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A Convex Powerdomain over Lattices: its Logic and λ-Calculus

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Abstract. To model at the same time parallel and non-deterministic functional calculi we define a powerdomain functor $\mathcal{P}$ such that it is an endofunctor over the category of algebraic lattices. $\mathcal{P}$ is locally continuous and we study the initial solution D^∞ of the domain equation $D = \mathcal{P}([D \to D]_\perp)$. We derive from the algebras of $\mathcal{P}$ the logic of D^∞, that is the axiomatic description of its compact elements. We then define a λ-calculus and a type assignment system using the logic of D^∞ as the related type theory. We prove that the filter model of this calculus, which is isomorphic to D^∞, is fully abstract with respect to the observational pre-order of the λ-calculus.

Keywords: λ-calculus, Non-determinism, Full Abstraction, Powerdomain Construction, Intersection Type Disciplines.

1. Introduction

One of the main issues in the design of programming languages is the achievement of a good compromise between the multiplicity of control structures and data types and the unicity of the mathematical semantics. This is especially compelling when combinations of different features induce an intricate syntax. In particular the development of functional languages endowed with primitives for parallelism and concurrency, like Concurrent Clean [33], CML [36] and FACILE [16, 17], calls for a mathematical treatment of functions and their extensions to "sets" of values (for a survey about the subject see e.g. [39]). The aim is to give a semantics more abstract than term models, and logical principles to reason about programs.

When dealing with these problems, it is usual to start with a calculus, and then to discover an abstract model explaining the underlying mathematical ideas. We go in the opposite direction: given a difficult problem, such as assessing in the same language call-by-name and call-by-value functional abstraction, non-deterministic and parallel evaluation, we concentrate on those functors over domains that seem to be good candidates for a mathematical formalizations of these notions. In this perspective the desired "combination" is realized by functorial composition in a system of domain equations. Provided that the system has a solution, this determines both a domain of interpretation and, what is more relevant, the guidelines of the definition of the syntax.

In the present paper we address the question of combining functionality, non-determinism and parallelism. In [11] and [14, 15] the problem was solved using domains which were algebraic lattices, but this prevents from the use of the convex powerdomain [35], as the category of algebraic lattices is not closed under that functor. A non-deterministic λ-calculus modeled by convex powerdomain in the category of SFP has been studied by Ong [32], but it remains unclear how to model parallelism in that setting.

In the present paper we propose a solution based on a new powerdomain functor. This functor is close to Plotkin's convex powerdomain functor, but it works within the category of algebraic lattices, so that parallelism can be understood as a join operator.

An extended abstract of the present paper appeared as [5].

1.1. Powerdomains in the category of algebraic lattices

Set theoretically a many-valued function is a function with values in some powerset. If one takes domains and continuous functions as abstract counterparts of data types and computable functions, the problem of a theory of many-valued functions reduces to the problem of a theory of powerdomains. Different constructions are possible, essentially because of different treatments of the undefined object. In the theory of powerdomains (see [35, 40, 24] and [20] for an elementary presentation) three main constructions have been devised (for other constructions, see e.g. [18, 19, 21]): the lower, upper and convex powerdomains. Roughly speaking, the lower powerdomain identifies those sets which have the same defined objects, while the upper powerdomain identifies with the totally undefined set any set having the undefined object among its elements. The convex powerdomain has a finer order on sets, which is obtained by combining those of both lower and upper powerdomains. A consequence is that the presence of the undefined object within a set is neither ignored nor collapsing the set to the undefined object itself.

It could be asked, however, what are the right powerdomains to model parallelism and non-determinism respectively. There is a general agreement in considering the parallel composition of two functions as the best behaved among them with respect to the partial ordering of their ranges. In case of functions this leads to the interpretation of the parallel operator as a join: in case of many-valued functions the lower powerdomain is the right choice. These two things match in the case of prime algebraic lattices: indeed any such structure is isomorphic to the lower powerdomain of its compact coprime elements; therefore the continuous union operation of the powerdomain coincides with the join. This reinforces the intuition that lower powerdomain and lattices are natural models for such "parallel functions" (see e.g. [11] for an application to parallel extensions of both the lazy, call-by-name and call-by-value lambda calculus).

Things are less clear when modeling non-determinism. In this case, if we want to keep it distinct from parallelism (see [27] for some reasons to do this), we are left either with a theory of total sets (i.e. not including $\perp$) based on the upper powerdomain (for which see [14, 15]), or with some theory of partial sets. Unfortunately the category **ALG** of algebraic lattices is not closed under the convex powerdomain, which is a serious drawback if we wish to model by partial sets a calculus in which both non-determinism and parallelism are present (like in [31, 14]).

To solve this problem we introduce a new powerdomain functor $\mathcal{P}$ such that the category of algebraic lattices is closed under $\mathcal{P}$. Let D be an algebraic lattice; following a general pattern, we consider the set $M(D)$ of finite non-empty sets of compacts in D, and define a pre-order over this set by

$$u \preceq v \;\Leftrightarrow\; [\forall e \in v\; \exists d \in u.\; d \sqsubseteq e] \;\&\; \bigsqcup u \sqsubseteq \bigsqcup v, \tag{1}$$

By contrast we recall the definition of the preorder on $M(D)$ inducing the convex (or Plotkin) powerdomain:

$$u \preceq^{\natural} v \;\Leftrightarrow\; [\forall e \in v\; \exists d \in u.\; d \sqsubseteq e] \;\&\; [\forall d \in u\; \exists e \in v.\; d \sqsubseteq e]$$

Clause $\forall e \in v\; \exists d \in u.\; d \sqsubseteq e$ is the same in both definitions, and it is exactly the definition of Smyth preorder, inducing the upper powerdomain. Clause $\bigsqcup u \sqsubseteq \bigsqcup v$ in the definition of $\preceq$ is weaker than $[\forall d \in u\; \exists e \in v.\; d \sqsubseteq e]$: actually the latter implies the former. In this sense our powerdomain is a weakening of convex powerdomain, retaining, however, the property of convex sets: indeed if $d_1 \sqsubseteq d_2 \sqsubseteq d_3$, then $\{d_1, d_2, d_3\}$ is equivalent to $\{d_1, d_3\}$ under both preorders.

The quotient of $M(D)$ under $\preceq$ is a sup-semilattice. It follows that, if we take $\mathcal{P}(D)$ as its ideal completion, then we obtain a complete lattice, which is algebraic. Singleton $\{\!|\_|\!\}$ and union $\uplus$ operations are defined in the standard way, and the action of $\mathcal{P}$ over a continuous map $f : D \to E$ of algebraic lattices is then given by the unique continuous extension of the map

$$f^{\dagger}(\{\!|d_1|\!\} \uplus \cdots \uplus \{\!|d_n|\!\}) = \{\!|f(d_1)|\!\} \uplus \cdots \uplus \{\!|f(d_n)|\!\} \uplus \{\!|f(\textstyle\bigsqcup_{i \le n} d_i)|\!\}. \tag{2}$$

It follows that $\mathcal{P}$ is a functor and that $\mathcal{P}(D)$ is actually a free construction, and indeed it is initial in the category of $\mathcal{P}$-algebras, here called **oper-ALG**. This consists in the subcategory of **ALG**, whose objects are *oper-lattices* (introduced in this paper), that is lattices endowed with a continuous binary operation $*$ which is idempotent, commutative and associative, and which satisfies the law

$$(d_1 * d_2) \sqcup d_3 = (d_1 \sqcup d_3) * (d_2 \sqcup d_3).$$

Morphisms of oper-lattices are continuous maps satisfying the law

$$f(d_1 * d_2) = f(d_1) * f(d_2) * f(d_1 \sqcup d_2).$$

1.2. A notion of call-by-value functions

$\mathcal{P}(D)$ is a powerdomain of partial sets, in the sense of the discussion of the previous paragraph, and it is similar to the convex powerdomain. More precisely, the closure operator associated to $\mathcal{P}$ has as fixed points exactly those subsets of D which are convex (X is convex if $X \ni x \sqsubseteq y \sqsubseteq z \in X$ implies $y \in X$) and closed under arbitrary join.

Let us concentrate on the set of compact elements $K(\mathcal{P}(D))$ of a lattice $\mathcal{P}(D)$. There are three kinds of objects: the bottom element, that is $\bot_{\mathcal{P}(D)} = \{\!|\bot_D|\!\}$; total objects of the shape $\{\!|d_1, \ldots, d_n|\!\}$ (an abbreviation for $\{\!|d_1|\!\} \uplus \cdots \uplus \{\!|d_n|\!\}$), where $d_i \neq \bot$ for all i; finally objects of the shape $\{\!|d_1, \ldots, d_n, \bot|\!\}$, the partial objects different from $\bot$. Two main choices for the set of values are possible: either a value is any object which is different from $\bot_{\mathcal{P}(D)}$, call this set $\mathcal{V}^{\flat}$; or the set of values coincides with the set of total objects, call this set $\mathcal{V}^{\sharp}$ (strictly speaking, these definitions concern just compact values, but they uniquely extend to the whole $\mathcal{P}(D)$).

In order to compare these two possibilities, let us consider the spaces of call-by-value functions that they determine. First of all these have to be strict functions and, since they are defined from powerdomains to powerdomains, they are taken as extensions of functions between domains underlying the powerdomains. More precisely, we consider functions $g : \mathcal{P}(D) \to \mathcal{P}(E)$ such that $g = \mathcal{P}(f_{\bot})$ for some $f : D \to E$. As usual, $f_{\bot} = \lambda d.$ if $d = \bot$ then $\bot$ else $f(d)$. Thus, for instance, $g(\{\!|d, \bot|\!\}) = \{\!|f(d), \bot|\!\}$.

We now parametrize the above definition with respect to a set $\mathcal{V} \subseteq \mathcal{P}(D)$ of values. If $f : D \to E$ then it determines the call-by-value function $f^{\mathcal{V}} : \mathcal{P}(D) \to \mathcal{P}(E)$, relative to $\mathcal{V}$:

$$f^{\mathcal{V}} = \lambda s \in \mathcal{P}(D).\ \text{if } s \in \mathcal{V} \text{ then } \mathcal{P}(f_\perp)(s) \text{ else } \perp. \tag{3}$$

It is easy to see that $f^\flat(\{\!| d, \perp |\!\}) = \{\!| f(d), \perp |\!\}$, while $f^\sharp(\{\!| d, \perp |\!\}) = \perp$ (where $f^\flat = f^{\mathcal{V}^\flat}$ and $f^\sharp = f^{\mathcal{V}^\sharp}$). Moreover, if both d_1 and d_2 are different from $\perp$, then $f^\flat(\{\!| d_1, d_2 |\!\}) = \{\!| f(d_1), f(d_2), f(d_1 \sqcup d_2) |\!\}$. In general this amounts to say that, for any $s, t \in \mathcal{P}(D)$,

$$f^\flat(s \uplus t) \in \mathcal{V}^\flat \ \Leftrightarrow\ f^\flat(s) \in \mathcal{V}^\flat \text{ or } f^\flat(t) \in \mathcal{V}^\flat \ \Leftrightarrow\ f^\flat(s \vee t) \in \mathcal{V}^\flat,$$

where $\vee$ is the join in $\mathcal{P}(D)$. This implies that call-by-value functions do not discriminate the union and the join when the set of values is $\mathcal{V}^\flat$. On the other hand choosing $\mathcal{V}^\sharp$ we have

$$\begin{aligned} f^\sharp(s \uplus t) \in \mathcal{V}^\sharp \ &\Leftrightarrow\ f^\sharp(s) \in \mathcal{V}^\sharp \text{ and } f^\sharp(t) \in \mathcal{V}^\sharp \\ f^\sharp(s \vee t) \in \mathcal{V}^\sharp \ &\Leftrightarrow\ f^\sharp(s) \in \mathcal{V}^\sharp \text{ or } f^\sharp(t) \in \mathcal{V}^\sharp. \end{aligned}$$

Note that, if e.g. $s = \{\!| d_1, \perp |\!\}$ and $t = \{\!| d_2, \perp |\!\}$, then $s \vee t = \{\!| d_1, d_2, d_1 \sqcup d_2, \perp |\!\}$. Since we want to preserve the distinction between many-valued functions arising from non-determinism and those which model parallel computations, our choice has to be $\mathcal{V} = \mathcal{V}^\sharp$.

1.3. The domain equation

We choose a type-free universe to work with. To treat functions as objects in the universe we endow the powerdomain with an applicative structure. Much like in case of models of type-free λ-calculus, objects are at the same time functions and elements in the domain of these functions: in the present case they should be viewed also as sets of functions.

Moreover all functions are basic values (that is non-bottom elements of the domain D of which we shall take the powerdomain $\mathcal{P}(D)$), hence the space of functions has to be lifted, so that even the everywhere undefined function will be different from the bottom (and hence from the singleton of the bottom). This leads to the following system of domain equations:

$$\begin{cases} D = \mathcal{P}(N_\perp) \\ N = [D \to D]. \end{cases} \tag{4}$$

If we want to underline that the objects we are dealing with are "sets", then we have to solve the equation

$$D = \mathcal{P}([D \to D]_\perp). \tag{5}$$

Otherwise, if the first class objects are functions, we get

$$N = [\mathcal{P}(N_\perp) \to \mathcal{P}(N_\perp)]. \tag{6}$$

Both solutions exist, and we call them D^∞ and N^∞ respectively, since $\mathcal{P}$ turns out to be locally continuous, so that they can be obtained as direct limits.

Considering the equation (5), application $(\_ \cdot \_)$ is naturally defined (restricting ourselves again to the compact elements) as follows:

$$\{\!| f_1, \ldots, f_m |\!\} \cdot d = \begin{cases} f_1(d) \uplus \cdots \uplus f_m(d) & \text{if } \{f_1, \ldots, f_m\} \subseteq [D \to D] \\ f_1(d) \uplus \cdots \uplus f_{m-1}(d) \uplus \perp & \text{if } \{f_1, \ldots, f_{m-1}\} \subseteq [D \to D] \text{ and } f_m = \perp. \end{cases} \tag{7}$$

On the other hand, if we consider (6), we have a particular case of the construction in [29]. Indeed it turns out that the functor $T = \mathcal{P} \circ (\_)_\perp$ is a strong monad, and that the solution N^∞ is a call-by-name T-reflexive object. Following [29], we can define an application morphism which, up to the usual identification of an element d of $D^\infty = T(N^\infty)$ with the corresponding "point" $d : 1 \to T(N^\infty)$ (where 1 is the terminal object in the category), turns out to coincide with $(\_ \cdot \_)$.

1.4. Representation of D^∞

Working inside the category of algebraic lattices, with Scott continuous functions as morphisms, has also the technical advantage that the domain can be described using the theory of Extended Abstract Type Structures (see [12]): this is simpler than the theory of Domain Prelocales (see [2]) which is used for describing SFP domains. EATS are isomorphic to the (dual of the) join-semilattice of compact elements of ω-algebraic lattices. A minimal EATS is generated from a countable set of type constants closed under a binary operator $\wedge$. Types are taken with a pre-order $\leq$ such that, taking the quotient, $\sigma \wedge \tau$ is the meet of σ and τ. Given an EATS, the whole domain is recovered from filters of types, ordered by subset inclusion.

