box$

It is well known that $\mathcal{F}^\bigtriangledown$ is a ω-algebraic cpo, whose compact (or finite) elements are the filters of the form $\uparrow A$ for some type A and whose bottom element is $\uparrow \Omega$.

Next we endow the space of filters with the notions of application and of λ-term interpretation. Let $\mathsf{Env}_{\mathcal{F}^\bigtriangledown}$ be the set of all mappings from the set of term variables to $\mathcal{F}^\bigtriangledown$.

Definition 7.

1. *Application $\cdot : \mathcal{F}^\bigtriangledown \times \mathcal{F}^\bigtriangledown \to \mathcal{F}^\bigtriangledown$ is defined as*
$$X \cdot Y = \{B \mid \exists A \in Y. A \to B \in X\}.$$
2. *The interpretation function: $[\![\ ]\!]^\bigtriangledown : \Lambda \times \mathsf{Env}_{\mathcal{F}^\bigtriangledown} \to \mathcal{F}^\bigtriangledown$ is defined by*
$$[\![M]\!]^\bigtriangledown_\rho = \{A \in \mathbb{T}^\bigtriangledown \mid \exists \Gamma \models \rho.\ \Gamma \vdash^\bigtriangledown M : A\},$$
where ρ ranges over $\mathsf{Env}_{\mathcal{F}^\bigtriangledown}$ and $\Gamma \models \rho$ if and only if $(x : B) \in \Gamma$ implies $B \in \rho(x)$.
3. *The triple $\langle \mathcal{F}^\bigtriangledown, \cdot, [\![\ ]\!]^\bigtriangledown \rangle$ is called the* filter model *over $\Sigma^\bigtriangledown$.* $\Box$

Notice that previous definition is sound, since it is easy to verify that $X \cdot Y$ is a $\bigtriangledown$-filter and moreover:

Theorem 4.
The filter model $\langle \mathcal{F}^\bigtriangledown, \cdot, [\![\ ]\!]^\bigtriangledown \rangle$ is a λ-model, in the sense of Hindley-Longo [18], that is:

1. $[\![x]\!]^\bigtriangledown_\rho = \rho(x)$;
2. $[\![MN]\!]^\bigtriangledown_\rho = [\![M]\!]^\bigtriangledown_\rho \cdot [\![N]\!]^\bigtriangledown_\rho$;
3. $[\![\lambda x.M]\!]^\bigtriangledown_\rho \cdot X = [\![M]\!]^\bigtriangledown_{\rho[X/x]}$;
4. $(\forall x \in FV(M).\ [\![x]\!]^\bigtriangledown_\rho = [\![x]\!]^\bigtriangledown_{\rho'}) \Rightarrow [\![M]\!]^\bigtriangledown_\rho = [\![M]\!]^\bigtriangledown_{\rho'}$;

5. $[\![\lambda x.M]\!]^{\triangledown}_{\rho} = [\![\lambda y.M[y/x]]\!]^{\triangledown}_{\rho}$*, if* $y \notin FV(M)$*;*
6. $(\forall X \in \mathcal{F}^{\triangledown}.[\![M]\!]^{\triangledown}_{\rho[X/x]} = [\![N]\!]^{\triangledown}_{\rho[X/x]}) \Rightarrow [\![\lambda x.M]\!]^{\triangledown}_{\rho} = [\![\lambda x.N]\!]^{\triangledown}_{\rho}$.

Moreover it is extensional - that is $[\![\lambda x.Mx]\!]^{\triangledown}_{\rho} = [\![M]\!]^{\triangledown}_{\rho}$ *when* $x \notin FV(M)$ *- iff* $\iota \notin \mathbb{C}$*.*

Proof. (1) We show $A \in [\![x]\!]^{\triangledown}_{\rho}$ iff $A \in \rho(x)$. The case $A \sim_{\triangledown} \Omega$ is immediate. If $A \not\sim_{\triangledown} \Omega$ and $A \in [\![x]\!]^{\triangledown}_{\rho}$, then $\Gamma \vdash^{\triangledown} x : A$ for some $\triangledown$-basis Γ such that $\Gamma \models \rho$. So there exists a premise $x : A'$ in Γ such that $A' \in \rho(x)$. By Theorem 3(1), $A' \leq_{\triangledown} A$, hence $A \in \rho(x)$. The vice versa is trivial.

(2) Let $A \in [\![MN]\!]^{\triangledown}_{\rho}$. Then there exists $\Gamma \models \rho$ such that $\Gamma \vdash^{\triangledown} MN : A$. By Theorem 3(2), there exists $B \in \mathbb{T}^{\triangledown}$ such that $\Gamma \vdash^{\triangledown} M : B \to A$ and $\Gamma \vdash^{\triangledown} N : B$, hence $B \in [\![N]\!]^{\triangledown}_{\rho}$ and $B \to A \in [\![M]\!]^{\triangledown}_{\rho}$. By definition of application it follows $A \in [\![M]\!]^{\triangledown}_{\rho} \cdot [\![N]\!]^{\triangledown}_{\rho}$.
Let now $A \in [\![M]\!]^{\triangledown}_{\rho} \cdot [\![N]\!]^{\triangledown}_{\rho}$. Then there exists $B \in \mathbb{T}^{\triangledown}$ such that $B \to A \in [\![M]\!]^{\triangledown}_{\rho}$ and $B \in [\![N]\!]^{\triangledown}_{\rho}$, hence there exist two $\triangledown$-bases, Γ and Γ', such that $\Gamma \models \rho$, $\Gamma' \models \rho$, and moreover $\Gamma \vdash^{\triangledown} M : B \to A$, $\Gamma' \vdash^{\triangledown} N : B$. Consider the basis $\Gamma'' = \Gamma \uplus \Gamma'$. We have $\Gamma'' \models \rho$, $\Gamma'' \vdash^{\triangledown} M : B \to A$ and $\Gamma'' \vdash^{\triangledown} N : B$. From the last two judgments we deduce $\Gamma'' \vdash^{\triangledown} MN : A$, which, along with the first judgment, implies $A \in [\![MN]\!]^{\triangledown}_{\rho}$.

(3) Let $A \in [\![M]\!]^{\triangledown}_{\rho[X/x]}$. Then there exists $\Gamma \models \rho[X/x]$ such that $\Gamma \vdash^{\triangledown} M : A$. Let $\Gamma = \Gamma', x{:}B$, then, by rule ($\to$ I), we get $\Gamma' \vdash^{\triangledown} \lambda x.M : B \to A$. This implies $B \to A \in [\![\lambda x.M]\!]^{\triangledown}_{\rho}$ since from $\Gamma \models \rho[X/x]$ we have $\Gamma' \models \rho$. Being $B \in X$ (because $\Gamma \models \rho[X/x]$), we get $A \in [\![\lambda x.M]\!]^{\triangledown}_{\rho} \cdot X$.
Let $A \in [\![\lambda x.M]\!]^{\triangledown}_{\rho} \cdot X$. Then there exists $\Gamma \models \rho$ and $B \in X$ such that $\Gamma \vdash^{\triangledown} \lambda x.M : B \to A$. Since $x \notin \mathrm{FV}(\lambda x.M)$ by Lemma 1(1) we can assume $x \notin \Gamma$. By Theorem 3(5) it follows $\Gamma, x{:}B \vdash^{\triangledown} M : A$. Since $B \in X$ we have $\Gamma, x{:}B \models \rho$, hence $A \in [\![M]\!]^{\triangledown}_{\rho}$.

(4) easily proven by induction on the structure of M.

(5) trivial.