EATS's are instances of Information Systems (see [37]), and represent several domain constructors. As usual, intersection represents join: no compatibility restriction is required since we work with lattices (for more details see [13]). The constructors involved in equation (5) are lifting, exponentiation and the powerdomain functor $\mathcal{P}$. The lifting is easily represented by adding a constant ω as the top, so that the filter $\uparrow \omega$ will be the newly added bottom of the domain of filters. The space of continuous functions is represented by arrow types $\sigma \to \tau$. The axioms concerning the arrow are the usual contravariance-covariance law plus

$$(\sigma \to \tau) \wedge (\sigma \to \rho) \leq \sigma \to \tau \wedge \rho$$
$$\sigma \to \omega \leq \omega \to \omega.$$

The former is needed to have a unique representation of step functions via principal filters of arrow types. The latter implies that the set of arrow types has a maximum, namely $\omega \to \omega$, so that, dually, the function space has a minimum: $\lambda x.\ \perp$. It is then clear why we have to rule out the inequation $\omega \leq \omega \to \omega$ (which is on the contrary included e.g. in [9]), since this would collapse the bottom of the space $[D \to D]$ with the bottom of $[D \to D]_\perp$.

We are left with the functor $\mathcal{P}$. The compact elements of $\mathcal{P}([D \to D]_\perp)$ can be identified with the set $M(\mathcal{P}([D \to D]_\perp))$, therefore we need a type connective which is the EATS counter-part of the union. To this aim we introduce a binary operator $\oplus$, and the axioms involving $\oplus$ follow directly from the definition and properties of $\preceq$. $\oplus$ is idempotent, commutative, associative and distributes over meet. This, together with the covariance of $\oplus$ in both its arguments, implies that the following equation holds:

$$\sigma \oplus \tau = \sigma \oplus \tau \oplus (\sigma \wedge \tau),$$

which represents the property that "sets" in our powerdomain are closed under joins of their elements. We just remark that no special constructor is needed for the singleton operation, since all "singletons" will be represented exactly by those types that cannot be non-trivially equated to any type whose outermost operator is $\oplus$.

We can now face the problem of representing the initial solution of equation (5). Indeed, applying to the case of EATS the technique to solve domain equations using Information Systems (see [26]), we know that it suffices, for each domain D_n in the direct limit $D^\infty = \lim_{\leftarrow} D_n$ (that gives the solution of (5)), to put into a single bag the types that represent its compacts, and then to take the space of filters. Now D_0 is the one element domain, hence no basic type constant but the constant ω is needed, and the representation of the domain D^∞ will simply be the set $\mathcal{F}$ of all filters of types generated from ω closing under $\wedge$, $\to$ and $\oplus$, which are (pre)-ordered as described above.

1.5. A λ-calculus of call-by-name and call-by-value many-valued functions

Usually one begins with a syntax, that is a language and a notion of evaluation, and looks for a model and an interpretation equating equivalent terms. In our exposition we go in the opposite direction. First an abstract structure of objects has been devised, and then we look for a syntax which is expressive with respect to that structure. The criterion of expressibility, as we shall see in the next sections, is adequacy and completeness, i.e. full abstraction.

To fix ideas, we start with the syntax of type free λ-calculus. We have to define both call-by-name and call-by-value abstractions, and we choose to introduce two sorts of variables, namely $x, y \ldots$ and $v, w, \ldots$ so that $\lambda x.M$ and $\lambda v.M$ are call-by-name and call-by-value abstractions, respectively.

Then we enrich this syntax (see e.g. [31, 14]) by adding binary operators for set construction (implicitly representing the non-deterministic internal choice) and parallel evaluation: "+" and "$\|$", respectively. The intended interpretations of these operators are the powerdomain union and the join of the lattice.

Turning to the reduction relation, usually written $\longrightarrow$, we consider rules (β), (μ) and (ν) from the classical λ-calculus (see [8]), but not rule (ξ). This last rule allows to reduce under λ-abstractions. Now a λ-abstraction always denotes a function (to be identified with the singleton of that function), then a total object, that is an element of $\mathcal{V}$, so that it doesn't make sense to evaluate further its body (see [34]).

To cope with commutativity, associativity and idempotence of $+$ and $\|$ we introduce a congruence relation $\approx$ such that e.g. $M + N \approx N + M$, and then we put

$$M \approx M',\ M' \longrightarrow N',\ N' \approx N \ \Rightarrow\ M \longrightarrow N. \tag{8}$$

Beside that, the reduction rules for $+$ and $\|$ will be the same: first if $M \longrightarrow M'$, then both $M + N \longrightarrow M' + N$ and $M\|N \longrightarrow M'\|N$. Second, since anything can be applied to anything else, we have to add rules for reducing an application where the leftmost term is either a sum or a parallel composition. These are

$$(M+N)L \longrightarrow (ML) + (NL) \qquad \text{and} \qquad (M\|N)L \longrightarrow (ML)\|(NL).$$

To understand these rules one may think, for the leftmost one, of the semantic definition of application (7). The rightmost one is explained by the fact that the ordering of the function space is the pointwise ordering, so that the join of two functions f and g is the map $\lambda x.\ f(x) \sqcup g(x)$. We account for the distributivity of the join over the union in the domain D by adding the rule

$$(M+N)\|L \longrightarrow (M\|L) + (N\|L). \tag{9}$$

Although union also distributes over join, we do not have the rule

$$(M\|N) + L \longrightarrow (M+L)\|(N+L), \tag{10}$$

essentially because, as it will appear clear, the calculus requires to bubble sums, and not to nest them.

The point of having the same operational semantics for both $+$ and $\|$ is that, as it is the case for denotational semantics, we expect that they are discriminated by call-by-value functions. It is then crucial to give the definition of the set $\mathbf{V}$ of syntactical values and the corresponding rule of call-by-value β-contraction.

The following grammar for the set $\mathbf{V}$ of syntactical values is justified thinking to the set $\mathcal{V}$ of semantical values:

$$V ::= v \mid \lambda x.M \mid \lambda v.M \mid V + V \mid V\|M \mid M\|V. \tag{11}$$

Indeed a call-by-value variable is meant to be interpreted into (a subset of) $\mathcal{V}$, hence it is a syntactical value. Abstractions, both call-by-name and call-by-value, are objects in $[D \to D]$, and therefore, as elements of D, singletons of non-bottom elements. If V_1 and V_2 are values, hence sets of non-bottom elements, their union is such, therefore $V_1 + V_2$ is a value. Finally, it is not difficult to show that $\mathcal{V}$ is an upper-closed set, hence $V\|M$ and $M\|V$ are values for any M. Any other term, that is something of the shape MN or of the shape $M+N$ and $M\|N$ but not generated by the grammar (11), is not immediately recognizable as a value (even if it possibly will reduce to a value).

We come to the definition of the call-by-value contraction. This rule usually takes the form

$$(\lambda v.M)V \longrightarrow M[V/v] \qquad \text{if } V \in \mathbf{V}. \tag{12}$$

In view of (3), however, this rule is sound just in case V is a syntactical value that denotes a singleton. Indeed, if V has the shape $V_1 + \cdots + V_n$ (where, say, each V_i is a singleton value), then (3) and (2) show that the effect of reducing $(\lambda v.M)V$ should be the following. First compute the "distribution" of the function denoted by $\lambda v.M$ over the elements of the "set" V, including their join:

$$(\lambda v.M)(V_1 + \cdots + V_n) \longrightarrow (\lambda v.M)V_1 + \cdots + (\lambda v.M)V_n + (\lambda v.M)(V_1\| \cdots \|V_n) \tag{13}$$

and then apply (12).

To realize this, it is essential to discriminate among values in general and singleton values. Tentatively one just drops $V+V$ from (11). Unfortunately this is not enough. Consider the term $(M+N)\|V$, where V represents a singleton. By (9) this evaluates to $(M\|V)+(N\|V)$, which doesn't denote a singleton in general. Clearly we require that, whenever $V \xrightarrow{*} V'$, $(\lambda v.M)V$ and $(\lambda v.M)V'$ are semantically equivalent (for now: denote the same object).

Even worse, we have to consider the term $L\|V$, where L is not a sum itself, but just reduces to a sum: something which cannot be effectively foretold. This says that we cannot have a simple syntactical definition (like a grammar) which discriminates between terms denoting singletons and terms denoting sets of more than one element.

In order to approximate the solution we observe that this problem is similar to the general one of recognizing whether a term denotes a value or not in the classical call-by-value λ-calculus (see [34]). In that case to evaluate $(\lambda v.M)N$ one reduces N until (eventually) a value is obtained, and then the contraction takes place. Here we do the same until a value is reached from N (if any), but then we have to distinguish among several cases. If V is a closed term in the subset $\mathbf{W}$ of $\mathbf{V}$ generated by the grammar

$$W ::= v \mid \lambda x.M \mid \lambda v.M \mid W\|W \tag{14}$$

then rule (12) can be safely applied, since V surely denotes a singleton value. We observe that in such a case V can never reduce to a sum. If V is in the subset $\mathbf{U}$ of $\mathbf{V}$ generated by the grammar

$$U ::= W \mid U\|M \mid M\|U \tag{15}$$

but $V \notin \mathbf{W}$, then V is not a normal form, even if it is always equivalent via $\approx$ to a term of the shape $W\|N$. For a similar notion of value which is not irreducible see [11, 14]. In this case we cannot simply substitute V for v in M as in (12), since V could evaluate to a sum. Indeed, if $V \in \mathbf{U}\setminus\mathbf{W}$, then $V \approx W\|N$ for some $W \in \mathbf{W}$ and $N \notin \mathbf{U}$. It is clear that W is not guaranteed to be the "best" among W and N, since N could evaluate to something which is better behaved than W, at least in some contexts. Therefore we replace v by W and we put the result in parallel with $(\lambda v.M)V'$, where V' is obtained from V by one step of reduction.

$$(\lambda v.M)V \longrightarrow M[W/v]\|(\lambda v.M)V' \qquad \text{if } V \approx W\|N \ \&\ W \in \mathbf{W} \ \&\ N \notin \mathbf{U} \ \&\ V \longrightarrow V'. \tag{16}$$

Finally, if $V \in \mathbf{V}\setminus\mathbf{U}$ (in such a case it is a proper sum), then rule (13) applies.

1.6. The convergence predicate and the observational pre-order

To compare the operational semantics with the denotational semantics and hence to check whether the expressibility criterion has been satisfied or not, we introduce a (pre)-congruence among terms which is based on the notion of convergence that the previous definition of $\longrightarrow$ naturally induces. Let the predicate $M\Downarrow^{must}$, to be read as "M must converge", be defined as follows:

$$M\Downarrow^{must} \;\Leftrightarrow\; \exists V \in \mathbf{V}.\; M \stackrel{*}{\longrightarrow} V. \tag{17}$$

At first sight this looks like a *may* convergence predicate. Indeed "*must* converge" should be reserved to those terms such that any reduction out of them produces a value. However, a notion of must convergence related to considering all possible reductions of a term is problematic, since, as it will be shown in Remark 4.2, the reduction relation informally presented in the above subsection is not Church-Rosser.

However there are two relevant facts: first we shall prove that, if $M\Downarrow^{must}$ and $M \longrightarrow N$, then $N\Downarrow^{must}$, which is a strong property especially because of lack of confluence. Second our values are sets, representing all possible outputs of a non-deterministic computation; but they are also, as values, total objects, hence the convergence notion that they induce is a must convergence predicate.

At the same time we introduce a *may* convergence predicate, which is motivated by the possible choice of $\mathcal{V}^\flat$ as the set of values. Its definition is (without changing the definition of syntactical values)

$$M\Downarrow^{may} \Leftrightarrow \exists V \in \mathbf{V}, N.\; M \stackrel{*}{\longrightarrow} V + N.$$

Note that if $M\Downarrow^{must}$ then $M\Downarrow^{may}$: indeed if $M \stackrel{*}{\longrightarrow} V$ for some $V \in \mathbf{V}$, then, by (8), it is also the case that $M \stackrel{*}{\longrightarrow} V + V$.

Any convergence predicate $\Downarrow$ induces a theory which is a Morris theory (see [30]). The theory is the set of the following inequations:

$$M \sqsubseteq^{\mathcal{O}} N \;\Leftrightarrow\; \forall \text{ context } C[\,] \text{ closing both } M, N.\; C[M]\Downarrow \;\Rightarrow\; C[N]\Downarrow . \tag{18}$$

Otherwise stated it is the theory of the observational pre-order where the only observable is convergence (see [3] and a long sequence of papers thereafter).

We shall study the theory induced by the predicate $\Downarrow^{must}$ for the same reason we chose $\mathcal{V}^\sharp$ instead of $\mathcal{V}^\flat$ as our semantical notion of value. Indeed with the second choice union and join collapse; similarly, choosing $\Downarrow^{may}$ as our convergence predicate, $+$ and $\|$ collapse.

1.7. The type assignment system and full abstraction theorem

Let $[\![\cdot]\!] : Env \times \Lambda_{+\|} \to D^\infty$ be the interpretation map formalizing the denotation of terms as informally described in the previous paragraphs. The statement assuring that the calculus correctly describes the structure D^∞ is:

$$M \sqsubseteq^{\mathcal{O}} N \;\Leftrightarrow\; \forall \eta \in Env.\; [\![M]\!]_\eta \sqsubseteq [\![N]\!]_\eta.$$

To establish this equivalence we use a standard methodology. We introduce a formal system assigning types to terms, starting from type assignments for (term) variables. The judgments of this system are of the shape

$$\Gamma \vdash M : \tau,$$

where Γ is a mapping of variables into the set of types and it is called the "basis" of the judgment. Deduction rules in the system will describe the structure of types according to the structure of terms, with three relevant exceptions:

$$(\omega)\quad \Gamma \vdash M : \omega \qquad (\leq)\quad \frac{\Gamma \vdash M : \sigma \qquad \sigma \leq \tau}{\Gamma \vdash M : \tau} \qquad (\wedge\,\mathrm{I})\quad \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \wedge \tau}.$$

These rules ensure that the set of types that can be given to any term M is a filter.

Now the map $[\![\cdot]\!]$ induces an interpretation $[\![\cdot]\!]^{\mathcal{F}}$ sending terms into $\mathcal{F}$ via the isomorphism between D^∞ and $\mathcal{F}$. Then we prove that

$$[\![M]\!]^{\mathcal{F}}_\eta = \{\sigma \mid \exists \Gamma.\; \Gamma \models \eta \;\&\; \Gamma \vdash M : \sigma\},$$

where η associates to each variable a filter, and $\Gamma \models \eta$ is true if and only if $\Gamma(\chi) \in \eta(\chi)$ for all (term) variable χ.

This theorem says that types characterize the denotations of terms. The next step is to show that types characterize the operational behaviors of terms. This is accomplished first by showing that types are preserved by conversion (namely by the reflexive, symmetric and transitive closure of $\longrightarrow$), denoted by $=$:

$$\Gamma \vdash M : \sigma \quad \text{and} \quad M = N \quad \Rightarrow \Gamma \vdash N : \sigma.$$

Further that types characterize the convergence of (closed) terms, that is

$$M \Downarrow^{must} \Leftrightarrow \vdash M : \omega \to \omega \qquad \text{and} \qquad M \Downarrow^{may} \Leftrightarrow \vdash M : (\omega \to \omega) \oplus \omega.$$

Finally we have to show that the calculus is able to check internally the typeability of a term M by a type σ. This can be obtained by finding terms R_σ and T_σ such that

$$\vdash \mathsf{R}_\sigma : \tau \;\Leftrightarrow\; \sigma \leq \tau \quad \text{and} \quad \mathsf{T}_\sigma M \Downarrow^{must} \Leftrightarrow \vdash M : \sigma.$$

Intuitively, R_σ and T_σ represent respectively the "worst" term that has the type σ and the "test" term which, when applied, checks whether its argument has type σ. R_σ and T_σ will be defined by mutual induction.

It is at this point that one realizes that something is missing in the calculus, namely the definability of the join of the elements of a set, which is essential to test if a sum $M + N$ has a type $\sigma \oplus \tau$ exactly when either the first one has type σ and the second τ or viceversa. Therefore we introduce in the syntax of the calculus an unary operator $\coprod$ whose semantics is

$$\coprod(\{\!|d_1, \ldots, d_m|\!\}) = \{\!| \bigsqcup_{i \leq m} d_i |\!\},$$

and we add to the definition of $\longrightarrow$ suitable rules dealing with this operator.

2. The Powerdomain Construction

In order to meet the requirements illustrated in the introduction, we construct a new powerdomain functor, based on a pre-order which coincides with Egli-Milner pre-order on those (finite) sets which contain a maximum.

Next we study the solution of a domain equation which yields an applicative structure in which objects are "sets" of functions from objects to objects.

We recall some standard definitions and notions. A *complete lattice* is a poset $\langle D, \sqsubseteq \rangle$, in which every subset $X \subseteq D$ has a sup ($\bigsqcup X$) and, therefore, an inf ($\sqcap X$). An element $c \in D$ is *compact* if and only if $c \sqsubseteq \bigsqcup X$ implies $c \sqsubseteq \bigsqcup Y$ for some finite $Y \subseteq X$, where X is an arbitrary subset of D. As usual $K(D)$ denotes the set of compact elements of D. For $X \subseteq D$, we write $\downarrow X = \{d \in D \mid \exists d' \in X.\ d \sqsubseteq d'\}$ for the left-closure of X, and $K(d) = K(D) \cap \downarrow \{d\}$. A lattice is *algebraic* if it is complete, and for every $d \in D$ $K(d)$ is directed and $d = \bigsqcup K(d)$. D is ω-*algebraic* if it is algebraic and $K(D)$ is denumerable.

Let $M(D)$ be the set of all finite non-empty subsets of $K(D)$. We introduce three pre-orders on $M(D)$: Smyth's pre-order ($\sqsubseteq^S$), Egli-Milner pre-order ($\sqsubseteq^{EM}$) and the new pre-order ($\preceq$).

2.1. Definition. Let $u, v \in M(D)$:

(i) $u \sqsubseteq^S v$ if and only if for all $e \in v$ there is $d \in u$ such that $d \sqsubseteq e$;

(ii) $u \sqsubseteq^{EM} v$ if and only if $u \sqsubseteq^S v$ and for all $d \in u$ there is $e \in v$ such that $d \sqsubseteq e$;

(iii) $u \preceq v$ if and only if $\bigsqcup u \sqsubseteq \bigsqcup v$ and $u \sqsubseteq^S v$;

(iv) $\simeq^S$, $\simeq^{EM}$ and $\simeq$ are the equivalences over $M(D)$ induced by $\sqsubseteq^S$, $\sqsubseteq^{EM}$, and $\preceq$, respectively.

As an immediate consequence of this definition we have the following properties of $\preceq$.

2.1. LEMMA. For all $u, v \in M(D)$:

(i) $u \simeq u \cup \{\bigsqcup u\}$;

(ii) $u \preceq v \Leftrightarrow u \cup \{\bigsqcup u\} \sqsubseteq^{EM} v \cup \{\bigsqcup v\}$.

(iii) $(M(D), \preceq)$ is a sup-semilattice. The join is given by

$$u \tilde{\sqcup} v = \{a \sqcup b \mid a \in u \ \& \ b \in v\}.$$

Proof. We prove just (iii) The correctness of the definition of $u \tilde{\sqcup} v$ relies on the closure of $K(D)$ under finite join. Obviously $u, v \preceq u \tilde{\sqcup} v$. Let $u \preceq w$, $v \preceq w$, for $w \in M(D)$. Fix any element $c \in w$. Then $\exists a \in u, b \in v. a \sqsubseteq c \ \& \ b \sqsubseteq c$, hence $a \sqcup b \sqsubseteq c$. This proves $u \tilde{\sqcup} v \sqsubseteq^S w$. Moreover $(\bigsqcup u) \sqcup (\bigsqcup v) \sqsubseteq \bigsqcup w$. Therefore $u \tilde{\sqcup} v \preceq w$. □

A feature of this pre-order is that set theoretic union distributes over $\tilde{\sqcup}$ and viceversa, up to $\simeq$, that is

$$(u \tilde{\sqcup} v) \cup w \simeq (u \cup w) \tilde{\sqcup} (v \cup w) \quad \text{and} \quad (u \cup v) \tilde{\sqcup} w \simeq (u \tilde{\sqcup} w) \cup (v \tilde{\sqcup} w).$$

Recall that an *ideal* is a non-empty left-closed set, closed under joins of finite subsets. The *ideal completion* $Idl(P, \sqsubseteq)$ of the poset $(P, \sqsubseteq)$ is the set of all ideals ordered by subset inclusion. We now define our powerdomain constructor.

2.2. DEFINITION. Let D be an algebraic lattice:

(i) $\mathcal{P}(D) = Idl(M(D), \preceq)$;

(ii) $\{\!|\_|\!\} : D \to \mathcal{P}(D)$ is defined by: $\{\!|d|\!\} = \{v \in M(D) \mid \exists c \in K(d) \,.\, v \preceq \{c\}\}$;

(iii) $\uplus : \mathcal{P}(D) \times \mathcal{P}(D) \to \mathcal{P}(D)$ is defined by: $s \uplus t = \downarrow\{u \cup v \mid u \in s, \ v \in t\}$.

It is easy to check that $\{\!|\_|\!\}$ and $\uplus$ are well defined. To shorten notation $\{\!|d_1, \ldots, d_n|\!\}$ will abbreviate $\{\!|d_1|\!\} \uplus \ldots \uplus \{\!|d_n|\!\}$.

$\mathcal{P}(D)$ is naturally ordered by subset inclusion, and moreover it is a complete lattice.

$$\bigvee X = \{u \in M(D) \mid \exists\, u_1, \ldots, u_n \in \bigcup X. \ u \preceq u_1 \tilde{\sqcup} \cdots \tilde{\sqcup} u_n\} \quad \text{for an arbitrary } X \subseteq \mathcal{P}(D).$$

By construction $\mathcal{P}(D)$ is algebraic, with basis $K(\mathcal{P}(D)) = \{\downarrow u \mid u \in M(D)\}$, where $\downarrow u = \{v \in M(D) \mid v \preceq u\}$. Therefore, if D is ω-algebraic, then $\mathcal{P}(D)$ is as well.

The distributivity of join with respect to union in $M(D)$ induces that of join over $\uplus$ in $\mathcal{P}(D)$, and viceversa:

$$(r \vee s) \uplus t = (r \uplus t) \vee (s \uplus t) \quad \text{and} \quad (r \uplus s) \vee t = (r \vee t) \uplus (s \vee t).$$

The powerdomain $\mathcal{P}(D)$ enjoys some "set theoretical" properties, among which the most interesting is that the set of $\uplus$-irreducible elements coincides with the set of "singletons".

2.3. DEFINITION. Let D be an algebraic lattice, then:

(i) the set $\mathcal{C}(\mathcal{P}(D))$ of $\uplus$-*irreducible elements* of $\mathcal{P}(D)$ is defined by:

$$r \in \mathcal{C}(\mathcal{P}(D)) \ \Leftrightarrow \ \forall s, t \in \mathcal{P}(D). \ s \uplus t \subseteq r \Rightarrow s \vee t \subseteq r;$$

(ii) given $s \in \mathcal{P}(D)$ define $\mathcal{C}(s) = \{C \subseteq^{\text{fne}} \mathcal{C}(\mathcal{P}(D)) \mid \biguplus C \in K(s)\}$, where $\subseteq^{\text{fne}}$ is short for "is a finite non-empty subset of".

Notice that for all $s \in \mathcal{P}(D)$ and $C \in \mathcal{C}(s)$, $C \subseteq K(\mathcal{P}(D))$ (otherwise a contradiction immediately arises from $\biguplus C \in K(s)$).

2.2. LEMMA. Let D be an algebraic lattice, then:

(i) the set $\mathcal{C}(\mathcal{P}(D))$ is the image of D under the mapping $\{\!| \cdot |\!\}$;

(ii) for all $s \in K(\mathcal{P}(D))$, if $s \neq \bot$ then there exists $C \subseteq^{\text{fne}} \mathcal{C}(\mathcal{P}(D))$ such that $s = \biguplus C$;
(iii) for all $s \in \mathcal{P}(D)$, $s = \bigvee\{\biguplus C \mid C \in \mathcal{C}(s)\}$.

Proof.

(i) For any $d \in D$ and $s, t \in \mathcal{P}(D)$, we show that if $s \uplus t \subseteq \{\!|d|\!\}$, then $s \vee t \subseteq \{\!|d|\!\}$. Notice that by Definition 2.1 $u \preceq \{c\}$ if and only if $\bigsqcup u \sqsubseteq c$. Given any $w \in s \vee t$ there are $u, v \in M(D)$ such that $u \in s, v \in t$ and $w \preceq u \,\tilde{\cup}\, v$. This implies that

$$\bigsqcup w \sqsubseteq \bigsqcup(u \,\tilde{\cup}\, v) = \bigsqcup(u \cup v),$$

where the joins are taken in D. By the definition of $\{\!|d|\!\}$, $s \uplus t \subseteq \{\!|d|\!\}$ implies that there exists $c \in K(d)$ such that $u \cup v \preceq \{c\}$, and also that $\bigsqcup(u \cup v) \sqsubseteq c$. Therefore $\bigsqcup w \sqsubseteq c$, so we conclude $w \in \{\!|d|\!\}$. This shows that $s \vee t \subseteq \{\!|d|\!\}$, by the arbitrariness of w.

Vice-versa suppose that $s \in \mathcal{P}(D)$; if $u \in s$ and $u = \{e_1, \ldots, e_n\}$ then $\{\!|e_1|\!\} \uplus \cdots \uplus \{\!|e_n|\!\} \subseteq s$. If s is $\uplus$-irreducible, this implies $\{\!|e_1|\!\} \vee \cdots \vee \{\!|e_n|\!\} = \{\!| \bigsqcup_{i \leq n} e_i |\!\} \subseteq s$.

Given an $\uplus$-irreducible s, let $S = \bigcup\{u \mid u \in s\}$, and $d = \bigsqcup S$. We show that S is directed. If $c, c' \in S$ then there are $u, v \in s$ such that $c \in u, c' \in v$. Since s is an ideal with respect to $\preceq$, $u \,\tilde{\cup}\, v \in s$. Moreover $c \sqcup c' \in u \,\tilde{\cup}\, v$, and therefore $c \sqcup c' \in S$.

Now on one hand we have

$$u \in s \Rightarrow u \subseteq S \Rightarrow \bigsqcup u \in K(d) \Rightarrow u \in \{\!|d|\!\}.$$

On the other hand, using the $\uplus$-irreducibility of s, we have

$$\begin{aligned} w \in \{\!|d|\!\} &\Rightarrow \exists c \in K(d).\ w \preceq \{c\} \\ &\Rightarrow \exists c' \in S.\ w \preceq \{c'\} \\ &\Rightarrow \exists u \in s, c' \in u.\ w \preceq \{c'\} \\ &\Rightarrow \exists u \in s, c_1, \ldots, c_n \in K(D).\ u = \{c_1, \ldots, c_n\} \ \&\ w \in \{\!| \textstyle\bigsqcup_{i \leq n} c_i |\!\} \\ &\Rightarrow w \in s, \end{aligned}$$

where the second implication follows from the facts that c is compact and S directed. We conclude that $s = \{\!|d|\!\}$.

(ii) If $s \in K(\mathcal{P}(D))$, then it is in the image of $M(D)$ under the natural embedding of $M(D)$ into $\mathcal{P}(D)$. Therefore, for some $e_1, \ldots, e_m \in K(D)$, $s = \{\!|e_1, \ldots, e_m|\!\}$. Now the statement follows from (i).

(iii) Immediate from (ii) and the fact that $\mathcal{P}(D)$ is algebraic.

□

2.1. Corollary. Let r be a compact and $\uplus$-irreducible element of $\mathcal{P}(D)$, then $r = s \uplus t$, where s and t are compacts, implies $r = s = t$.

Proof. By Lemma 2.2 (i), there exists $e \in K(D)$ such that $r = \{\!|e|\!\}$. Let $s = \downarrow \{u\}$, $t = \downarrow \{v\}$, for $u, v \in M(D)$. Since $\{e\} \preceq u \cup v$, we have in particular

$$\forall d \in u.\ e \sqsubseteq d \quad \text{and} \quad \forall d' \in v.\ e \sqsubseteq d'.$$

This trivially implies $\{e\} \preceq u$ and $\{e\} \preceq v$. On the other hand, since $u \,\tilde{\cup}\, v \preceq \{e\}$, we have $\bigsqcup(u \cup v) \sqsubseteq e$, hence immediately we conclude $u \preceq \{e\}$ and $v \preceq \{e\}$. Thus it is the case $r = s = t$. □

$\mathcal{P}$ can be turned into a functor. For this aim we first introduce the category **oper-ALG** of oper-lattices (lattices with an operation).

2.4. Definition. The category **oper-ALG** of oper-lattices is the following.

(Objects) An object in **oper-ALG** is a ω-algebraic lattice D equipped with a continuous binary operation $*_D : D \times D \to D$ such that (omitting the subscript):

(i) $d_1 * d_2 = d_2 * d_1$ (commutativity),
(ii) $d_1 * (d_2 * d_3) = (d_1 * d_2) * d_3$ (associativity),
(iii) $d * d = d$ (idempotence),
(iv) $(d_1 * d_2) \sqcup d_3 = (d_1 \sqcup d_3) * (d_2 \sqcup d_3)$ (join distributivity).

(Morphisms) A morphism $f : D \to E$ from D to E is a continuous function such that

$$f(d *_D d') = f(d) *_E f(d') *_E f(d \sqcup d'), \tag{19}$$

for $d, d' \in D$.

2.1. Proposition. **oper-ALG** is a well defined category.

Proof. Let $(D, *)$ be an oper-lattice. For Id_D to be a morphism we have to prove $d * e = d * e * (d \sqcup e)$, for any $d, e \in D$. We have

$$\begin{aligned} d * e &= (d * e) \sqcup (d * e) && \text{(idempotency)} \\ &= (d \sqcup (d * e)) * (e \sqcup (d * e)) && \text{(distributivity)} \\ &= (d \sqcup d) * (d \sqcup e) * (e \sqcup e) * (d \sqcup e) && \text{(distributivity and associativity)} \\ &= d * e * (d \sqcup e) \end{aligned}$$

We prove that morphisms are closed under composition. First we note that

$$d_1 \sqsubseteq d_2 \sqsubseteq d_3 \Rightarrow d_1 * d_3 = d_1 * d_2 * d_3 \tag{20}$$

Indeed

$$\begin{aligned} d_1 * d_3 &= d_1 * d_1 * d_3 && \text{by idempotency of } * \\ &\sqsubseteq d_1 * d_2 * d_3 && \text{by monotonicity of } * \\ &\sqsubseteq d_1 * d_3 * d_3 && \text{by monotonicity of } * \\ &= d_1 * d_3 && \text{by idempotency of } * \end{aligned}$$

Now let $f : (D, *_D) \to (E, *_E)$, $g : (E, *_E) \to (C, *_C)$ be morphisms. Then (omitting subscripts)

$$\begin{aligned} (g \circ f)(d * e) &= g(f(d) * f(e) * f(d \sqcup e)) \\ &= g(f(d) * f(e)) * (g \circ f)(d \sqcup e) * g((f(d) * f(e)) \sqcup f(d \sqcup e)) \\ &= g(f(d) * f(e)) * (g \circ f)(d \sqcup e) && \text{(a)} \\ &= (g \circ f)(d) * (g \circ f)(e) * g(f(d) \sqcup f(e)) * (g \circ f)(d \sqcup e) \\ &= (g \circ f)(d) * (g \circ f)(e) * (g \circ f)(d \sqcup e) && \text{(b)} \end{aligned}$$

where in (a) we use the fact that, by the monotonicity of f, $f(d), f(e) \sqsubseteq f(d \sqcup e)$, so that $f(d) * f(e) \sqsubseteq f(d \sqcup e)$ by monotonicity and idempotency of $*$. In (b), since $f(d) \sqcup f(e) \sqsubseteq f(d \sqcup e)$, we know that $g(f(d) \sqcup f(e)) \sqsubseteq (g(f(d) \sqcup f(e))$ by the monotonicity of g. On the other hand $(g \circ f)(d) * (g \circ f)(e) \sqsubseteq g(f(d) \sqcup f(e))$ by a reasoning similar to that in (a); then (b) follows by (20).