(6) Suppose that the premise hold and $A \in [\![\lambda x.M]\!]^{\triangledown}_{\rho}$. Then there is $\Gamma \models \rho$ such that $\Gamma \vdash^{\triangledown} \lambda x.M : A$. Since $x \notin \mathrm{FV}(\lambda x.M)$ by Lemma 1(1) we can assume $x \notin \Gamma$. By Theorem 3(3) there exist I and $B_i, C_i \in \mathbb{T}^{\triangledown}$ such that $\Gamma \vdash^{\triangledown} \lambda x.M : B_i \to C_i$ for all $i \in I$ and $\bigcap_{i \in I}(B_i \to C_i) \leq_{\triangledown} A$. So we have, for each $i \in I$, by Theorem 3(5) $\Gamma, x{:}B_i \vdash^{\triangledown} M : C_i$. By the premise, we get, for each $i \in I$, $\Gamma, x{:}B_i \vdash^{\triangledown} N : C_i$, which implies $[\![\lambda x.M]\!]^{\triangledown}_{\rho} \subseteq [\![\lambda x.N]\!]^{\triangledown}_{\rho}$. Similarly one proves $[\![\lambda x.N]\!]^{\triangledown}_{\rho} \subseteq [\![\lambda x.M]\!]^{\triangledown}_{\rho}$.

We show now that the model is extensional when $\iota \notin \mathbb{C}$. Let $A \in [\![\lambda x.Mx]\!]^{\triangledown}_{\rho}$, with $x \notin \mathrm{FV}(M)$. Then there is $\Gamma \models \rho$ such that $\Gamma \vdash^{\triangledown} \lambda x.Mx : A$. Reasoning as in the proof of (6), we have that there exist I and $B_i, C_i \in \mathbb{T}^{\triangledown}$ such that for each $i \in I$, $\Gamma, x{:}B_i \vdash^{\triangledown} Mx : C_i$ and $\bigcap_{i \in I}(B_i \to C_i) \leq_{\triangledown} A$. By Theorem 3(2), it follows that there exists, for each $i \in I$, D_i such that $\Gamma, x{:}B_i \vdash^{\triangledown} M : D_i \to C_i$, and $\Gamma, x{:}B_i \vdash^{\triangledown} x : D_i$. We have $B_i \leq_{\triangledown} D_i$ for each $i \in I$ either by Theorem 3(1) if $D_i \not\sim_{\triangledown} \Omega$ or by axiom (Ω) and rule (trans) if $D_i \sim_{\triangledown} \Omega$. Hence we get, by

$(\leq_\nabla)$, $\Gamma, x{:}B_i \vdash^\nabla M : B_i \to C_i$, for each $i \in I$. Since $x \notin \mathrm{FV}(M)$ we can apply Lemma 1(1) and obtain, for each $i \in I$, $\Gamma \vdash^\nabla M : B_i \to C_i$, hence, by rules $(\cap\mathrm{I})$ and $(\leq_\nabla)$, $\Gamma \vdash^\nabla M : A$, which implies $A \in [\![M]\!]^\nabla_\rho$.
Suppose $A \in [\![M]\!]^\nabla_\rho$. Then there exists $\Gamma \models \rho$ such that $\Gamma \vdash^\nabla M : A$. Since $x \notin \mathrm{FV}(M)$ by Lemma 1(1) we can assume $x \notin \Gamma$. By Theorem 2 we have $A \sim_\nabla \bigcap_{i \in I}(B_i \to C_i)$ for suitable I and $B_i, C_i \in \mathbb{T}^\nabla$, so by applying rule $(\leq_\nabla)$ we get $\Gamma \vdash^\nabla M : B_i \to C_i$ for each $i \in I$. By Lemma 1(2), we have $\Gamma, x : B_i \vdash^\nabla M : B_i \to C_i$. This judgment, along with $\Gamma, x : B_i \vdash^\nabla x : B_i$ allows to obtain, by rule $(\to \mathrm{E})$, $\Gamma, x : B_i \vdash^\nabla Mx : C_i$, for each $i \in I$. By rule $(\to \mathrm{I})$ we deduce $\Gamma \vdash^\nabla \lambda x.Mx : B_i \to C_i$ for each $i \in I$, hence by rule $(\cap\mathrm{I})$ it follows $\Gamma \vdash^\nabla \lambda x.Mx : \bigcap_{i\in I}(B_i \to C_i)$, which implies, by rule $(\leq_\nabla)$, $\Gamma \vdash^\nabla \lambda x.Mx : A$, so we conclude $A \in [\![\lambda x.Mx]\!]^\nabla_\rho$.

Finally, if $\iota \in \mathbb{C}$ the model is non-extensional, since taking $\rho(x) = \uparrow \iota$, we get $[\![x]\!]^\nabla_\rho = \uparrow \iota$ while $[\![\lambda y.xy]\!]^\nabla_\rho = \uparrow \Omega$. $\square$

3 Semantical proof of the easiness of $\omega_2\omega_2$

Let ω_2 be the λ-term $\lambda x.xx$. For an arbitrary closed λ-term M we build a non-extensional filter model $\langle \mathcal{F}^{\nabla'}, \cdot, [\![\ ]\!]^{\nabla'} \rangle$ such that $[\![M]\!]^{\nabla'} = [\![\omega_2\omega_2]\!]^{\nabla'}$.

First we give a lemma which characterizes the types derivable for ω_2 and $\omega_2\omega_2$.

Lemma 1.

1. *$\vdash^\nabla \omega_2 : A \to B$ iff $A \leq_\nabla A \to B$;*
2. *$\vdash^\nabla \omega_2\omega_2 : B$ iff $A \leq_\nabla A \to B$ for some $A \in \mathbb{T}^\nabla$ such that $\vdash^\nabla \omega_2 : A$.*
3. *If $\vdash^\nabla \omega_2\omega_2 : B$ then there exists $A \in \mathbb{T}^\nabla$ such that $\#(A) = 0$, $A \leq_\nabla A \to B$ and $\vdash^\nabla \omega_2 : A$.*

Proof. (1) By a straightforward computation $A \leq_\nabla A \to B$ implies $\vdash^\nabla \omega_2 : A \to B$. Conversely, suppose $\vdash^\nabla \omega_2 : A \to B$. If $B \sim_\nabla \Omega$, then by axioms (Ω), $(\Omega\text{-}\eta)$, and rules (η), $(trans)$, we have $A \leq_\nabla A \to B$. Otherwise, by Theorem 3(5) it follows $x : A \vdash^\nabla xx : B$. By Theorem 3(2) there exists a type $C \in \mathbb{T}^\nabla$ such that $x : A \vdash^\nabla x : C \to B$ and $x : A \vdash^\nabla x : C$. Notice that $B \not\sim_\nabla \Omega$ implies $C \to B \not\sim_\nabla \Omega$, since from $C \to B \sim_\nabla \Omega$ we get $C \to B \sim_\nabla \Omega \to \Omega$ by axiom $(\Omega\text{-}\eta)$ and rule $(trans)$ and this implies $B \sim_\nabla \Omega$ by Theorem 1. So by Theorem 3(1), we get $A \leq_\nabla C \to B$. We have $A \leq_\nabla C$ either by Theorem 3(1) if $C \not\sim_\nabla \Omega$ or by axiom (Ω) and rule (trans) if $C \sim_\nabla \Omega$. From $A \leq_\nabla C \to B$ and $A \leq_\nabla C$ by rule (η) it follows $A \leq_\nabla A \to B$.

(2) The case $B \sim_\nabla \Omega$ is trivial. Otherwise, if $\vdash^\nabla \omega_2\omega_2 : B$, by Theorem 3(2) it follows that there exists $A \in \mathbb{T}^\nabla$ such that $\vdash^\nabla \omega_2 : A$ and $\vdash^\nabla \omega_2 : A \to B$. We conclude by (1).

(3) Let $\vdash^\nabla \omega_2\omega_2 : B$. Then, by Point (2), there exists $A \in \mathbb{T}^\nabla$ such that $\vdash^\nabla \omega_2 : A$ and $A \leq_\nabla A \to B$. We prove the thesis by induction on $\#(A)$. If $\#(A) = 0$ we are done. Suppose now $\#(A) \geq 1$. First by Theorem 3(4) A is functional. By

applying Lemma 2, we obtain a type A' such that $A' \sim_{\triangledown} A$, $A' \equiv \bigcap_{i\in I}(C_i \to D_i)$ and $\#(A') = \#(A)$. From $A \leq_{\triangledown} A \to B$ we have $\bigcap_{i\in I}(C_i \to D_i) \leq_{\triangledown} A \to B$, hence, by Theorem 1, there exists $J \subseteq I$ such that $A \leq_{\triangledown} \bigcap_{i\in J} C_i$ and $\bigcap_{i\in J} D_i \leq_{\triangledown} B$. Since $\vdash^{\triangledown} \omega_2 : A$, by $(\leq_{\triangledown})$ it follows $\vdash^{\triangledown} \omega_2 : C_i \to D_i$ for all $i \in J$ and $\vdash^{\triangledown} \omega_2 : \bigcap_{i\in J} C_i$. By Point (1) it follows $\forall i \in J.C_i \leq_{\triangledown} C_i \to D_i$. By axiom $(\to$-$\cap)$ and rule (η) we get $C \leq_{\triangledown} C \to \bigcap_{i\in J} D_i$, and also $C \leq_{\triangledown} C \to B$, where $C \equiv \bigcap_{i\in J} C_i$. We have obtained:

$$\begin{array}{l} \vdash^{\triangledown} \omega_2 : C; \\ C \leq_{\triangledown} C \to B; \\ \#(C) < \#(A') = \#(A). \end{array}$$

The thesis follows by applying the induction properties. □

We build the desired model by taking the union of a suitable countable sequence of eitt's $\Sigma^{\triangledown_n}$ defined in such a way that the final interpretation of M coincides with the interpretation of $\omega_2\omega_2$. In the following $\langle\cdot,\cdot\rangle$ denotes any bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Definition 8.

1. *The eitt's $\Sigma^{\triangledown_n}$ are defined inductively on n as follows:*
   - $\mathbb{C}^{\triangledown_1} = \{\Omega, \iota\}$;
   - $\triangledown_1 = \triangledown_0$;
   - $\mathbb{C}^{\triangledown_{n+1}} = \mathbb{C}^{\triangledown_n} \cup \{\varsigma_{\langle n,m\rangle} \mid m \in \mathbb{N}\}$;
   - $\triangledown_{n+1} = \triangledown_n \cup \{\varsigma_{\langle n,m\rangle} \sim \varsigma_{\langle n,m\rangle} \to W_{\langle n,m\rangle} \mid m \in \mathbb{N}\}$,

   where $\langle W_{\langle n,m\rangle}\rangle_{m\in\mathbb{N}}$ is any enumeration of the set $\{A \mid \vdash^{\triangledown_n} M : A\}$.
2. *We define $\Sigma^{\triangledown'}$ as follows:*

$$\mathbb{C}^{\triangledown'} = \bigcup_{n\in\mathbb{N}} \mathbb{C}^{\triangledown_n};\ \triangledown' = \bigcup_{n\in\mathbb{N}} \triangledown_n. \ \square$$

Since $\Sigma^{\triangledown'}$ is an eitt and $\iota \in \mathbb{C}^{\triangledown'}$, by Theorem 4, it follows that it induces a non-extensional λ-model.

Theorem 5.
The triple $\langle \mathcal{F}^{\triangledown'}, \cdot, [\![\ ]\!]^{\triangledown'}\rangle$ is a non-extensional λ-model.

We need also a negative result on the typing of ω_2.

Lemma 2.

$$\nvdash^{\triangledown'} \omega_2 : \iota \text{ and } \nvdash^{\triangledown'} \omega_2 : \iota \to \iota.$$

Proof. From Theorem 3(4) we get $\nvdash^{\triangledown'} \omega_2 : \iota$, since the type ι is not functional.

To show $\nvdash^{\triangledown'} \omega_2 : \iota \to \iota$ we define the sets $\mathcal{E}_\Omega$, $\mathcal{G}_\tau$ as the minimal sets such that:

$$\begin{array}{l} \Omega \in \mathcal{E}_\Omega;\ \ A \in \mathbb{T}^{\triangledown'}, B \in \mathcal{E}_\Omega \Rightarrow A \to B \in \mathcal{E}_\Omega; \\ A, B \in \mathcal{E}_\Omega \Rightarrow A \cap B \in \mathcal{E}_\Omega;\ \ W_i \in \mathcal{E}_\Omega \Rightarrow \varsigma_i \in \mathcal{E}_\Omega; \\ \iota \in \mathcal{G}_\iota;\ \ \mathcal{E}_\Omega \subseteq \mathcal{G}_\iota;\ \ A, B \in \mathcal{G}_\iota \Rightarrow A \cap B \in \mathcal{G}_\iota. \end{array}$$

It is easy to check by induction on $\leq_{\nabla'}$ that for all $A \in \mathbb{T}^{\nabla'}$:

$$\begin{array}{l} B \leq_{\nabla'} A \text{ and } B \in \mathcal{E}_\Omega \text{ imply } A \in \mathcal{E}_\Omega; \\ B \leq_{\nabla'} A \text{ and } B \in \mathcal{G}_\iota \text{ imply } A \in \mathcal{G}_\iota. \end{array}$$

This implies that $\iota \not\leq_{\nabla'} \iota \to \iota$ since $\iota \in \mathcal{G}_\iota$ and $\iota \to \iota \notin \mathcal{G}_\iota$.
Suppose by contradiction that $\vdash^{\nabla'} \omega_2 : \iota \to \iota$: we get $\iota \leq_{\nabla'} \iota \to \iota$ by Lemma 1(1). □

We are now able to prove the first main theorem.

Theorem 6.
The filter model $\langle \mathcal{F}^{\nabla'}, \cdot, [\![\ ]\!]^{\nabla'} \rangle$ is not trivial and $[\![M]\!]^{\nabla'} = [\![\omega_2\omega_2]\!]^{\nabla'}$.

Proof. The model is not trivial since it is easy to derive $\vdash^{\nabla'} \mathsf{I} : \iota \to \iota$ while $\not\vdash^{\nabla'} \omega_2 : \iota \to \iota$ by Lemma 2.
The inclusion $[\![M]\!]^{\nabla'} \subseteq [\![\omega_2\omega_2]\!]^{\nabla'}$ is almost immediate by construction. By easy calculation, ω_2 may be given type ς_j for any integer j. In fact, since $\varsigma_j \sim_{\nabla'} \varsigma_j \to W_j$, it follows $x : \varsigma_j \vdash^{\nabla'} xx : W_j$, hence $\vdash^{\nabla'} \lambda x.xx : \varsigma_j \to W_j \sim_{\nabla'} \varsigma_j$. From this last fact, by applying ($\to$E), we obtain $\vdash^{\nabla'} \omega_2\omega_2 : W_j$ for all j. This proves $[\![M]\!]^{\nabla'} \subseteq [\![\omega_2\omega_2]\!]^{\nabla'}$.