□

2.1. Theorem. Let **ALG** be the category of algebraic lattices with Scott continuous functions as morphisms. Then the forgetful functor F : **oper-ALG** $\to$ **ALG** has a left adjoint whose action on objects is given by $\mathcal{P}$ (more precisely, each lattice D is transformed into the oper-lattice $(\mathcal{P}(D), \uplus)$).

Proof. We take the characterization of adjunction given in [28], Theorem IV.2.(ii). $\{\!| \cdot |\!\} : D \to F(\mathcal{P}(D))$ is an universal arrow from D to F. In fact, let E be any oper-lattice and $f : D \to F(E)$ a continuous function. Then there exists a unique morphism $f^\dagger : \mathcal{P}(D) \to E$ such that $f = F(f^\dagger) \circ \{\!| \cdot |\!\}$, namely the unique continuous extension of

$$f^\dagger(\{\!|d_1, \ldots, d_n|\!\}) = f(d_1) * \ldots * f(d_n) * f(\bigsqcup_{i \leq n} d_i).$$

This defines the action of $\mathcal{P}$ on morphisms: if D' is an algebraic lattice and $g : D \to D'$ a continuous function, then $\mathcal{P}(g) = (\{\!| \cdot |\!\} \circ g)^\dagger$.

□

In order to get a domain suitable for our purposes we look for the initial solution, D^∞, of the domain equation

$$D = \mathcal{P}([D \to D]_\perp), \tag{21}$$

where we identify $\mathcal{P}$ and $F \circ \mathcal{P}$. As seen in the Introduction, the initial solution N^∞ of

$$N = [\mathcal{P}(N_\perp) \to \mathcal{P}(N_\perp)]. \tag{22}$$

is immediately recovered from D^∞ as $N^\infty = [D^\infty \to D^\infty]$. The existence of the domain D^∞ is assured by general results on fixpoint domain equations, provided that we can prove that $\mathcal{P}$ is locally continuous, that is for any chain $f_n : D \to E$ it holds $\mathcal{P}(\bigsqcup_{n\in\mathbb{N}} f_n) = \bigsqcup_{n\in\mathbb{N}} \mathcal{P}(f_n)$. Let $f = \bigsqcup_{n\in\mathbb{N}} f_n$. Then $\mathcal{P}(f)$ is the unique continuous extension of

$$\{\!|d_1, \ldots, d_m|\!\} \mapsto \{\!|f(d_1), \ldots, f(d_m), f(\bigsqcup_{i\leq m} d_i)|\!\}.$$

Since

$$\begin{aligned}\{\!|f(d_1), \ldots, f(d_m), f(\textstyle\bigsqcup_{i\leq m} d_i)|\!\} &= \{\!| \textstyle\bigsqcup_{n\in\mathbb{N}} f_n(d_1), \ldots, \bigsqcup_{n\in\mathbb{N}} f_n(d_m), \bigsqcup_{n\in\mathbb{N}} f_n(\bigsqcup_{i\leq m} d_i)|\!\} \\ &= \textstyle\bigsqcup_{n\in\mathbb{N}} \{\!|f_n(d_1), \ldots, f_n(d_m), f_n(\bigsqcup_{i\leq m} d_i)|\!\},\end{aligned}$$

we have that $\mathcal{P}(f)$ extends

$$\{\!|d_1, \ldots, d_m|\!\} \mapsto \textstyle\bigsqcup_{n\in\mathbb{N}} \{\!|f_n(d_1), \ldots, f_n(d_m), f_n(\bigsqcup_{i\leq m} d_i)|\!\},$$

hence $\mathcal{P}$ is locally continuous.

Because of the continuity of $\mathcal{P}$, the direct limit technique to compute the initial solution D^∞ of equation (21) works (since $(\_)_\perp$ is locally continuous as well). D^∞ is the colimit of the ω-chain $(D_i, \varepsilon_i)_{i\in\mathbb{N}}$, where $D_0 = \{\perp\}$, $D_{n+1} = \mathcal{P}([D_n \to D_n]_\perp)$. The embedding ε_0 is $\lambda x.\ \perp$. To construct ε_{n+1} suppose $\varepsilon_n : D_n \to D_{n+1}$ is given; then $\varepsilon_{n+1} = \mathcal{P}(h_\perp)$ where $h = \lambda f \in [D_n \to D_n].\ \varepsilon_n \circ f \circ \pi_n$ and π_n is the projection from D_{n+1} to D_n determined by ε_n.

As in the case of the simpler equation modeling the lazy λ-calculus, this is a non-trivial solution because of the lifting functor.

We call Ψ the isomorphism $D^\infty \simeq \mathcal{P}([D^\infty \to D^\infty]_\perp)$; however we shall identify elements of D^∞ and $\mathcal{P}([D^\infty \to D^\infty]_\perp)$ when no ambiguity arises.

Because of Ψ we have a "union" operation over D^∞, formally defined by

$$d \uplus e = \Psi^{-1}(\Psi(d) \uplus \Psi(e)).$$

Compact elements of domains constructed by direct limit are images of compact elements of the approximate domains, so that, up to embedding-projection of each D_n into D^∞, we have $K(D^\infty) = \bigcup_{n\in N} K(D_n)$. They are inductively characterized by the following lemma, whose proof is immediate from the construction of D_n.

2.3. Lemma.

(i) $K(D_0) = \{\perp\}$;

(ii) $K(D_{n+1}) = \{\{\!| f_1, \ldots, f_m |\!\} \mid 1 \leq m,\ f_1, \ldots, f_m \in K([D_n \to D_n]_\perp)\}$;

(iii) $K([D_{n+1} \to D_{n+1}]_\perp) = \{\bigsqcup_{1\leq k}(c_i \Rightarrow d_i) \mid c_i, d_i \in K(D_n)\} \cup \{\perp\}$,

where $(c \Rightarrow d)(e)$ is the step function: if $c \sqsubseteq e$ then d else $\perp$.

Note that we drop the usual consistency condition in taking finite joins of step functions. This is sound since we work with lattices.

To finish up the present section, we study D^∞ as an applicative structure. Equation (22) clearly falls into the general framework studied in [29]; therefore in the following we concentrate on the categorical side by illustrating how our construction can be seen as an instance of the categorical concept of *monad*. In detail, we first prove that $\mathcal{P} \circ (\cdot)_\perp$ induces a monad over **ALG**. Then we illustrate how it is possible to define an application between elements of D^∞ by exploiting the fact that $\mathcal{P} \circ (\_)_\perp$ has a *tensorial strength*.

2.5. Definition. A *monad* over a category $\mathcal{C}$ is a structure (T, η, μ), where $T : \mathcal{C} \to \mathcal{C}$ is a functor, $\eta : Id \to T$ and $\mu : T^2 \to T$ are natural transformations such that the following diagrams commute:

$$\begin{tikzcd} T^3A \arrow[r, "\mu_{TA}"'] \arrow[d, "T(\mu_A)"'] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "Id_A"'] & T^2A \arrow[d, "\mu_A"] & TA \arrow[l, "T(\eta_A)"'] \arrow[dl, "Id_A"] \\ & TA & \end{tikzcd}$$

The following definitions of *tensorial strength* and of *strong monads* are taken from [29].

2.6. Definition. Let (T, η, μ) be a monad over a category $\mathcal{C}$, which has all finite products.

(i) A *tensorial strength* t of T is a family of morphisms $t_{A,B} : A \times T(B) \to T(A \times B)$ indexed over pairs of objects of $\mathcal{C}$, such that for all points $a : 1 \to A$ and $b : 1 \to T(B)$ it holds

$$t_{A,B} \circ \langle a, b\rangle = T(\langle a \circ !_B, Id_B\rangle) \circ b, \tag{23}$$

where $!_B : B \to 1$ is the unique morphism from B to the terminal object.

(ii) If t is a tensorial strength of (T, η, μ), then (T, η, μ, t) is called a *strong monad*.

In [29] it is shown that, if $\mathcal{C}$ has enough points, then the tensorial strength of a monad T (if it exists) is uniquely determined. The notion of exponents in a cartesian category can be specialized to a strong monad over that category as follows.

2.7. Definition. Let $\mathcal{C}$ be a cartesian category and (T, η, μ, t) a strong monad over $\mathcal{C}$. Then a T-*exponential* is a pair $\langle T(B)^A, eval^T_{A,B} : T(B)^A \times A \to T(B)\rangle$, with the universal property:

$$\forall f : C \times A \to T(B)\, \exists! \Lambda^T(f) : C \to T(B)^A.\ f = eval^T_{A,B} \circ \Lambda^T(f) \times Id_A;$$

that is the following diagram commutes:

$$\begin{tikzcd} C \times A \arrow[r, "f"] \arrow[d, "\Lambda^T(f) \times Id_A"'] & T(B) \\ T(B)^A \times A \arrow[ur, "eval^T_{A,B}"'] & \end{tikzcd}$$

A *call-by-name T-reflexive object* is an object N of $\mathcal{C}$ such that there is a retraction $T(N)^{T(N)} \lhd N$, say via (ψ_N, φ_N). Let $\kappa_{A,B} : A \times B \to B \times A$ be the obvious isomorphism, and define $\tilde{t}_{A,B} : T(A) \times B \to T(A \times B)$ by

$$\tilde{t}_{A,B} = T(\kappa) \circ t_{B,A} \circ \kappa.$$

Then the following diagram defines the *call-by-name T-application morphism* $app_N : T(N) \times T(N) \to T(N)$:

$$\begin{tikzcd}
T(N) \times T(N) \arrow[rr, dashed, "app_N"] \arrow[d, "\tilde{t}_{N,T(N)}"'] & & T(N) \\
T(N \times T(N)) \arrow[r, dashed, "T(eval^T_{T(N),N} \circ \varphi_N \times Id_{TN})"] \arrow[d, "T(\varphi_N \times Id_{TN})"'] & T^2(N) \arrow[ur, "\mu_N"] & \\
T(T(N)^{T(N)} \times T(N)) \arrow[ur, "T(eval^T_{T(N),N})"'] & &
\end{tikzcd}$$

2.2. PROPOSITION. Let T be the functor $\mathcal{P} \circ (\_)_\perp$.

The quadruple (T, η^T, μ^T, t^T) is a strong monad over the category **ALG**, where:

(i) $\eta^T_D : D \to T(D)$ is the map $d \mapsto \{\!|(d,0)|\!\}$;

(ii) $\mu^T_D : T^2(D) \to T(D)$ is the unique continuous strict morphism of oper-lattices which extends

$$\{\!|\{\!|(d_1,0),\ldots,(d_n,0)|\!\}|\!\} \mapsto \{\!|(d_1,0),\ldots,(d_n,0)|\!\},$$
$$\{\!|\{\!|(d_1,0),\ldots,(d_n,0),\perp|\!\}|\!\} \mapsto \{\!|(d_1,0),\ldots,(d_n,0),\perp|\!\}.$$

(iii) $t^T_{D,E}$ is the unique continuous extension of the map

$$(d, \{\!|(e_1,0),\ldots,(e_n,0)|\!\}) \mapsto \{\!|((d,e_1),0),\ldots,((d,e_n),0)|\!\},$$
$$(d, \{\!|(e_1,0),\ldots,(e_n,0),\perp|\!\}) \mapsto \{\!|((d,e_1),0),\ldots,((d,e_n),0),\perp|\!\}.$$

Proof. Consider the subcategory $\mathcal{A}'$ of **oper-ALG** whose objects are oper-lattices and whose morphisms are *strict* morphisms of oper-lattices. Then the forgetful functor $F : \mathcal{A}' \to$ **ALG** has T as a left adjoint: for each D, $\eta^T_D : D \to T(D)$ is an universal arrow from D to F. In fact for each $f : D \to E$, where E is an oper-lattice, there exists a unique strict morphism of oper-lattices $f' : T(D) \to E$ such that $f' \circ \eta^T_D = f$. Indeed f' extends the morphism which sends $\{\!|(d_1,0),\ldots,(d_n,0)|\!\}$ to $f(d_1) * \ldots * f(d_n) * f(\bigsqcup_{i \le n} d_i)$ and $\{\!|(d_1,0),\ldots,(d_n,0),\perp|\!\}$ to $f(d_1) * \ldots * f(d_n) * f(\bigsqcup_{i \le n} d_i) * \perp$. The thesis follows from the fact that each adjunction determines a monad (see [28], Section VI.1). In particular the definition of μ^T satisfies $\mu^T = \epsilon \circ T$, where ϵ is the counit of the adjunction.

As regards t^T, consider that, since **ALG** is a concrete category, it has enough points. Moreover it has all finite products. Therefore the tensorial strength is uniquely characterized by equation (23).

□

We give an explicit definition of the application over D^∞, and we show that it coincides with the call-by-name T-application morphism determined by the monad $\mathcal{P} \circ (\cdot)_\perp$.

2.3. Proposition. Consider the map $\cdot : \mathcal{P}([D^\infty \to D^\infty]_\perp) \times D^\infty \to D^\infty$ which is the unique continuous extension of the map from $K(\mathcal{P}([D^\infty \to D^\infty]_\perp) \times D^\infty) = K(\mathcal{P}([D^\infty \to D^\infty]_\perp)) \times K(D^\infty)$ to $K(D^\infty)$ defined by

$$\{\!|f_1, \ldots, f_m|\!\} \cdot d = \begin{cases} f_1(d) \uplus \cdots \uplus f_m(d) & \text{if } \forall i \leq m.\ f_i \neq \perp \\ f_1(d) \uplus \cdots \uplus f_{m-1}(d) \uplus \perp & \text{if } \forall i \leq m-1.\ f_i \neq \perp \text{ and } f_m = \perp. \end{cases}$$

Then for all $d, e \in D^\infty$ and taking $N = N^\infty = [D^\infty \to D^\infty]$,

$$d \cdot e = app_N(d, e).$$

Proof. The call-by-name T-reflexive object of our construction is the initial solution N^∞ of the equation (22). On the other hand we have that, up to isomorphism, the following equations hold (writing simply D for D^∞ and N for N^∞)

$$T(N) = \mathcal{P}(N_\perp) = \mathcal{P}([D \to D]_\perp) = D,$$

so that

$$T(N) \times T(N) = \mathcal{P}([D \to D]_\perp) \times D.$$

If $u = \{\!|f_1, \ldots, f_m|\!\} \in K(\mathcal{P}([D \to D]_\perp)) = K(T(N))$ and $d \in D$, then

$$\tilde{t}_{N,T(N)}(u, d) = \begin{cases} \{\!|(f_1, d), \ldots, (f_m, d)|\!\} & \text{if } f_1, \ldots, f_m \neq \perp \\ \{\!|(f_1, d), \ldots, (f_{m-1}, d), \perp|\!\} & \text{if } f_1, \ldots, f_{m-1} \neq \perp \text{ and } f_m = \perp. \end{cases}$$

In any case this "set" is an element of $T(N \times T(N)) = T(T(N)^{T(N)} \times T(N))$, and the T-evaluation morphism $eval^T_{T(N),N} : T(N)^{T(N)} \times T(N) \to T(N)$ is given by

$$\begin{cases} eval^T_{T(N),N}(f, d) = f(d) \text{ if } f \in N = [D \to D] = T(N)^{T(N)} \\ eval^T_{T(N),N}(\perp, d) = \perp. \end{cases}$$

Therefore we have, when $f_1, \ldots, f_m \neq \perp$, that (ignoring the isomorphism $\varphi_N = \Psi$):

$$\begin{aligned} app_N(u, d) &= \mu_N \circ T(eval^T_{T(N),N}) \circ \tilde{t}_{N,T(N)}(u, d) \\ &= \mu_N \circ T(eval^T_{T(N),N})(\{\!|(f_1, d), \ldots, (f_m, d)|\!\}) \\ &= \mu_N(\{\!|eval^T_{T(N),N}(f_1, d), \ldots, eval^T_{T(N),N}(f_m, d)|\!\}) \\ &= \mu(\{\!|f_1(d), \ldots, f_m(d)|\!\}) \\ &= f_1(d) \uplus \cdots \uplus f_m(d) \\ &= \{\!|f_1, \ldots, f_m|\!\} \cdot d. \end{aligned}$$

Similarly, if $f_1, \ldots, f_{m-1} \neq \perp$ and $f_m = \perp$ then $app_N(u, d) = f_1(d) \uplus \cdots \uplus f_{m-1}(d) \uplus \perp$. □

We finally include the definition of an operator $\coprod$ which roughly computes the join of the elements of a set.