On the other hand, let $\vdash^{\nabla'} \omega_2\omega_2 : B$. Then applying Lemma 1(3), it follows that there exists A such that $\#(A) = 0$, $\vdash^{\nabla'} \omega_2 : A$ and $A \leq_{\nabla'} A \to B$. Let $A \equiv \bigcap_{i\in I} \psi_i$. By Lemma 2, for all $i \in I$ we get that ψ_i is either Ω or ς_j for some integer j. This implies either $A \sim_{\nabla'} \Omega \to \Omega$ or $A \sim_{\nabla'} \bigcap_{j\in J}(\varsigma_j \to W_j)$ for some J. Therefore from $A \leq_{\nabla'} A \to B$ either $\Omega \leq_{\nabla'} B$ or $\exists L \subseteq J$ such that $\bigcap_{j\in L} W_j \leq_{\nabla'} B$, by Theorem 1. Since each W_j is in $[\![M]\!]^{\nabla'}$, we have $B \in [\![M]\!]^{\nabla'}$ and we are done. □

4 Semantical proof of the easiness of $\omega_3\omega_3\mathsf{I}$

Let ω_3 be the λ-term $\lambda x.xxx$. For an arbitrary closed λ-term M we build a non-extensional filter model $\langle \mathcal{F}^{\nabla''}, \cdot, [\![\ ]\!]^{\nabla''} \rangle$ such that $[\![M]\!]^{\nabla''} = [\![\omega_3\omega_3\mathsf{I}]\!]^{\nabla''}$.

First we give a lemma, which characterizes the types deducible for ω_3 and $\omega_3\omega_3\mathsf{I}$.

Notation.
In the following $\bar{Q}$ will denote types of the shape $\bigcap_{k\in K}(Q_k \to Q_k)$, that is the minimal types which I can receive. □

Lemma 3.

1. $\vdash^{\nabla} \omega_3 : A \to B$ *iff* $A \leq_{\nabla} A \to A \to B$;
2. $\vdash^{\nabla} \omega_3\omega_3\mathsf{I} : B$ *iff there exist* $A, \bar{Q}$ *such that* $A \leq_{\nabla} A \to A \to \bar{Q} \to B$ *and* $\vdash^{\nabla} \omega_3 : A$.

Proof. Throughout this proof we freely (and heavily!) use Theorem 3. We show just ($\Rightarrow$), since ($\Leftarrow$) are obtained simply by applying the typing rules. The case $B \sim_{\nabla} \Omega$ is trivial.

(1) $\vdash^{\nabla} \omega_3 : A \to B$ implies $x : A \vdash^{\nabla} xxx : B$, hence there exist C, D such that $x : A \vdash^{\nabla} x : C \to D \to B$, $x : A \vdash^{\nabla} x : C$ and $x : A \vdash^{\nabla} x : D$. The first of these three last judgments implies $A \leq_{\nabla} C \to D \to B$, while the last two imply $A \leq_{\nabla} C$ and $A \leq_{\nabla} D$, hence, by rule (η), we obtain $A \leq_{\nabla} A \to A \to B$.

(2) If $\vdash^{\nabla} \omega_3\omega_3\mathsf{I} : B$, then there exists C such that $\vdash^{\nabla} \omega_3\omega_3 : C \to B$ and $\vdash^{\nabla} \mathsf{I} : C$. By straightforward calculation it must exist $\bar{Q}$ such that $\bar{Q} \leq_{\nabla} C$. Moreover this implies $C \to B \leq_{\nabla} \bar{Q} \to B$, hence by rule ($\leq_{\nabla}$), $\vdash^{\nabla} \omega_3\omega_3 : \bar{Q} \to B$. This last judgment holds only if there exists A such that $\vdash^{\nabla} \omega_3 : A \to \bar{Q} \to B$ and $\vdash^{\nabla} \omega_3 : A$. We conclude by using Point (1). $\square$

We define the eitt $\Sigma^{\nabla''}$ as the union of a countable sequence of eitt's Σ^{∇_n} similarly to $\Sigma^{\nabla'}$.

Definition 9.

1. *Σ^{∇_n} are defined inductively on n as the eitt's generated by the following sets $\mathbb{C}^{\nabla_n}$ and ∇_n:*
   - $\mathbb{C}^{\nabla_1} = \{\Omega, \iota\}$;
   - $\nabla_1 = \nabla_0$;
   - $\mathbb{C}^{\nabla_{n+1}} = \mathbb{C}^{\nabla_n} \cup \{\alpha_{\langle n,m\rangle}, \beta_{\langle n,m\rangle}, \gamma_{\langle n,m\rangle} \mid m \in \mathbb{N}\}$;
   - $\nabla_{n+1} = \nabla_n \cup \{\alpha_{\langle n,m\rangle} \leq \beta_{\langle n,m\rangle}, \alpha_{\langle n,m\rangle} \leq \gamma_{\langle n,m\rangle}, \beta_{\langle n,m\rangle} \leq \gamma_{\langle n,m\rangle},$
     $\alpha_{\langle n,m\rangle} \sim (\gamma_{\langle n,m\rangle} \to \alpha_{\langle n,m\rangle}) \cap (\beta_{\langle n,m\rangle} \to \tau \to W_{\langle n,m\rangle}),$
     $\beta_{\langle n,m\rangle} \sim \gamma_{\langle n,m\rangle} \to \alpha_{\langle n,m\rangle},$
     $\gamma_{\langle n,m\rangle} \sim \gamma_{\langle n,m\rangle} \to \beta_{\langle n,m\rangle} \mid m \in \mathbb{N}\}$,

   where $\tau \equiv \iota \to \iota$ and $\langle W_{\langle n,m\rangle}\rangle_{m \in \mathbb{N}}$ is any enumeration of the set $\{A \mid \vdash^{\nabla_n} M : A\}$.
2. *We define $\Sigma^{\nabla''}$ as follows:*

$$\mathbb{C}^{\nabla''} = \bigcup_{n \in \mathbb{N}} \mathbb{C}^{\nabla_n}; \; \nabla'' = \bigcup_{n \in \mathbb{N}} \nabla_n . \square$$

From Definition 9 we immediately have that $\Sigma^{\nabla''}$ is an eitt and $\iota \in \mathbb{C}^{\nabla''}$, hence it induces a non-extensional λ-model.

Theorem 7.
The triple $\langle \mathcal{F}^{\nabla''}, F^{\nabla''}, G^{\nabla''} \rangle$ is a non-extensional λ-model. $\square$

Notation. Sometimes we will omit subscript $\langle n, m\rangle$ for α, β, γ, W. $\square$

The proof of $[\![M]\!]^{\nabla''} = [\![\omega_3\omega_3\mathsf{I}]\!]^{\nabla''}$ is done in three steps. First we show that some subtypings do not hold (Lemma 5). The second step is a characterization of the constants which can be deduced for ω_3 (Lemma 6). Lastly we use these results in order to obtain properties for the types which satify particular subtyping relations and which contain proper subtypes deducible for ω_3 (Lemma 7).

For the first step we introduce four subsets of $\mathbb{T}^{\nabla''}$, whose meaning is the following (recall that $\tau \equiv \iota \to \iota$)[1]:

1. $\mathcal{E}_\Omega$ is the set of all types which are equivalent to Ω;
2. $\mathcal{G}_\iota$ is the set of all types which are greater than or equivalent to ι;
3. $\mathcal{L}_\iota$ is the set of all types which are less than or equivalent to ι;
4. $\mathcal{G}_\tau$ is the set of all types which are greater than or equivalent to τ.

Definition 10. *The sets $\mathcal{E}_\Omega$, $\mathcal{G}_\iota$, $\mathcal{L}_\iota$, $\mathcal{G}_\tau$, are defined as the minimal sets such that:*

$$\begin{array}{l}
\Omega \in \mathcal{E}_\Omega; \quad A \in \mathbb{T}^{\nabla''}, B \in \mathcal{E}_\Omega \;\Rightarrow\; A \to B \in \mathcal{E}_\Omega; \quad A, B \in \mathcal{E}_\Omega \;\Rightarrow\; A \cap B \in \mathcal{E}_\Omega;\\
\iota \in \mathcal{G}_\iota; \quad \mathcal{E}_\Omega \subseteq \mathcal{G}_\iota; \quad A, B \in \mathcal{G}_\iota \;\Rightarrow\; A \cap B \in \mathcal{G}_\iota;\\
\iota \in \mathcal{L}_\iota; \quad A \in \mathbb{T}^{\nabla''}, B \in \mathcal{L}_\iota \;\Rightarrow\; A \cap B, B \cap A \in \mathcal{L}_\iota;\\
\mathcal{E}_\Omega \subseteq \mathcal{G}_\tau; \quad A \in \mathcal{L}_\iota, B \in \mathcal{G}_\iota \;\Rightarrow\; A \to B \in \mathcal{G}_\tau; \quad A, B \in \mathcal{G}_\tau \;\Rightarrow\; A \cap B \in \mathcal{G}_\tau. \;\Box
\end{array}$$

From this definition we easily get:

Lemma 4.
1. $\mathcal{E}_\Omega = \{A \in \mathbb{T}^{\nabla''} \mid \Omega \leq_{\nabla''} A\}$; *3.* $\mathcal{L}_\iota = \{A \in \mathbb{T}^{\nabla''} \mid A \leq_{\nabla''} \iota\}$;
2. $\mathcal{G}_\iota = \{A \in \mathbb{T}^{\nabla''} \mid \iota \leq_{\nabla''} A\}$; *4.* $\mathcal{G}_\tau = \{A \in \mathbb{T}^{\nabla''} \mid \tau \leq_{\nabla''} A\}$.

Proof. It is easy to check by induction on the definition of $\mathcal{E}_\Omega$ that $A \in \mathcal{E}_\Omega \Rightarrow A \sim_{\nabla''} \Omega$, and similarly for the other sets.
Vice versa one can show by induction on $\leq_{\nabla''}$ that:

$$\begin{array}{l}
B \leq_{\nabla''} A \text{ and } B \in \mathcal{E}_\Omega \text{ imply } A \in \mathcal{E}_\Omega;\\
B \leq_{\nabla''} A \text{ and } B \in \mathcal{G}_\iota \text{ imply } A \in \mathcal{G}_\iota;\\
B \leq_{\nabla''} A \text{ and } A \in \mathcal{L}_\iota \text{ imply } B \in \mathcal{L}_\iota;\\
B \leq_{\nabla''} A \text{ and } B \in \mathcal{G}_\tau \text{ imply } A \in \mathcal{G}_\tau.
\end{array}$$

We consider just the case of application of rule (η) for $B \in \mathcal{G}_\tau$. Let $A \equiv A' \to A''$, $B \equiv B' \to B''$ and let the judgment $B \leq_{\nabla''} A$ be obtained by applying rule (η) from the premises $A' \leq_{\nabla''} B'$ and $B'' \leq_{\nabla''} A''$. Since $B \in \mathcal{G}_\tau$, it must hold $B' \in \mathcal{L}_\iota$ and $B'' \in \mathcal{G}_\iota$. By (3) and (2) we obtain $A' \in \mathcal{L}_\iota$ and $A'' \in \mathcal{G}_\iota$, hence A belongs to $\mathcal{G}_\tau$. $\Box$

Next lemma ensures that the types α, β, γ are not equivalent when $W \not\sim_{\nabla''} \Omega$.

Lemma 5. *If $W_i \not\sim_{\nabla''} \Omega$ then $\gamma_i \not\leq_{\nabla''} \beta_i \not\leq_{\nabla''} \alpha_i$ and $\bar{Q} \not\leq_{\nabla''} \gamma_i$.*

Proof. We prove $\gamma_i \not\leq_{\nabla''} \beta_i \not\leq_{\nabla''} \alpha_i$ by contradiction. By Theorem 1 $\gamma_i \leq_{\nabla''} \beta_i$ holds iff $\beta_i \leq_{\nabla''} \alpha_i$, since $\gamma_i \sim_{\nabla''} \gamma_i \to \beta_i$, $\beta_i \sim_{\nabla''} \gamma_i \to \alpha_i$ and $\alpha_i \not\sim_{\nabla''} \Omega$ by Lemma 4(1). So it is sufficient to prove $\beta_i \not\leq_{\nabla''} \alpha_i$. Since $\alpha_i \sim_{\nabla''} \beta_i \cap (\beta_i \to \tau \to W_i)$, we are done if we can prove that it is impossible to have

[1] The symbols $\mathcal{E}_\Omega$, etc. are overloaded, since they were used already in the proof of Lemma 2 and they will be used in Section 5, but no confusion can arise, since it is always clear from the context the eitt we are considering.

$\beta_i \leq_{\nabla''} \beta_i \to \tau \to W_i$, that is $\gamma_i \to \alpha_i \leq_{\nabla''} \beta_i \to \tau \to W_i$. By Theorem 1 being $W_i \not\sim_{\nabla''} \Omega$ this last judgment is equivalent to the pair of judgments $\beta_i \leq_{\nabla''} \gamma_i$ and $\alpha_i \leq_{\nabla''} \tau \to W_i$. Consider this last judgment, that is equivalent to $(\gamma_i \to \alpha_i) \cap (\beta_i \to \tau \to W_i) \leq_{\nabla''} \tau \to W_i$. By Theorem 1, it should hold $\tau \leq_{\nabla''} \gamma_i$, which is impossible by Lemma 4(4), since $\gamma_i \notin \mathcal{G}_\tau$.
Finally we prove $\bar{Q} \not\leq_{\nabla''} \gamma_i$. If by contradiction $\bar{Q} \leq_{\nabla''} \gamma_i$, by Theorem 1 (since $\gamma_i \sim_{\nabla''} \gamma_i \to \beta_i$ and $\beta_i \not\sim_{\nabla''} \Omega$ by Lemma 4(1)), there exists $H \subseteq K$ such that $\gamma_i \leq_{\nabla''} \bigcap_{h \in H} Q_h$ and $\bigcap_{h \in H} Q_h \leq_{\nabla''} \beta_i$, which implies $\gamma_i \leq_{\nabla''} \beta_i$. This is a contradiction by above. $\square$

The definition of $\Sigma^{\nabla''}$ is tailored so that, as far as constants are concerned, ω_3 can receive just α's, and hence β, γ, and Ω, but not ι.