2.8. Definition. Define $\coprod : D^\infty \to D^\infty$ as the unique continuous extension of

$$\coprod(\perp) = \perp$$
$$\coprod(\{\!|d_1, \ldots, d_n|\!\}) = \{\!|\bigsqcup_{i \leq n} d_i|\!\}.$$

3. A Logical Presentation

Logical presentations are useful tools for defining and studying models [2]. We use intersection types, which provide simple descriptions of ω-algebraic lattices. Types are formed by starting with the universal type ω and closing under the arrow, which is used for describing the function space, under the intersection, which corresponds to the semantic join (as in [11, 15, 14]), and under a new type constructor $\oplus$, which represents the restriction of the semantic operation $*$ to compacts.

In determining the structure of the space of filters, i.e. the structure of the domain of the model, the choice of the ordering over types is crucial. We describe the order and its properties in subsection 3.1. Subsection 3.2 gives the isomorphism between the space of filters and the initial solution of our powerdomain equation.

3.1. The Set of Types

The set *Type* of types is defined by adding the type constructor "sum" $\oplus$ to the intersection types [9, 12]:

$$\sigma ::= \omega \mid \sigma \to \tau \mid \sigma \wedge \tau \mid \sigma \oplus \tau.$$

In writing types we assume that $\wedge$ and $\oplus$ take precedence over $\to$.

We look for a partial order over types which corresponds in a natural way to the (quotient of the) pre-order $\preceq$ defined on $M(D)$ in Section 2. This will be clarified by the properties we will show of this order.

3.1. Definition (Preorder on Types). Let $\sigma \leq \tau$ be the least pre-order over *Type* such that:

(i) $\sigma \leq \omega$;
(ii) $\sigma \to \omega \leq \omega \to \omega$;
(iii) $\sigma \leq \sigma \wedge \sigma$;
(iv) $\sigma \wedge \tau \leq \sigma, \quad \sigma \wedge \tau \leq \tau$;
(v) $\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma \wedge \tau \leq \sigma' \wedge \tau'$;
(vi) $\sigma \leq \sigma \oplus \sigma \leq \sigma$;
(vii) $\sigma \oplus \tau \leq \tau \oplus \sigma$;
(viii) $\sigma \oplus (\tau \oplus \tau') \leq (\sigma \oplus \tau) \oplus \tau'$;
(ix) $\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma \oplus \tau \leq \sigma' \oplus \tau'$;
(x) $(\sigma \oplus \sigma') \wedge \tau \leq (\sigma \wedge \tau) \oplus (\sigma' \wedge \tau)$;
(xi) $(\sigma \to \tau) \wedge (\sigma \to \tau') \leq \sigma \to (\tau \wedge \tau')$;
(xii) $\sigma' \leq \sigma,\ \tau \leq \tau' \Rightarrow \sigma \to \tau \leq \sigma' \to \tau'$;

Let $\sigma = \tau$ be defined as "$\sigma \leq \tau$ and $\tau \leq \sigma$". Then the quotient structure $\langle Type_{/=}, \leq_{/=} \rangle$ is a topped inf-semilattice, where the top is $[\omega]$ and $[\sigma] \wedge_{/=} [\tau] = [\sigma \wedge \tau]$. As usual we identify *Type* with $Type_{/=}$ and $\leq$ with $\leq_{/=}$. We write $\sigma < \tau$ if and only if $\sigma \leq \tau$ and $\sigma \neq \tau$.

3.1. Proposition. The axiomatic presentation of 3.1 is consistent, and therefore $\langle Type, \leq \rangle$ is a non-trivial structure. In particular:

$$\omega \not\leq \omega \to \omega.$$

Proof. Let $f : Type \to \{\bot, \top\}$ be the following map into the two points lattice, ordered by $\bot \sqsubseteq \top$:

$$\begin{aligned} f(\omega) &= \top \\ f(\sigma \to \tau) &= \bot \\ f(\sigma \wedge \tau) &= f(\sigma \oplus \tau) = f(\sigma) \sqcap f(\tau). \end{aligned}$$

Checking through Definition 3.1 we see that

$$\sigma \leq \tau \;\Rightarrow\; f(\sigma) \sqsubseteq f(\tau),$$

for all σ and τ (actually f is a meet-semilattice morphism). Now

$$f(\omega) = \top \not\sqsubseteq \bot = f(\omega \to \omega)$$

implies the thesis by contradiction.

□

Notation. Because of its relevance and frequent occurrences in the technical development, we abbreviate $\omega \to \omega$ by ω_1.

We have the distributivity of $\wedge$ with respect to $\oplus$ and the viceversa.

3.1. Lemma. For all $\sigma, \sigma', \tau, \tau' \in \mathit{Type}$:
(i) $(\sigma \oplus \sigma') \wedge \tau = (\sigma \wedge \tau) \oplus (\sigma' \wedge \tau)$;
(ii) $(\sigma \wedge \sigma') \oplus \tau = (\sigma \oplus \tau) \wedge (\sigma' \oplus \tau)$.

Proof.
(i) Because of 3.1 (x), it suffices to show that $(\sigma \wedge \tau) \oplus (\sigma' \wedge \tau) \leq (\sigma \oplus \sigma') \wedge \tau$. This follows from $(\sigma \wedge \tau) \oplus (\sigma' \wedge \tau) \leq \sigma \oplus \sigma'$, by 3.1 (iv) and (ix), and from $(\sigma \wedge \tau) \oplus (\sigma' \wedge \tau) \leq \tau \oplus \tau \leq \tau$, by 3.1 (iv), (ix) and (vi).
(ii) $(\sigma \wedge \sigma') \oplus \tau \leq (\sigma \oplus \tau) \wedge (\sigma' \oplus \tau)$ follows, by 3.1 (v), from $(\sigma \wedge \sigma') \oplus \tau \leq \sigma \oplus \tau$ and $(\sigma \wedge \sigma') \oplus \tau \leq \sigma' \oplus \tau$, which in turn hold by 3.1 (iv) and (ix).
For the opposite inclusion, using (i) of the present Lemma, we have:

$$\begin{aligned}
(\sigma \oplus \tau) \wedge (\sigma' \oplus \tau) &= (\sigma \wedge (\sigma' \oplus \tau)) \oplus (\tau \wedge (\sigma' \oplus \tau)) \\
&= (\sigma \wedge \sigma') \oplus (\sigma \wedge \tau) \oplus (\sigma' \wedge \tau) \oplus (\tau \wedge \tau) \\
&\leq (\sigma \wedge \sigma') \oplus \tau \oplus \tau \oplus \tau \\
&= (\sigma \wedge \sigma') \oplus \tau.
\end{aligned}$$

□

Notation. Let $I, J, H, K, \ldots$ be finite sets of indexes. Since $\wedge$ and $\oplus$ are commutative and associative, we will freely use the following notations:

$$\bigwedge_{i \leq n} \sigma_i, \qquad \bigwedge_{i \in I} \sigma_i, \qquad \bigoplus_{i \leq n} \sigma_i, \qquad \bigoplus_{i \in I} \sigma_i,$$

with obvious meanings.

A key equation, derivable from the above axioms, is $\sigma \oplus \tau = \sigma \oplus \tau \oplus (\sigma \wedge \tau)$. This can be generalized as follows:

3.2. Proposition.

$$(\mathbf{P0}) \qquad \bigoplus_{i \leq n} \sigma_i = \bigoplus_{i \leq n} \sigma_i \oplus \left(\bigwedge_{i \leq n} \sigma_i\right).$$

Proof. By induction on n.

$$\begin{aligned}
\bigoplus_{i \leq n+1} \sigma_i \oplus \left(\bigwedge_{i \leq n+1} \sigma_i\right) &= \bigoplus_{i \leq n} \sigma_i \oplus \sigma_{n+1} \oplus \left(\bigwedge_{i \leq n} \sigma_i \wedge \sigma_{n+1}\right) \\
&= \left(\bigoplus_{i \leq n} \sigma_i \oplus \sigma_{n+1} \oplus \left(\bigwedge_{i \leq n} \sigma_i\right)\right) \wedge \left(\bigoplus_{i \leq n} \sigma_i \oplus \sigma_{n+1} \oplus \sigma_{n+1}\right) \\
&\qquad \text{by distributivity} \\
&= \left(\bigoplus_{i \leq n} \sigma_i \oplus \sigma_{n+1}\right) \wedge \left(\bigoplus_{i \leq n} \sigma_i \oplus \sigma_{n+1}\right) \\
&\qquad \text{by induction and idempotency of } \oplus \\
&= \bigoplus_{i \leq n+1} \sigma_i.
\end{aligned}$$

□

The following notion of irreducible types is the counter-part of the notion of singletons.

3.2. Definition. A type σ is *irreducible* if and only if

$$\forall \tau, \rho \in Type.\ \sigma = \tau \oplus \rho \Rightarrow \sigma = \tau = \rho.$$

IType is the set of irreducible types different from ω.

The exclusion of ω from *IType* is accounted for by the fact that ω itself is irreducible (see 3.1 (i) below), and because irreducible types different from ω play a central role in the semantic construction. We first prove that any type is equivalent to a finite sum of irreducible types (possibly including ω), and that the restriction of the inequality relation to irreducible types characterizes the relation itself over the whole set *Type*.

It is useful to introduce a map Θ which drops external $\oplus$; we will see that Θ returns finite sets of irreducible types.

3.3. Definition. Let $\Theta : Type \to \wp(Type)$ be defined as follows:

(i) $\Theta(\omega) = \{\omega\}$

(ii) $\Theta(\sigma \to \tau) = \{\sigma \to \tau\}$

(iii) $\Theta(\sigma \oplus \tau) = \Theta(\sigma) \cup \Theta(\tau)$

(iv) $\Theta(\sigma \wedge \tau) = \begin{cases} \{\sigma' \in \Theta(\sigma) \mid \omega \in \Theta(\tau)\} \cup \\ \{\tau' \in \Theta(\tau) \mid \omega \in \Theta(\sigma)\} \cup \\ \{\sigma' \wedge \tau' \mid \sigma', \tau' \not\equiv \omega \ \&\ \sigma' \in \Theta(\sigma) \ \&\ \tau' \in \Theta(\tau)\}. \end{cases}$

In words, we may compute $\Theta(\sigma)$ as follows:

Step 1: by distributing $\wedge$ over $\oplus$, put σ into a $\oplus$-normal form, i.e. a sum of conjunctions of arrow types and ω (this preserves equality by 3.1 (i));

Step 2: in each term of the above sum, erase any occurrence of ω in conjunction with some arrow type (which again is justified by the axioms of the equality);

Step 3: then $\Theta(\sigma)$ is just the finite set of conjunctions in the sum obtained from step 2.

3.1. Example.

$$\begin{aligned} \Theta((\omega \to \omega) \wedge \omega) &= \{\omega \to \omega\}, \\ \Theta((\omega \to \omega) \oplus \omega) &= \{\omega \to \omega, \omega\}, \\ \Theta((\omega \to \omega) \wedge ((\omega \to \omega \to \omega) \oplus \omega)) &= \{\omega \to \omega, (\omega \to \omega) \wedge (\omega \to \omega \to \omega)\} \\ &= \Theta(((\omega \to \omega) \wedge (\omega \to \omega \to \omega)) \oplus ((\omega \to \omega) \wedge \omega)). \end{aligned}$$

3.3. Proposition. For all $\sigma, \tau \in Type$:

(i) $\sigma = \bigoplus_{\sigma' \in \Theta(\sigma)} \sigma'$;

(ii) $\sigma \leq \tau$ if and only if $\bigwedge_{\sigma' \in \Theta(\sigma)} \sigma' \leq \bigwedge_{\tau' \in \Theta(\tau)} \tau'$ and $\forall \sigma' \in \Theta(\sigma)\ \exists \tau' \in \Theta(\tau).\ \sigma' \leq \tau'$;

(iii) σ is irreducible if and only if $\forall \sigma' \in \Theta(\sigma).\ \sigma = \sigma'$;

(iv) *IType* and *IType* $\cup \{\omega\}$ are closed under intersection;

(v) for all I, σ_i, τ_i we have $\bigwedge_{i \in I}(\sigma_i \to \tau_i) \in IType$;

(vi) $\sigma' \in \Theta(\sigma)$ implies either that $\sigma' \equiv \omega$ or $\sigma' \equiv \bigwedge_{i \in I}(\rho_i \to \tau_i)$ for some ρ_i, τ_i and I;

(vii) $\Theta(\sigma)$ is a set of irreducible types.

Proof.

(i) By easy induction on the definition of Θ using the distributivity of $\oplus$ over $\wedge$.

(ii) $(\Rightarrow)$. By induction on the presentation of $\leq$.

$(\Leftarrow)$. Let $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$ and $\Theta(\tau) = \{\tau_j \mid j \in J\}$. By hypothesis for all $i \in I$ there exists $j_i \in J$ such that $\sigma_i \leq \tau_{j_i}$ and $\bigwedge_{i \in I} \sigma_i \leq \bigwedge_{j \in J} \tau_j$. Now

$$\begin{aligned} \sigma &= \textstyle\bigoplus_{i \in I} \sigma_i \oplus (\bigwedge_{i \in I} \sigma_i) && \text{by (i) and } \mathbf{(P0)} \\ &\leq \textstyle\bigoplus_{i \in I} \tau_{j_i} \oplus (\bigwedge_{j \in J} \tau_j) && \text{by hypothesis} \\ &\leq \textstyle\bigoplus_{j \in J} \tau_j && \text{by monotonicity and idempotency of } \oplus \\ &= \tau. \end{aligned}$$

(iii) ($\Rightarrow$) follows by definition of $IType$ and by (i). ($\Leftarrow$). Let $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$, $\Theta(\tau) = \{\tau_j \mid j \in J\}$ and $\Theta(\rho) = \{\rho_k \mid k \in K\}$. If $\sigma = \tau \oplus \rho$ then

$$\sigma = \bigwedge_{i \in I} \sigma_i = \bigwedge_{j \in J} \tau_j \wedge \bigwedge_{k \in K} \rho_k,$$

by the hypothesis that $\sigma = \sigma_i$ for all $i \in I$, and by (ii): therefore $\sigma \leq \tau$ and $\sigma \leq \rho$. On the other hand $\tau \oplus \rho \leq \sigma$ and the hypothesis on σ imply, by (ii), that $\tau_j \leq \sigma$ for all $j \in J$ and $\rho_k \leq \sigma$ for all $k \in K$. We conclude that $\tau \leq \sigma$ and $\rho \leq \sigma$, so that $\sigma = \tau = \rho$.