Lemma 6. *Let $\psi \in \mathbb{C}^{\nabla''}$, then $\vdash^{\nabla''} \omega_3 : \psi$ iff $\alpha_i \leq_{\nabla''} \psi$ for some i. Moreover $\not\vdash^{\nabla''} \omega_3 : \tau$.*

Proof. We know, from Theorem 3(4), that $\not\vdash^{\nabla''} \omega_3 : \iota$. We show:

$$(1) \vdash^{\nabla''} \omega_3 : \alpha_i \text{ for all } i;\ (2) \not\vdash^{\nabla''} \omega_3 : \tau.$$

(1) Since $\alpha_i \sim_{\nabla''} (\gamma_i \to \alpha_i) \cap (\beta_i \to \tau \to W_i)$, by Lemma 3(1), it is sufficient to prove that $\gamma_i \leq_{\nabla''} \gamma_i \to \gamma_i \to \alpha_i$ and $\beta_i \leq_{\nabla''} \beta_i \to \beta_i \to \tau \to W_i$. The first judgment is immediate by the equivalences in ∇'' on γ_i and β_i. The second one follows by using rule (η) twice:

$$\begin{aligned} \beta_i &\sim_{\nabla''} \gamma_i \to \alpha_i \\ &\leq_{\nabla''} \gamma_i \to \beta_i \to \tau \to W_i \text{ (since } \alpha_i \leq_{\nabla''} \beta_i \to \tau \to W_i) \\ &\leq_{\nabla''} \beta_i \to \beta_i \to \tau \to W_i \text{ (since } \beta_i \leq_{\nabla''} \gamma_i). \end{aligned}$$

(2) If $\vdash^{\nabla''} \omega_3 : \tau$, then by Lemma 3(1), it should be $\iota \leq_{\nabla''} \iota \to \iota \to \iota$, which is impossible by Lemma 4(2) since $\iota \to \iota \to \iota \notin \mathcal{G}_\iota$. $\square$

Lemma 7.

1. *If $\bigcap_{i \in I} \alpha_i \leq_{\nabla''} A \to B \to C$ and $\vdash^{\nabla''} \omega_3 : B$ then there is $J \subseteq I$ such that $\bigcap_{i \in J} \alpha_i \leq_{\nabla''} B \to C$.*
2. *If $A \leq_{\nabla''} A \to A \to B$ and $\vdash^{\nabla''} \omega_3 : A$ then there are I, $\alpha_i \in \mathbb{C}^{\nabla''}$, and n such that $C \leq_{\nabla''} C \to C \to A^n \to B$, where $C \equiv \bigcap_{i \in I} \alpha_i$.*

Proof. (1) The case $C \sim_{\nabla''} \Omega$ is trivial. Otherwise by Theorem 1 from $\bigcap_{i \in I} \alpha_i \leq_{\nabla''} A \to B \to C$ we get $(\bigcap_{i \in J} \alpha_i) \cap (\bigcap_{i \in H} (\tau \to W_i)) \leq_{\nabla''} B \to C$ for some $J, H \subseteq I$, since $\alpha_i \sim_{\nabla''} (\gamma_i \to \beta_i) \cap (\beta_i \to \tau \to W_i)$. It is impossible to have $B \leq_{\nabla''} \tau$, since this implies $\vdash^{\nabla''} \omega_3 : \tau$, which contradicts Lemma 6. So, by Theorem 1 it must hold $\bigcap_{i \in J} \alpha_i \leq_{\nabla''} B \to C$.

(2) The case $B \sim_{\nabla''} \Omega$ is trivial. Otherwise the proof is by induction on $\#(A)$. Notice that A is functional by Theorem 3(4).
If $\#(A) = 0$, then A is an intersection of constants which cannot contain ι. Thus we have $A \equiv \bigcap_{i \in J} \psi_i$ such that ψ_i is α, β, γ, or Ω, for all $i \in J$ (notice that

we cannot have $\psi_i \equiv \Omega$ for all $i \in J$ by Lemma 4(1) since $A \to A \to B \notin \mathcal{E}_\Omega$ when $B \not\sim_{\nabla''} \Omega$). Consider now the type C obtained from A by replacing every constant β_l and γ_l occurring in A with α_l. By Lemma 6 $\vdash^{\nabla''}\omega_3 : C$. Moreover $C \leq_{\nabla''} A$, hence $C \leq_{\nabla''} C \to C \to B$. The type C has the required shape since by construction $C \equiv \bigcap_{i\in I} \alpha_i$, where $I = \{i \in J \mid \psi_i \not\equiv \Omega\}$, hence the thesis is satisfied (in this case $n = 0$).

Let us suppose now $\#(A) > 0$. Then, by Lemma 2, there exist J, D_j, E_j such that $\bigcap_{j\in J}(D_j \to E_j) \sim_{\nabla''} A$ and moreover $\#(\bigcap_{j\in J}(D_j \to E_j)) = \#(A)$. By Theorem 1, from $\bigcap_{j\in J}(D_j \to E_j) \leq_{\nabla''} A \to A \to B$ there exists $H \subseteq J$ such that $A \leq_{\nabla''} \bigcap_{j\in H} D_j$ and $\bigcap_{j\in H} D_j \leq_{\nabla''} A \to B$. Let $D \equiv \bigcap_{j\in H} D_j$. Since $\vdash^{\nabla''}\omega_3 : A$, it follows by $(\leq_\nabla)$ $\vdash^{\nabla''}\omega_3 : D$. By the same reason, for all $j \in H$, $\vdash^{\nabla''}\omega_3 : D_j \to E_j$. Hence, by Lemma 3(1), it follows, for all $j \in H$, $D_j \leq_{\nabla''} D_j \to D_j \to E_j$, which implies $\bigcap_{j\in H} D_j \leq_{\nabla''} \bigcap_{j\in H}(D_j \to D_j \to E_j)$. Using ($\to$-$\cap$) and ($\eta$) we obtain $D \leq_{\nabla''} D \to D \to \bigcap_{j\in H} E_j$, which implies $D \leq_{\nabla''} D \to D \to A \to B$. Since $\#(D) < \#(A)$, we can now apply the inductive hypothesis and deduce that there exists $C \equiv \bigcap_{i\in I} \alpha_i$ and n such that $C \leq_{\nabla''} C \to C \to D^n \to A \to B$, which implies $C \leq_{\nabla''} C \to C \to A^{n+1} \to B$, since $A \leq_{\nabla''} D$. □

Lastly we can prove the second main theorem.

Theorem 8.
The filter model $\langle \mathcal{F}^{\nabla''}, \cdot, [\![\,]\!]^{\nabla''}\rangle$ is not trivial and $[\![M]\!]^{\nabla''} = [\![\omega_3\omega_3\mathsf{I}]\!]^{\nabla''}$.

Proof. The model is not trivial since it is easy to derive $\vdash^{\nabla'} \mathsf{I} : \tau$ while $\not\vdash^{\nabla'} \omega_3 : \tau$ by Lemma 6.

($\subseteq$) By Lemma 6 we have $\vdash^{\nabla''}\omega_3 : \alpha_i$ for all i, hence, by $(\leq_\nabla)$, $\vdash^{\nabla''}\omega_3 : \beta_i$ and $\vdash^{\nabla''}\omega_3 : \beta_i \to \tau \to W_i$. By rule ($\to$ E) it follows $\vdash^{\nabla''}\omega_3\omega_3 : \tau \to W_i$. Since $\vdash^{\nabla''}\mathsf{I} : \tau$, we conclude by rule ($\to$ E) $\vdash^{\nabla''}\omega_3\omega_3\mathsf{I} : W_i$ for all i, which implies $[\![M]\!]^{\nabla''} \subseteq [\![\omega_3\omega_3\mathsf{I}]\!]^{\nabla''}$.