(iv) Immediate by (iii) and by definition of $\Theta(\sigma \wedge \tau)$.

(v) It is easy to verify that $\Theta(\bigwedge_{i \in I}(\sigma_i \to \tau_i)) = \{\bigwedge_{i \in I}(\sigma_i \to \tau_i)\}$, so we are done by (iii).

(vi) By a straightforward induction on the definition of Θ.

(vii) Immediate by (v) and (vi).

□

3.1. REMARK. The key point in Proposition 3.3 is (ii). Comparing this statement with the definition of the (pre)order $\preceq$ over $M(D)$ it is apparent that $\leq$ and $\preceq$ are dual to each other. A stronger correspondence is property (**P2**) in Lemma 3.2.

Point (vi) clearly gives a characterization of irreducible types, while (i) assures that the function Θ is the right tool to compute type inequalities.

3.1. COROLLARY.

(i) ω is irreducible;

(ii) $\omega_1 < \omega_1 \oplus \omega < \omega$;

(iii) $\sigma \leq \omega_1 \;\Leftrightarrow\; \omega \notin \Theta(\sigma) \;\Leftrightarrow\; \sigma \neq \sigma \oplus \omega$;

(iv) $\sigma = \sigma \oplus \omega \;\&\; \sigma \neq \omega \Rightarrow \sigma \leq \omega_1 \oplus \omega$ and $\omega \in \Theta(\sigma)$.

Proof.

(i) For any $\sigma, \tau \in Type$,

$$\begin{aligned} \omega \leq \sigma \oplus \tau &\Rightarrow \omega \leq \textstyle\bigwedge_{\sigma' \in \Theta(\sigma)} \sigma' \wedge \bigwedge_{\tau' \in \Theta(\tau)} \tau' && \text{by 3.3 (ii)} \\ &\Rightarrow \forall \sigma' \in \Theta(\sigma)\, \forall \tau' \in \Theta(\tau).\ \omega \leq \sigma' \;\&\; \omega \leq \tau' && \\ &\Rightarrow \omega = \sigma = \tau && \text{by 3.3 (i).} \end{aligned}$$

(ii) By 3.1 (i), (vi) and (ix) we prove

$$\omega_1 = \omega_1 \oplus \omega_1 \leq \omega_1 \oplus \omega \leq \omega \oplus \omega = \omega.$$

$\omega_1 \oplus \omega \not\leq \omega_1$ and $\omega \not\leq \omega_1 \oplus \omega$ are consequences of 3.3 (ii) using the fact that $\omega \not\leq \omega_1$, by 3.1.

(iii) By 3.3 (i) and (ii) we know that $\sigma \neq \sigma \oplus \omega$ if and only if $\omega \notin \Theta(\sigma)$. This implies by 3.3 (vi) that all $\sigma' \in \Theta(\sigma)$ have the shape $\bigwedge_{i \in I}(\rho_i \to \tau_i)$, for some I, ρ_i, τ_i. By this, 3.1 (i), 3.1 (ii), 3.1 (xii) 3.3 (i) and 3.1 (vi), the thesis follows.

(iv)

$$\begin{aligned} \sigma = \sigma \oplus \omega \;\&\; \sigma \neq \omega \Rightarrow \sigma &= \textstyle\bigoplus_{\sigma' \in \Theta(\sigma) \setminus \{\omega\}} \sigma' \oplus \omega \\ &\Rightarrow \sigma \leq \omega_1 \oplus \omega \end{aligned}$$

using 3.3 (i), 3.3 (vi), and point (iii) of this Corollary.
$\omega \in \Theta(\sigma \oplus \omega)$ by definition, so $\omega \in \Theta(\sigma)$ by 3.3 (ii).

□

We finish this section with two technical lemmas stating the main properties of $\leq$.

3.2. LEMMA.

(P1) $\sigma \oplus \tau \leq \rho$ implies that for some $\mu \geq \sigma$ one has $\rho = \mu \oplus \rho$;

(P2) Let σ_i, τ_j be irreducible types for $i \in I,\ j \in J$. $\bigoplus_{i\in I} \sigma_i \leq \bigoplus_{j\in J} \tau_j$ if and only if $\bigwedge_{i\in I} \sigma_i \leq \bigwedge_{j\in J} \tau_j$ and, for all $i \in I$, there is $j \in J$ such that $\sigma_i \leq \tau_j$;

(P3) $\bigoplus_{i\in I} \sigma_i \oplus \omega = (\bigwedge_{i\in I} \sigma_i) \oplus \omega$;

(P4) $\bigwedge_{i\in I}(\sigma_i \to \tau_i) \leq \bigwedge_{j\in J}(\mu_j \to \nu_j)$ if and only if, for all $j \in J$ such that $\nu_j \neq \omega$, there is $I_j \subseteq I$ such that $\mu_j \leq \bigwedge_{i\in I_j} \sigma_i$ and $\bigwedge_{i\in I_j} \tau_i \leq \nu_j$;

Proof.

(P1) Let $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$ and $\Theta(\rho) = \{\rho_h \mid h \in H\}$, for some I, σ_i, H, ρ_h. By 3.3 (ii) we know that

$$\sigma \oplus \tau \leq \rho \Rightarrow \forall i \in I\ \exists\, h_i \in H.\ \sigma_i \leq \rho_{h_i}.$$

Set $\mu = \bigoplus_{i\in I} \rho_{h_i}$: then $\sigma \leq \mu$ by construction and $\rho = \mu \oplus \rho$ by the idempotence, commutativity and associativity of $\oplus$.

(P2) It follows from 3.3 (ii), considering that by definition $\Theta(\bigoplus_{i\in I} \sigma_i) = \bigcup_{i\in I} \Theta(\sigma_i)$, and that $\Theta(\bigoplus_{j\in J} \tau_j) = \bigcup_{j\in J} \Theta(\tau_j)$. Moreover $\sigma_i = \sigma'_i$ for all $\sigma'_i \in \Theta(\sigma_i)$ and $\tau_j = \tau'_j$ for all $\tau'_j \in \Theta(\tau_j)$ by 3.3 (iii), since σ_i and τ_j are irreducible by hypothesis.

(P3) Immediate from **(P2)**.

(P4) Define the map θ: $Type \to \wp(IType)$ as follows:

(i) $\theta(\omega) = \emptyset$;
(ii) $\theta(\sigma \to \tau) = \{\sigma \to \tau\}$;
(iii) $\theta(\sigma \oplus \tau) = \theta(\sigma \wedge \tau) = \theta(\sigma) \cup \theta(\tau)$.

Let $\theta(\alpha) = \{\sigma_i \to \tau_i \mid i \in I\}$ and $\theta(\beta) = \{\mu_j \to \nu_j \mid j \in J\}$ for some (possibly empty) I, J.

We establish the following more general statement, of which **(P4)** is a particular case:

$$\begin{array}{l}\alpha \leq \beta \ \Leftrightarrow \\ \ [\theta(\beta) = \emptyset \text{ or } \theta(\alpha) \neq \emptyset]\ \&\ \forall j \in J.\ \nu_j \neq \omega \Rightarrow\ \exists\, I_j \subseteq I.\ \mu_j \leq \bigwedge_{i\in I_j} \sigma_i\ \&\ \bigwedge_{i\in I_j} \tau_i \leq \nu_j.\end{array}$$

The proof is by induction on the definition of $\leq$. All cases are immediate except transitivity. Suppose that $\alpha \leq \beta$ because of $\alpha \leq \gamma \leq \beta$ for some γ. Let $\theta(\gamma) = \{\xi_h \to \rho_h \mid h \in H\}$. If $\theta(\beta) \neq \emptyset$ then, by induction, $\theta(\gamma) \neq \emptyset$. The induction hypothesis gives:

$$\forall h \in H.\ \rho_h \neq \omega \Rightarrow \exists\, I_h \subseteq I.\ \xi_h \leq \bigwedge_{i\in I_h} \sigma_i\ \&\ \bigwedge_{i\in I_h} \tau_i \leq \rho_h, \tag{24}$$

$$\forall j \in J.\ \nu_j \neq \omega \Rightarrow \exists\, H_j \subseteq H.\ \mu_j \leq \bigwedge_{h\in H_j} \xi_h\ \&\ \bigwedge_{h\in H_j} \rho_h \leq \nu_j. \tag{25}$$

For any $\nu_j \neq \omega$ set $H'_j = \{h \in H_j \mid \rho_h \neq \omega\}$, which is nonempty as a consequence of (25). Then

$$\mu_j \leq \bigwedge_{h\in H'_j} \xi_h \text{ and } \bigwedge_{h\in H'_j} \rho_h \leq \nu_j. \tag{26}$$

By (24) we know that

$$h \in H'_j \Rightarrow \xi_h \leq \bigwedge_{i\in I_h} \sigma_i\ \&\ \bigwedge_{i\in I_h} \tau_i \leq \rho_h, \tag{27}$$

therefore

$$\begin{array}{l}\mu_j \leq \bigwedge_{h\in H'_j} \xi_h \leq \bigwedge_{h\in H'_j}(\bigwedge_{i\in I_h} \sigma_i) \text{ by (26) and (27)} \\ \bigwedge_{h\in H'_j}(\bigwedge_{i\in I_h} \tau_i) \leq \bigwedge_{h\in H'_j} \rho_h \leq \nu_j \text{ by (27) and (26)}\end{array}$$

so that we set $I_j = \bigcup_{h\in H'_j} I_h$.

□

Property **(P2)** explicitly relates $\leq$ to the ordering in the powerdomain constructed in Section 2. More precisely the ordering $\leq$ on types is dual to the pre-order $\preceq$ on $M(D)$, when we interpret $\oplus$ as union operator ($\uplus$) and $\wedge$ as join ($\tilde{\sqcup}$). In fact, in Section 6 we will see that the quotient of types under $=$ gives the dual of an oper-lattice.

Property **(P4)** assures the representability of continuous functions. Let us assume that σ, τ, μ, ν correspond to the domain elements s, t, m, n (possibly with indexes). If we regard $\sigma \to \tau$ as corresponding to the step function $s \Rightarrow t$, we can rewrite **(P4)** as the following standard implication:

$$\bigsqcup_{i \in I}(s_i \Rightarrow t_i) \preceq \bigsqcup_{j \in J}(m_j \Rightarrow n_j) \text{ if and only if}$$

$$\forall i \in I.\, \exists J' \subseteq J.\, \bigsqcup_{j \in J'} m_j \preceq s_i \,\&\, t_i \preceq \bigsqcup_{j \in J'} n_j,$$

which implies the usual representability of continuous functions as joins of step functions (see also the proof of 3.1 and [12, 13]).

3.2. Remark. The hypothesis $\nu_j \neq \omega$ cannot be dropped in **(P4)**. Indeed, by 3.1 (ii) and (xii) we have, for any σ,

$$\sigma \to \omega = \omega \to \omega.$$

This implies that, for any choice of μ, ν and σ,

$$\mu \to \nu \leq \omega \to \omega = \sigma \to \omega.$$

So that **(P4)** would fail as soon as $\sigma \not\leq \mu$.

3.3. Lemma. Suppose that $\rho \wedge \xi \leq \bigoplus_{i \in I}(\sigma \to \tau_i)$, then:

(P5) if $\rho, \xi \leq \omega_1$ then there exist $\{\mu_j \mid j \in J\}$ and $\{\nu_l \mid l \in L\}$ such that

$$\rho \leq \bigoplus_{j \in J}(\sigma \to \mu_j), \quad \xi \leq \bigoplus_{l \in L}(\sigma \to \nu_l), \quad \text{and} \quad (\bigoplus_{j \in J} \mu_j) \wedge (\bigoplus_{l \in L} \nu_l) \leq \bigoplus_{i \in I} \tau_i;$$

(P6) if $\rho \leq \omega_1$ and $\xi = \xi \oplus \omega \neq \omega$ then there exist $\{\mu_j \mid j \in J\}$ and $\{\nu_l \mid l \in L\}$ such that

$$\rho \leq \bigoplus_{j \in J}(\sigma \to \mu_j), \quad \xi \leq \bigoplus_{l \in L}(\sigma \to \nu_l) \oplus \omega \quad \text{and}$$
$$(\bigoplus_{j \in J} \mu_j) \wedge (\bigoplus_{l \in L} \nu_l \oplus \omega) \leq \bigoplus_{i \in I} \tau_i.$$

If instead $\rho \wedge \xi \leq \bigoplus_{i \in I}(\sigma \to \tau_i) \oplus \omega$, then:

(P7) if $\rho, \xi \neq \omega$ then there exist $\{\mu_j \mid j \in J\}$ and $\{\nu_l \mid l \in L\}$ such that

$$\rho \leq \bigoplus_{j \in J}(\sigma \to \mu_j) \oplus \omega \quad \xi \leq \bigoplus_{l \in L}(\sigma \to \nu_l) \oplus \omega \quad \text{and}$$
$$[(\bigoplus_{j \in J} \mu_j) \wedge (\bigoplus_{l \in L} \nu_l)] \oplus \omega \leq \bigoplus_{i \in I} \tau_i \oplus \omega.$$

Proof.

(P5) Let $\Theta(\rho) = \{\rho_h \mid h \in H\}$ and $\Theta(\xi) = \{\xi_k \mid k \in K\}$. As $\rho, \xi \leq \omega_1$, $\omega \notin \Theta(\rho) \cup \Theta(\xi)$ by 3.1 (iii). Therefore $\Theta(\rho \wedge \xi) = \{\rho_h \wedge \xi_k \mid h \in H, k \in K\}$. Moreover let $\rho_h = \bigwedge_{r \in R_h}(\alpha_{h,r} \to \beta_{h,r})$ and $\xi_k = \bigwedge_{s \in S_k}(\gamma_{k,s} \to \delta_{k,s})$. Define $R'_h = \{r \mid \sigma \leq \alpha_{h,r}\}$ and $S'_k = \{s \mid \sigma \leq \gamma_{k,s}\}$. Clearly $R'_h \subseteq R_h$, $S'_k \subseteq S_k$ and some of them may be empty. We set intersections indexed by the empty set to ω.

$$\begin{array}{lll} \rho \wedge \xi \leq \bigoplus_{i \in I}(\sigma \to \tau_i) \;\Rightarrow & & (28) \\ \Rightarrow \quad \forall h \in H, k \in K.\ \exists f(h,k) \in I. \rho_h \wedge \xi_k \leq \sigma \to \tau_{f(h,k)} & \text{by } \mathbf{(P2)} & \\ \Rightarrow\ \forall h \in H, k \in K.\ \exists f(h,k) \in I. \bigwedge_{r \in R'_h} \beta_{h,r} \wedge \bigwedge_{s \in S'_k} \delta_{k,s} \leq \tau_{f(h,k)} & \text{by } \mathbf{(P4)}. & (29) \end{array}$$

$$\begin{array}{llll} \rho \wedge \xi \leq \bigoplus_{i \in I}(\sigma \to \tau_i) & \Rightarrow & \bigwedge_{h \in H, k \in K}(\rho_h \wedge \xi_k) \leq \bigwedge_{i \in I}(\sigma \to \tau_i) & \text{by } \mathbf{(P2)} \\ & \Rightarrow & \forall i \in I.\ \bigwedge_{h \in H, k \in K}(\rho_h \wedge \xi_k) \leq \sigma \to \tau_i & \\ & \Rightarrow & \forall i \in I.\ \bigwedge_{h \in H, k \in K}(\bigwedge_{r \in R'_h} \beta_{h,r} \wedge \bigwedge_{s \in S'_k} \delta_{k,s}) \leq \tau_i & \text{by } \mathbf{(P4)} \\ & \Rightarrow & \bigwedge_{h \in H, k \in K}(\bigwedge_{r \in R'_h} \beta_{h,r} \wedge \bigwedge_{s \in S'_k} \delta_{k,s}) \leq \bigoplus_{i \in I} \tau_i. & (30) \end{array}$$