($\supseteq$) Let $B \in [\![\omega_3\omega_3\mathsf{I}]\!]^{\nabla''}$. The only interesting case is $B \not\sim_{\nabla''} \Omega$. We get $\vdash^{\nabla''}\omega_3\omega_3\mathsf{I} : B$ and by Lemma 3(2), there exist $A, \bar{Q}$ such that $A \leq_{\nabla''} A \to A \to \bar{Q} \to B$ and $\vdash^{\nabla} \omega_3 : A$. By Lemma 7(2) there are I, $\alpha_i \in \mathbb{C}^{\nabla''}$, and n such that $C \leq_{\nabla''} C \to C \to A^n \to \bar{Q} \to B$, where $C \equiv \bigcap_{i\in I} \alpha_i$. By applying $n+1$ times Lemma 7(1) we get $\bigcap_{i\in J} \alpha_i \leq_{\nabla''} D \to \bar{Q} \to B$ for some $J \subseteq I$, where $D \equiv C$ if $n = 0$ and $D \equiv A$ otherwise. By Theorem 1 this gives $(\bigcap_{i\in H} \alpha_i) \cap (\bigcap_{i\in K}(\tau \to W_i)) \leq_{\nabla''} \bar{Q} \to B$ for some $H, K \subseteq J$, being $\alpha_i \sim_{\nabla''} (\gamma_i \to \alpha_i) \cap (\beta_i \to \tau \to W_i)$. Let $L = \{i \in H \mid W_i \sim_{\nabla''} \Omega\}$. Since, by Lemma 5 $W_i \not\sim_{\nabla''} \Omega$ implies $\bar{Q} \not\leq_{\nabla''} \gamma_i, \beta_i$, if we apply Theorem 1 to $(\bigcap_{i\in H} \alpha_i) \cap (\bigcap_{i\in K}(\tau \to W_i)) \leq_{\nabla''} \bar{Q} \to B$ we have that there exist $L' \subseteq L$ and $K' \subseteq K$ such that $(\bigcap_{i\in L'} \alpha_i) \cap (\bigcap_{i\in K'} W_i) \leq_{\nabla''} B$. Notice that $W_i \sim_{\nabla''} \Omega$ gives $\alpha_i \sim_{\nabla''} \alpha_i \to \alpha_i$. The proof by structural induction on terms that $\{x{:}A \mid x \in \mathrm{FV}(N)\} \vdash^{\nabla} N : A$ whenever $A \sim_{\nabla''} A \to A$ is easy. This implies $\vdash^{\nabla''} M : \alpha_i$ being M closed, i.e. $\alpha_i \in [\![M]\!]^{\nabla''}$ for all $i \in L'$. Moreover by construction $W_i \in [\![M]\!]^{\nabla''}$ for all $i \in K'$. Since $[\![M]\!]^{\nabla''}$ is a filter, $(\bigcap_{i\in L'} \alpha_i) \cap (\bigcap_{i\in K'} W_i)$ is in it, hence B too. □

5 Extensional models

In this section we will show how to modify the model construction of previous sections in order to obtain extensional filter models. In this way we will prove the consistency of $\lambda\beta\eta + \{\omega_2\omega_2 = M\}$ and of $\lambda\beta\eta + \{\omega_3\omega_3\mathsf{I} = M\}$.

5.1 Consistency of $\lambda\beta\eta + \{\omega_2\omega_2 = M\}$

If in Definition 8 we put:

- $\mathbb{C}^{\nabla_1} = \{\Omega, \chi\}$;
- $\nabla_1 = \nabla_0 \cup \{\chi \sim \Omega \to \chi\}$;

we obtain an eitt we call $\Sigma^{\nabla'_\eta}$. By Theorem 4, $\langle \mathcal{F}^{\nabla'_\eta}, \cdot, [\![\ ]\!]^{\nabla'_\eta}\rangle$ is an extensional λ-model. It is essentially the model of [20].

In $\Sigma^{\nabla'_\eta}$ the type χ plays the role of ι in $\Sigma^{\nabla'}$. So instead of Lemma 2 we need:

Lemma 8.

$$\nvdash^{\nabla'} \omega_2 : \chi \textit{ and } \nvdash^{\nabla'} \omega_2 : (\chi \to \chi) \to \chi \to \chi.$$

Proof. Define the set $\mathcal{E}_\Omega$ as the minimal set such that:

$$\Omega \in \mathcal{E}_\Omega; \quad A \in \mathbb{T}^{\nabla'_\eta}, B \in \mathcal{E}_\Omega \;\Rightarrow\; A \to B \in \mathcal{E}_\Omega;$$
$$A, B \in \mathcal{E}_\Omega \;\Rightarrow\; A \cap B \in \mathcal{E}_\Omega; \quad W_i \in \mathcal{E}_\Omega \;\Rightarrow\; \varsigma_i \in \mathcal{E}_\Omega.$$

It is easy to check by induction on $\leq_{\nabla'_\eta}$ that for all $A \in \mathbb{T}^{\nabla'_\eta}$:

$$B \leq_{\nabla'_\eta} A \;\&\; B \in \mathcal{E}_\Omega \;\Rightarrow\; A \in \mathcal{E}_\Omega.$$

This implies that $\Omega \not\leq_{\nabla'_\eta} \chi$ since $\Omega \in \mathcal{E}_\Omega$ and $\chi \notin \mathcal{E}_\Omega$.
Suppose by contradiction that $\vdash^{\nabla'_\eta} \omega_2 : \chi$, i.e. $\vdash^{\nabla'_\eta} \omega_2 : \Omega \to \chi$: we get $\Omega \leq_{\nabla'_\eta} \Omega \to \chi \sim_{\nabla'_\eta} \chi$ by Lemma 1(1).
Similarly from $\vdash^{\nabla'_\eta} \omega_2 : (\chi \to \chi) \to \chi \to \chi$ by Lemma 1(1) we get $\chi \to \chi \leq_{\nabla'_\eta} (\chi \to \chi) \to (\chi \to \chi) \to \chi \to \chi$, which implies by Theorem 1 $\chi \to \chi \leq_{\nabla'_\eta} \chi$. Applying again Theorem 1 to this last judgment we get $\Omega \leq_{\nabla'_\eta} \chi$. □

The model $\langle \mathcal{F}^{\nabla'_\eta}, \cdot, [\![\ ]\!]^{\nabla'_\eta}\rangle$ is not trivial since it is easy to derive $\vdash^{\nabla'_\eta} \mathsf{I} : (\chi \to \chi) \to \chi \to \chi$ while $\nvdash^{\nabla'_\eta} \omega_2 : (\chi \to \chi) \to \chi \to \chi$ by Lemma 8. The proof that this model equates $\omega_2\omega_2$ and M is just the proof of Theorem 6 using Lemma 8 instead of Lemma 2.

5.2 Consistency of $\lambda\beta\eta + \{\omega_3\omega_3\mathsf{I} = M\}$

If in Definition 9 we put:

- $\mathbb{C}^{\nabla_1} = \{\Omega, \sigma, \tau\}$;
- $\nabla_1 = \nabla_0 \cup \{\sigma \sim \tau \to \sigma, \tau \sim \sigma \to \sigma\}$;

and we erase the condition $\tau \equiv \iota \to \iota$, we obtain an eitt that we call $\Sigma^{\nabla''_\eta}$.[2] Notice that in this case it is not possible to start from a simpler theory Σ^{∇_1} built on just two constants, say Ω and χ, as in the case of $\Sigma^{\nabla'_\eta}$. If Σ^{∇_1} contains only two constants, we are forced to define $\tau \equiv \chi \to \chi$. Whatever equivalence we choose to make each constant equivalent to a suitable intersection of arrow, we derive type τ for ω_3, which contradicts Lemma 10(2) below.

By Theorem 4 $\langle \mathcal{F}^{\nabla''_\eta}, \cdot, [\![\;]\!]^{\nabla''_\eta} \rangle$ is an extensional λ-model. We prove that this model equates $\omega_3\omega_3\mathsf{I}$ to M just mimicking the same proof for the model $\langle \mathcal{F}^{\nabla''}, \cdot, [\![\;]\!]^{\nabla''} \rangle$.