Therefore we can put $J = H$, $L = K$, $\mu_j \equiv \bigwedge_{r \in R'_j} \beta_{j,r}$, and $\nu_l \equiv \bigwedge_{s \in S'_l} \delta_{l,s}$, where $j \in J$ and $l \in L$ (always setting intersections on the empty set to ω).
Now (29) and (30) can be rewritten respectively as:

$$\forall j \in J, l \in L.\ \exists f(j,l) \in I. \mu_j \wedge \nu_l \leq \tau_{f(j,l)}, \tag{31}$$

$$\bigwedge_{j \in J, l \in L}(\mu_j \wedge \nu_l) \leq \bigoplus_{i \in I} \tau_i. \tag{32}$$

It is easy to verify that $\rho_j \leq \sigma \to \mu_j$ and $\xi_l \leq \sigma \to \nu_l$. This implies $\rho \leq \bigoplus_{j \in J}(\sigma \to \mu_j)$ and $\xi \leq \bigoplus_{l \in L}(\sigma \to \nu_l)$. Moreover we have:

$$\begin{array}{rll} (\bigoplus_{j \in J} \mu_j) \wedge (\bigoplus_{l \in L} \nu_l) & = \bigoplus_{j \in J, l \in L}(\mu_j \wedge \nu_l) & \\ & = \bigoplus_{j \in J, l \in L}(\mu_j \wedge \nu_l) \oplus (\bigwedge_{j \in J, l \in L}(\mu_j \wedge \nu_l)) & \text{by } \mathbf{(P0)} \\ & \leq \bigoplus_{j \in J, l \in L} \tau_{f(j,l)} \oplus \bigoplus_{i \in I} \tau_i & \text{by (31) and (32)} \\ & = \bigoplus_{i \in I} \tau_i. & \end{array}$$

(P6) By 3.3 (i) and **(P3)** $\xi = (\bigwedge_{\xi' \in \Theta(\xi)} \xi') \oplus \omega$, which implies $\rho \wedge \xi = (\rho \wedge \bigwedge_{\xi' \in \Theta(\xi)} \xi') \oplus \rho$. Let $\bigwedge_{\xi' \in \Theta(\xi)} \xi' = \bigwedge_{k \in K}(\gamma_k \to \delta_k)$, $K' = \{k \mid \sigma \leq \gamma_k\}$, and ρ_h, R'_h be as in the proof of **(P5)**.

$$\begin{array}{rll} \rho \wedge \xi \leq \bigoplus_{i \in I}(\sigma \to \tau_i) & \Rightarrow \forall h \in H. \exists g(h) \in I.\ \rho_h \leq \sigma \to \tau_{g(h)} & \text{by } \mathbf{(P2)} \\ & \Rightarrow \forall h \in H. \exists g(h) \in I.\ \bigwedge_{r \in R'_h} \beta_{h,r} \leq \tau_{g(h)} & \text{by } \mathbf{(P4)} \quad (33) \end{array}$$

and

$$\bigwedge_{h \in H}(\bigwedge_{r \in R'_h} \beta_{h,r}) \wedge \bigwedge_{k \in K'} \delta_k \leq \bigoplus_{i \in I} \tau_i. \tag{34}$$

We choose J, μ_j, L as in the proof of **(P5)**, $\nu_l = \delta_l$ if $l \in K'$, $\nu_l = \omega$ otherwise; then (33) and (34) can be rewritten as follows:

$$\forall j \in J. \exists g(j) \in I.\ \mu_j \leq \tau_{g(j)}, \tag{35}$$

$$\bigwedge_{j \in J, l \in L}(\mu_j \wedge \nu_l) \leq \bigoplus_{i \in I} \tau_i. \tag{36}$$

It is easy to verify that $\rho \leq \bigoplus_{j \in J}(\sigma \to \mu_j)$ and $\xi \leq \bigoplus_{l \in L}(\sigma \to \nu_l) \oplus \omega$. We conclude observing that:

$$\begin{array}{rll} (\bigoplus_{j \in J} \mu_j) \wedge [\bigoplus_{l \in L} \nu_l \oplus \omega] & = [\bigoplus_{j \in J, l \in L}(\mu_j \wedge \nu_l)] \oplus [\bigoplus_{j \in J} \mu_j] & \\ & = [\bigoplus_{j \in J, l \in L}(\mu_j \wedge \nu_l)] \oplus [\bigoplus_{j \in J} \mu_j] \oplus [\bigwedge_{j \in J, l \in L}(\mu_j \wedge \nu_l)] & \text{by } \mathbf{(P0)} \\ & \leq \bigoplus_{j \in J} \tau_{g(j)} \oplus \bigoplus_{j \in J} \tau_{g(j)} \oplus \bigoplus_{i \in I} \tau_i & \text{by (35), (36)} \\ & = \bigoplus_{i \in I} \tau_i. & \end{array}$$

(P7) Notice that $\rho \wedge \xi \leq \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega$ implies $(\rho \wedge \xi) \oplus \omega \leq \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega$. Now, let $\bigwedge_{\rho'\in\Theta(\rho)} \rho' = \bigwedge_{h\in H}(\alpha_h \to \beta_h)$, and $\bigwedge_{\xi\in\Theta(\xi)} \xi' = \bigwedge_{k\in K}(\gamma_k \to \delta_k)$. By **(P3)** and **(P2)**

$$(\rho \wedge \xi) \oplus \omega \leq \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega \Rightarrow \bigwedge_{h\in H}(\alpha_h \to \beta_h) \wedge \bigwedge_{k\in K}(\gamma_k \to \delta_k) \leq \bigwedge_{i\in I}(\sigma \to \tau_i).$$

Let again $H' = \{h \mid \sigma \leq \alpha_h\}$, $K' = \{k \mid \sigma \leq \gamma_k\}$. We get, by **(P4)**,

$$\bigwedge_{h\in H}(\alpha_h \to \beta_h) \wedge \bigwedge_{,k\in K}(\gamma_k \to \delta_k) \leq \bigwedge_{i\in I}(\sigma \to \tau_i) \Rightarrow$$

$$\bigwedge_{h\in H'} \beta_h \wedge \bigwedge_{k\in K'} \delta_k \leq \bigwedge_{i\in I} \tau_i.$$

Therefore we can choose $J = H$, $L = K$, $\mu_j = \beta_j$ if $j \in H'$, $\mu_j = \omega$ otherwise, $\nu_l = \delta_l$ if $l \in K'$, $\nu_l = \omega$ otherwise. In fact we have: $\rho \leq \bigwedge_{j\in J}(\sigma \to \mu_j)\oplus\omega$, $\xi \leq \bigwedge_{l\in L}(\sigma \to \nu_l)\oplus\omega$.

Finally $[(\bigoplus_{j\in J} \mu_j) \wedge (\bigoplus_{l\in L} \nu_l)] \oplus \omega \leq \bigoplus_{i\in I} \tau_i \oplus \omega$, since $\bigwedge_{j\in J} \mu_j \wedge \bigwedge_{l\in L} \nu_l \leq \bigwedge_{i\in I} \tau_i$.

□

3.2. Isomorphism between D^∞ and $\mathcal{F}$

We claim that the set $\mathcal{F}$ of filters over *Type*, ordered by subset inclusion, is isomorphic to D^∞ as oper-lattice. We first show that they are isomorphic as ω-algebraic lattices.

3.4. Proposition. The poset $(\mathcal{F}, \subseteq)$ is an ω-algebraic lattice where the meet is the intersection and the join is the least filter including the union, i.e. $F \,\dot\cup F' =\uparrow \{\sigma \wedge \tau \mid \sigma \in F, \tau \in F'\}$. The compact elements of $(\mathcal{F}, \subseteq)$ are the principal filters $\uparrow \sigma =^{\text{def}} \{\tau \in Type \mid \sigma \leq \tau\}$.

Proof. Standard.

□

By this proposition the isomorphism $\mathcal{F} \simeq D^\infty$ inside the category **ALG** is established if there is a one-to-one order preserving map between compact elements of these domains. This is equivalent to have a surjective map $(\_)^+ : Type \to K(D)$ such that $\sigma \leq \tau$ if and only if $(\tau)^+ \sqsubseteq (\sigma)^+$, for all types σ, τ.

3.4. Definition. (i) The map $(\_)^+ : Type \to K(D^\infty)$ is defined by:

$$\begin{array}{c}(\omega)^+ = \bot;\\ (\sigma \to \tau)^+ = \{\!|(\sigma)^+ \Rightarrow (\tau)^+|\!\};\\ (\sigma \wedge \tau)^+ = (\sigma)^+ \sqcup (\tau)^+;\\ (\sigma \oplus \tau)^+ = (\sigma)^+ \uplus (\tau)^+.\end{array}$$

(ii) $\Phi : \mathcal{F} \to D^\infty$ is the unique continuous extension of the map $(\ )^+$, that is:

$$\Phi(F) = \bigsqcup\{(\sigma)^+ \mid \sigma \in F\}.$$

The map is well defined by Lemma 2.3 and by the fact that the join of compact elements is compact.

3.1. Theorem. The map $(\ )^+$ is surjective and such that $(\sigma)^+ \sqsubseteq (\tau)^+$ if and only if $\tau \leq \sigma$. Therefore Φ is an isomorphism in the category **ALG**.

Proof. To see that $(\ )^+$ is surjective we show that for all n, $K(D_n) \subseteq (Type)^+$ by induction on n.

In fact $\{(\omega)^+\} = K(D_0) = \{\bot\}$ which establishes the thesis when $n = 0$.

Let $e = \{\!|f_1, \ldots, f_m|\!\} \in K(D_{n+1})$. Then for all $i \leq m$, either $f_i = \bigsqcup_{j\in J_i}(e_j^{(i)} \Rightarrow d_j^{(i)})$, where J_i is a finite set of indexes and $e_j^{(i)}, d_j^{(i)} \in K(D_n)$ for all $j \in J_i$, or $f_i = \bot$. In the latter case, by the idempotence of $\uplus$, we can suppose that there is just one $i \leq m$ such that $f_i = \bot$. By induction there are types $\mu_j^{(i)}, \nu_j^{(i)}$ such that $(\mu_j^{(i)})^+ = e_j^{(i)}$ and $(\nu_j^{(i)})^+ = d_j^{(i)}$. Then, if $f_i \neq \bot$ for all i, we have

$$e = [\textstyle\bigoplus_{i\leq m} (\bigwedge_{j\in J_i} \mu_j^{(i)} \to \nu_j^{(i)})]^+,$$

otherwise let $f_m = \bot$, then

$$e = [\textstyle\bigoplus_{i\leq m-1} (\bigwedge_{j\in J_i}(\mu_j^{(i)} \to \nu_j^{(i)})) \oplus \omega]^+.$$

We are left to show that

$$\sigma \leq \tau \;\Leftrightarrow\; (\tau)^+ \sqsubseteq (\sigma)^+.$$

The $(\Rightarrow)$ part can be easily proved by induction on the definition of $\leq$. The proof of the $(\Leftarrow)$ part is by structural induction on σ and τ. The more interesting case is when

$$\sigma = \bigoplus_{i\in I}\left[\bigwedge_{h\in H_i}(\mu_h^{(i)} \to \nu_h^{(i)})\right] \quad \text{and} \quad \tau = \bigoplus_{j\in J}\left[\bigwedge_{k\in K_j}(\rho_k^{(j)} \to \xi_k^{(j)})\right],$$

for suitable finite sets of indexes and types. Then, if $(\mu_h^{(i)})^+ = e_h^{(i)}$, $(\nu_h^{(i)})^+ = d_h^{(i)}$, $(\rho_k^{(j)})^+ = r_k^{(j)}$, and $(\xi_k^{(j)})^+ = s_k^{(j)}$, we have:

$$(\sigma)^+ = \{\!|\bigsqcup_{h\in H_i}(e_h^{(i)} \Rightarrow d_h^{(i)}) \mid i \in I|\!\} \quad \text{and} \quad (\tau)^+ = \{\!|\bigsqcup_{k\in K_j}(r_k^{(j)} \Rightarrow s_k^{(j)}) \mid j \in J|\!\}.$$

Now $\{\!|\bigsqcup_{k\in K_j}(r_k^{(j)} \Rightarrow s_k^{(j)}) \mid j \in J|\!\} \leq \{\!|\bigsqcup_{h\in H_i}(e_h^{(i)} \Rightarrow d_h^{(i)}) \mid i \in I|\!\}$ if and only if:

$$\bigsqcup_{j\in J}\left[\bigsqcup_{k\in K_j}(r_k^{(j)} \Rightarrow s_k^{(j)})\right] \sqsubseteq \bigsqcup_{i\in I}\left[\bigsqcup_{h\in H_i}(e_h^{(i)} \Rightarrow d_h^{(i)})\right] \tag{37}$$

$$\forall i \in I.\exists j \in J.\ \bigsqcup_{k\in K_j}(r_k^{(j)} \Rightarrow s_k^{(j)}) \sqsubseteq \bigsqcup_{h\in H_i}(e_h^{(i)} \Rightarrow d_h^{(i)}). \tag{38}$$

Inequation (37) implies that

$$\forall j \in J.\forall k \in K_j.\forall i \in I.\exists H_i' \subseteq H_i.\ \bigsqcup_{i\in I}\left[\bigsqcup_{h\in H_i'} e_h^{(i)}\right] \sqsubseteq r_k^{(j)} \;\&\; s_k^{(j)} \sqsubseteq \bigsqcup_{i\in I}\left[\bigsqcup_{h\in H_i'} d_h^{(i)}\right].$$

Notice that some H_i' may be empty, in which case the corresponding *sup* is equated to $\bot$. By induction we have

$$\forall j \in J.\forall k \in K_j.\forall i \in I.\exists H_i' \subseteq H_i.\ \rho_k^{(j)} \leq \bigwedge_{i\in I}\left(\bigwedge_{h\in H_i'} \mu_h^{(i)}\right) \;\&\; \bigwedge_{i\in I}\left(\bigwedge_{h\in H_i'} \nu_h^{(i)}\right) \leq \xi_k^{(j)},$$

therefore by **(P4)** we get

$$\bigwedge_{i\in I}\left[\bigwedge_{h\in H_i}(\mu_h^{(i)} \to \nu_h^{(i)})\right] \leq \bigwedge_{j\in J}\left[\bigwedge_{k\in K_j}(\rho_k^{(j)} \to \xi_k^{(j)})\right]. \tag{39}$$

Inequation (38) implies that

$$\forall i \in I. \exists j \in J. \forall k \in K_j. \exists H'_i \subseteq H_i.\ \bigsqcup_{h \in H'_i} e_h^{(i)} \sqsubseteq r_k^{(j)} \ \&\ s_k^{(j)} \sqsubseteq \bigsqcup_{h \in H'_i} d_h^{(i)}.$$

By induction we have

$$\forall i \in I. \exists j \in J. \forall k \in K_j\ \exists H'_i \subseteq H_i.\ \rho_k^{(j)} \leq \bigwedge_{h \in H'_i} \mu_h^{(i)} \ \&\ \bigwedge_{h \in H'_i} \nu_h^{(i)} \leq \xi_k^{(j)},$$

therefore by **(P4)** it follows that

$$\forall i \in I. \exists j \in J.\ \bigwedge_{h \in H_i} (\mu_h^{(i)} \to \nu_h^{(i)}) \leq \bigwedge_{k \in K_j} (\rho_k^{(j)} \to \xi_k^{(j)}). \tag{40}$$

Now (39) and (40) imply, by **(P2)**, that

$$\bigoplus_{i \in I} \left[\bigwedge_{h \in H_i} (\mu_h^{(i)} \to \nu_h^{(i)}) \right] \leq \bigoplus_{j \in J} \left[\bigwedge_{k \in K_j} (\rho_k^{(j)} \to \xi_k^{(j)}) \right],$$

that is $\sigma \leq \tau$.