First we introduce five subsets of $\mathbb{T}^{\nabla''_\eta}$, whose meaning is the following:

1. $\mathcal{E}_\Omega$ is the set of all types which are equivalent to Ω;
2. $\mathcal{G}_\sigma$ is the set of all types which are greater than or equivalent to σ;
3. $\mathcal{L}_\sigma$ is the set of all types which are less than or equivalent to σ;
4. $\mathcal{G}_\tau$ is the set of all types which are greater than or equivalent to τ;
5. $\mathcal{L}_\tau$ is the set of all types which are less than or equivalent to τ.

Definition 11. *The sets $\mathcal{E}_\Omega$, $\mathcal{G}_\sigma$, $\mathcal{L}_\sigma$, $\mathcal{G}_\tau$, $\mathcal{L}_\tau$, are defined as the minimal sets such that:*

$\Omega \in \mathcal{E}_\Omega$; $\;A \in \mathbb{T}^{\nabla''_\eta}, B \in \mathcal{E}_\Omega \Rightarrow A \to B \in \mathcal{E}_\Omega$; $\;A, B \in \mathcal{E}_\Omega \Rightarrow A \cap B \in \mathcal{E}_\Omega$;
$\sigma \in \mathcal{G}_\sigma$; $\;\mathcal{E}_\Omega \subseteq \mathcal{G}_\sigma$; $\;A \in \mathcal{L}_\tau, B \in \mathcal{G}_\sigma \Rightarrow A \to B \in \mathcal{G}_\sigma$; $\;A, B \in \mathcal{G}_\sigma \Rightarrow A \cap B \in \mathcal{G}_\sigma$;
$\sigma \in \mathcal{L}_\sigma$; $\;A \in \mathcal{G}_\tau, B \in \mathcal{L}_\sigma \Rightarrow A \to B \in \mathcal{L}_\sigma; A \in \mathbb{T}^{\nabla''_\eta}, B \in \mathcal{L}_\sigma \Rightarrow A \cap B, B \cap A \in \mathcal{L}_\sigma$;
$\tau \in \mathcal{G}_\tau$; $\;\mathcal{E}_\Omega \subseteq \mathcal{G}_\tau$; $\;A \in \mathcal{L}_\sigma, B \in \mathcal{G}_\sigma \Rightarrow A \to B \in \mathcal{G}_\tau$; $\;A, B \in \mathcal{G}_\tau \Rightarrow A \cap B \in \mathcal{G}_\tau$;
$\tau \in \mathcal{L}_\tau$; $\;A \in \mathcal{G}_\sigma, B \in \mathcal{L}_\sigma \Rightarrow A \to B \in \mathcal{L}_\tau$; $\;A \in \mathbb{T}^{\nabla''_\eta}, B \in \mathcal{L}_\tau \Rightarrow A \cap B, B \cap A \in \mathcal{L}_\tau$.□

Similarly to Lemma 4 we can easily show:

Lemma 9.

1. $\mathcal{E}_\Omega = \{A \in \mathbb{T}^{\nabla''_\eta} \mid \Omega \leq_{\nabla''_\eta} A\}$;
2. $\mathcal{G}_\sigma = \{A \in \mathbb{T}^{\nabla''_\eta} \mid \sigma \leq_{\nabla''_\eta} A\}$; 4. $\mathcal{G}_\tau = \{A \in \mathbb{T}^{\nabla''_\eta} \mid \tau \leq_{\nabla''_\eta} A\}$;
3. $\mathcal{L}_\sigma = \{A \in \mathbb{T}^{\nabla''_\eta} \mid A \leq_{\nabla''_\eta} \sigma\}$; 5. $\mathcal{L}_\tau = \{A \in \mathbb{T}^{\nabla''_\eta} \mid A \leq_{\nabla''_\eta} \tau\}$.

From previous lemma we immediately have that σ and τ are incomparable:

Corollary 1. *$\sigma \not\leq_{\nabla''_\eta} \tau$ and $\tau \not\leq_{\nabla''_\eta} \sigma$.*

In correspondence with Lemmas 5, 6, 7 we have:

[2] Remark that $\tau \equiv \iota \to \iota$ in $\Sigma^{\nabla''}$, while τ is an atom in $\Sigma^{\nabla''_\eta}$. We use the same name since this allows us to have the same defininition of $\alpha_{(n,m)}$.

Lemma 10.

1. *If* $W_i \not\sim_{\nabla''_\eta} \Omega$ *then* $\gamma_i \not\leq_{\nabla''_\eta} \beta_i \not\leq_{\nabla''_\eta} \alpha_i$ *and* $\bar{Q} \not\leq_{\nabla''_\eta} \gamma_i$.
2. *Let* $\psi \in \mathbb{C}^{\nabla''}$, *then* $\vdash^{\nabla''_\eta} \omega_3 : \psi$ *iff* $\alpha_i \leq_{\nabla''_\eta} \psi$ *for some* i.
3. *If* $\bigcap_{i\in I} \alpha_i \leq_{\nabla''_\eta} A \to B \to C$ *and* $\vdash^{\nabla''_\eta} \omega_3 : B$ *then there is* $J \subseteq I$ *such that* $\bigcap_{i\in J} \alpha_i \leq_{\nabla''_\eta} B \to C$.
4. *If* $A \leq_{\nabla''_\eta} A \to A \to B$ *and* $\vdash^{\nabla''_\eta} \omega_3 : A$ *then there are* I, $\alpha_i \in \mathbb{C}^{\nabla''_\eta}$, *and* n *such that* $C \leq_{\nabla''_\eta} C \to C \to A^n \to B$, *where* $C \equiv \bigcap_{i\in I} \alpha_i$.

Proof. Almost all proofs are the same as those of the corresponding lemmas in Section 4, by using Lemma 9 instead of Lemma 4. We only need to prove:

$$(a) \not\vdash^{\nabla''_\eta} \omega_3 : \tau; \; (b) \not\vdash^{\nabla''_\eta} \omega_3 : \sigma.$$

(a) If $\vdash^{\nabla''_\eta} \omega_3 : \tau \sim_{\nabla''_\eta} \sigma \to \sigma$, then by Lemma 3(1), it should be $\sigma \leq_{\nabla''_\eta} \sigma \to \sigma \to \sigma$, which implies $\tau \to \sigma \leq_{\nabla''_\eta} \sigma \to \tau$. In particular, by Theorem 1, being $\tau \not\sim_{\nabla''_\eta} \Omega$ by Lemma 9(1), it should be $\sigma \leq_{\nabla''_\eta} \tau$, which is impossible by Corollary 1.
(b) is proven similarly to previous Point. If $\vdash^{\nabla''_\eta} \omega_3 : \sigma$, it should hold $\tau \leq_{\nabla''_\eta} \tau \to \tau \to \sigma$. Since $\tau \sim_{\nabla''_\eta} \sigma \to \sigma$, this should implies $\tau \leq_{\nabla''_\eta} \sigma$, which is impossible by Corollary 1. $\Box$

The model $\langle \mathcal{F}^{\nabla''_\eta}, \cdot, [\![\,]\!]^{\nabla''_\eta} \rangle$is not trivial since it is easy to derive $\vdash^{\nabla''_\eta} \mathsf{I} : \tau$ while $\not\vdash^{\nabla''_\eta} \omega_3 : \tau$ by the proof of Lemma 10(2). The proof that this model equates $\omega_3\omega_3\mathsf{I}$ and M is just the proof of Theorem 8 using Lemma 10 instead of Lemmas 5, 6, 7.

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