□

The isomorphism Φ and the union operation $\uplus$ over D^∞ induce on $\mathcal{F}$ a continuous binary map $* : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$. Indeed, since $\Phi(\uparrow\sigma) = (\sigma)^+$, we immediately have from Definition 3.4

$$\Phi(\uparrow\sigma) \uplus \Phi(\uparrow\tau) = (\sigma)^+ \uplus (\tau)^+ = (\sigma \oplus \tau)^+ = \Phi(\uparrow(\sigma \oplus \tau)).$$

This extends by continuity to

$$F * F' = \Phi^{-1}(\Phi(F) \uplus \Phi(F')) = \uparrow \{\sigma \oplus \tau \mid \sigma \in F \ \&\ \tau \in F'\}.$$

Therefore $\mathcal{F}$ is turned into an oper-lattice by Φ, so that $*$ satisfies all conditions of Proposition 2.4 and we conclude:

3.2. Corollary. Let $* : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ be defined by:

$$F * F' = \uparrow \{\sigma \oplus \tau \mid \sigma \in F \ \&\ \tau \in F'\}.$$

Then the structures $(\mathcal{F}, *, \subseteq)$ and $(D^\infty, \uplus, \sqsubseteq)$ are isomorphic in **oper-ALG**.

We say that a filter F is $\oplus$-*complete* if and only if $\sigma \oplus \tau \in F$ implies $\sigma \wedge \tau \in F$. Let $\mathcal{F}_C$ be the set of $\oplus$-complete filters. Clearly $\oplus$-complete filters are related to $\uplus$-irreducible elements of D^∞. Recall the operator $\mathcal{C}$ defined in 2.3 (i): then, abusing notation, we define

$$\mathcal{C}(D^\infty) = \Psi^{-1}(\mathcal{C}(\mathcal{P}([D^\infty \to D^\infty]_\perp))).$$

3.5. Proposition. $\mathcal{C}(D^\infty) = \Phi(\mathcal{F}_C)$, therefore $\mathcal{F}_C$ is the set of "singletons" in $\mathcal{F}$.

Proof. It is straightforward to show that

$$G \in \mathcal{F}_C \ \Leftrightarrow\ \forall F, F' \in \mathcal{F}.\ F * F' \subseteq G \Rightarrow F \,\bar{\cup} F' \subseteq G.$$

Now the thesis follows from Definition 2.3 (i), since Φ, Ψ are isomorphisms.

□

As a consequence, we immediately have that $\Phi^{-1}(\mathcal{C}(D^\infty) \cap K(D^\infty))$ is the set of the principal filters $\uparrow\sigma$ such that σ is irreducible. Indeed, if $\mu \oplus \nu \in \uparrow\sigma$ and σ is irreducible, then

$$\sigma = \sigma \wedge (\mu \oplus \nu) = (\sigma \wedge \mu) \oplus (\sigma \wedge \nu),$$

which implies, by irreducibility, $\sigma = \sigma \wedge \mu = \sigma \wedge \nu$, that is $\sigma \leq \mu \wedge \nu$.

Lastly we examine the application between filters induced by the application in D^∞.

3.4. Lemma. (i) The functional application over D^∞ induces, via the isomorphism, an application operation over $\mathcal{F}$. More precisely, for all $F, F' \in \mathcal{F}$, if $F \cdot F'$ is defined as $\Phi^{-1}(\Phi(F) \cdot \Phi(F'))$ then we have:

$$\begin{aligned} F \cdot F' = \{\textstyle\bigoplus_{i\in I} \tau_i \mid \exists \sigma \in F'.\ \bigoplus_{i\in I}(\sigma \to \tau_i) \in F\} \ \cup \\ \{\textstyle\bigoplus_{i\in I} \tau_i \oplus \omega \mid \exists \sigma \in F'.\ \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega \in F\} \cup \uparrow \omega. \end{aligned}$$

Moreover the application distributes to the left over the "union" operation $*$, that is, for any $F, F', G \in \mathcal{F}$, we have:

$$(F * F') \cdot G = (F \cdot G) * (F' \cdot G).$$

(ii) Let $\coprod\ : D^\infty \to D^\infty$ be defined as in 2.3. If $\coprod(F) = \Phi^{-1}(\coprod \Phi(F))$ is the induced operation over $\mathcal{F}$, then

$$\coprod(F) = \uparrow \{\sigma \wedge \tau \mid \sigma \oplus \tau \in F\}.$$

Proof.

(i) Once the main formula has been established, the left distributivity is shown by straightforward calculations.

To prove the main formula it suffices to consider F and F' to be principal filters.

Let $\Theta(\sigma) = \{\sigma_1, \ldots, \sigma_m\}$. We consider two cases, according to whether $\Theta(\sigma)$ contains ω or not.

If $\omega \notin \Theta(\sigma)$, then by 3.3 (iii) and 3.3 (vii) $\sigma_i = \bigwedge_{j\in J_i}(\mu_j^{(i)} \to \nu_i^{(i)})$ for some $J_i, \mu_j^{(i)}, \nu_i^{(i)}$. Therefore $\Phi(\uparrow \sigma) = \{\!| f_1, \ldots, f_m |\!\}$, where $f_i = (\sigma_i)^+ = (\bigwedge_{j\in J_i}(\mu_j^{(i)} \to \nu_i^{(i)}))^+$.

Let $\Phi(\uparrow \tau) = d$, $\Phi(\uparrow \mu_j^{(i)}) = e_j^{(i)}$, $\Phi(\uparrow \nu_j^{(i)}) = g_j^{(i)}$.

$$\begin{aligned} \Phi(\uparrow \sigma) \cdot \Phi(\uparrow \tau) &= \left(\textstyle\biguplus_{i\le m} \Phi(\uparrow \sigma_i)\right) \cdot \Phi(\uparrow \tau) \\ &= \textstyle\biguplus_{i\le m} \{\!| f_i(d) |\!\} \\ &= \textstyle\biguplus_{i\le m} \bigsqcup_{j\in J_i'} g_j^{(i)} \\ &= \textstyle\biguplus_{i\le m} \Phi(\uparrow \bigwedge_{j\in J_i'} \nu_j^{(i)}) \\ &= \Phi(\uparrow \textstyle\bigoplus_{i\le m}(\bigwedge_{j\in J_i'} \nu_j^{(i)})) \end{aligned}$$

where $J_i' = \{j \in J_i \mid e_j^{(i)} \sqsubseteq d\} = \{j \in J_i \mid \tau \le \mu_j^{(i)}\}$.

Define $\mu_i = \bigwedge_{j\in J_i'} \mu_j^{(i)}$, $\nu_i = \bigwedge_{j\in J_i'} \nu_j^{(i)}$ and $\mu = \bigwedge_{i\le m} \mu_i$. Then

$$\begin{aligned} \sigma &= \textstyle\bigoplus_{i\le m}(\bigwedge_{j\in J_i}(\mu_j^{(i)} \to \nu_j^{(i)})) \\ &\le \textstyle\bigoplus_{i\le m}(\bigwedge_{j\in J_i'}(\mu_j^{(i)} \to \nu_j^{(i)})) \\ &\le \textstyle\bigoplus_{i\le m}(\bigwedge_{j\in J_i'}(\mu \to \nu_j^{(i)})) \\ &= \textstyle\bigoplus_{i\le m}(\mu \to \nu_i). \end{aligned}$$

Hence we conclude

$$\uparrow\sigma \cdot \uparrow\tau \ = \uparrow\left\{ \bigoplus_{i\le m} \nu_i \mid \exists \mu \in \uparrow\tau.\ \bigoplus_{i\le m}(\mu \to \nu_i) \in \uparrow\sigma \right\}.$$

If $\omega \in \Theta(\sigma)$, say $\sigma_m = \omega$, then $\Phi(\uparrow \sigma) = \{\!| f_1, \ldots, f_{m-1}, \perp |\!\}$ and $\uparrow\sigma \cdot \uparrow\tau$ becomes

$$\uparrow\left\{ \bigoplus_{i\le m} \nu_i \oplus \omega \mid \exists \mu \in \uparrow\tau.\ \bigoplus_{i\le m}(\mu \to \nu_i) \oplus \omega \in \uparrow\sigma \right\}.$$

Finally, if $\sigma = \omega$, then $\Phi(\uparrow \sigma) = \perp = \{\!| \perp |\!\}$ so that $\uparrow\sigma \cdot \uparrow\tau$ has to be $\uparrow \omega$. Since in any other case ω is in the set $\uparrow\sigma \cdot \uparrow\tau$, we conclude that in the general case the formula stated in this Lemma is valid.

(ii) Easy.

□

4. The Calculus and its Operational Semantics

In this section we introduce a calculus expressing parallel and non-deterministic constructors on higher-order functions. As explained in the Introduction, the semantic operators are the guidelines for this definition. We extend the syntax of λ-calculus with a parallel operator $\|$, a non-deterministic choice operator $+$, and a may-must operator $\coprod$. The operator $\|$ corresponds to the semantic join and therefore to the intersection type constructor. The operator $+$ corresponds to the semantic operator $*$ and therefore to the type constructor $\oplus$. Finally the unary operator $\coprod$ corresponds to the semantic operator which, applied to a set, gives the join of its elements.

We use two sorts of variables, namely the set Vn of call-by-name variables ranged over by x and the set Vv of call-by-value variables ranged over by v. The symbol χ will range over the set $\mathsf{Vn} \cup \mathsf{Vv}$.

$$M ::= x \mid v \mid \lambda\chi.M \mid (MM) \mid (M\|M) \mid (M+M) \mid (\coprod M).$$

We call $\Lambda_{+\|}$ the set of terms; $\Lambda^0_{+\|}$ is the set of closed terms. For any $M \in \Lambda_{+\|}$, we denote the set of free variables of M by $FV(M)$.

We shall write $M \equiv N$ for syntactical equality up to α-congruence, that is up to renaming of bound variables (of course any bound variable will be renamed by a variable of the same sort). By $M[N/\chi]$ is meant the result of substituting in M all free occurrences of χ by N, renaming bound variables of M that occur free in N.

We assume the following precedence among operators: application, abstraction, $\coprod$, $\|$, $+$.

The abstractions $\lambda x.M$ and $\lambda v.M$ express in our formalism non-strict and strict functions respectively, using a notion of value which has been discussed in the introduction and is specified in the following definition.

4.1. Definition (Values). We define the sets $\mathbf{W} \subseteq \mathbf{U} \subseteq \mathbf{V} \subseteq \Lambda_{+\|}$ as follows:

(i) $\mathbf{W}$ is the set of terms generated by the following grammar:

$$W ::= v \mid \lambda\chi.M \mid W\|W$$

(ii) $\mathbf{U}$ is the set of terms generated by the following grammar:

$$U ::= W \mid U\|M \mid M\|U$$

(iii) $\mathbf{V}$ is the set of terms generated by the following grammar:

$$V ::= U \mid V + V.$$

$\mathbf{W}^0 = \mathbf{W} \cap \Lambda^0_{+\|}$, and similarly for $\mathbf{U}^0$ and $\mathbf{V}^0$.

Let us introduce the congruence relation $\approx$ which accounts for the idempotence, commutativity and associativity of $\|$ and $+$.

4.2. Definition. The binary relation $\approx$ is the minimal congruence over $\Lambda_{+\|}$ such that:

$$\begin{array}{cc} M + M \approx M & M\|M \approx M \\ M + N \approx N + M & M\|N \approx N\|M \\ M + (N + L) \approx (M + N) + L & M\|(N\|L) \approx M\|(N\|L). \end{array}$$

Since the reduction and conversion relations that we are going to define are up to $\approx$, we shall abstract away from the bracketing of sums or parallels, and we shall freely make use of the following abbreviations:

$$\sum_{i\in I} M_i \equiv M_1 + \cdots + M_n \qquad \text{and} \qquad \|_{i\in I} M_i \equiv M_1\| \cdots \|M_n$$

where $I = \{1, \ldots, n\}$.

The following map extracts from a term in $\mathbf{U}$ the largest subterm belonging to $\mathbf{W}$, up to $\approx$.

4.3. Definition. The map $m : \mathbf{U} \to \mathbf{W}$ is defined as follows
(i) $m(W) \equiv W$ if $W \in \mathbf{W}$;
(ii) $m(U \| U') \equiv m(U) \| m(U')$ if $U, U' \in \mathbf{U}$;
(iii) $m(U \| M) \equiv m(M \| U) \equiv m(U)$ if $U \in \mathbf{U}$ and $M \notin \mathbf{U}$.

4.1. Remark. $m(U)$ is either U itself, when $U \in \mathbf{W}$, or there exist $W \in \mathbf{W}$ and $M \notin \mathbf{U}$ such that $U \approx W \| M$ and $m(U) \equiv W$.

4.4. Definition (Reduction and Conversion Relations).
(i) The reduction relation $\longrightarrow$ is the least binary relation over $\Lambda^0_{+\|}$ such that:

$$(\beta) \quad (\lambda x.M)N \longrightarrow M[N/x] \qquad (\mu) \quad \frac{N \longrightarrow N'}{MN \longrightarrow MN'} \qquad (\nu) \quad \frac{M \longrightarrow M'}{MN \longrightarrow M'N}$$

$$(\beta_v) \quad \frac{W \in \mathbf{W}}{(\lambda v.M)W \longrightarrow M[W/v]} \qquad (\beta_v\|) \quad \frac{U \longrightarrow V \quad U \in \mathbf{U}}{(\lambda v.M)U \longrightarrow M[m(U)/v] \| (\lambda v.M)V}$$

$$(\beta_v+) \quad \frac{V \approx \sum_{i\in I} U_i \qquad \forall i \in I \quad U_i \in \mathbf{U}}{(\lambda v.M)V \longrightarrow \sum_{i\in I}(\lambda v.M)U_i \ + \ (\lambda v.M)(\|_{i\in I} U_i)}$$

$$(\|_a) \quad \frac{M \longrightarrow M'}{M \| N \longrightarrow M' \| N} \qquad (+_a) \quad \frac{M \longrightarrow M'}{M + N \longrightarrow M' + N}$$

$$(\|_{app}) \ (M \| N)L \longrightarrow (ML) \| (NL) \qquad (+_{app}) \ (M + N)L \longrightarrow (ML) + (NL)$$

$$(+\|) \ (M + N) \| L \longrightarrow (M \| L) + (N \| L) \qquad (\approx) \quad \frac{M \approx M' \qquad M' \longrightarrow N' \qquad N' \approx N}{M \longrightarrow N}$$

$$(\amalg_+) \ \amalg(M + N) \longrightarrow (\amalg M) \| (\amalg N) \qquad (\amalg_\|) \ \amalg(M \| N) \longrightarrow (\amalg M) \| (\amalg N)$$

$$(\amalg_{TV}) \ \frac{W \in \mathbf{W}}{\amalg W \longrightarrow W} \qquad (\amalg_{red}) \ \frac{M \longrightarrow N}{\amalg M \longrightarrow \amalg N.}$$

(ii) $\xrightarrow{*}$ is the reflexive and transitive closure of $\longrightarrow \cup \approx$, and $=$ is the symmetric closure of $\xrightarrow{*}$.

4.2. Remark.

(i) The relation $\xrightarrow{*}$ is not simply the reflexive and transitive closure of $\longrightarrow$ since we think of congruent terms as denoting the same object: therefore if $M \approx N$ then $M \xrightarrow{*} N$ (much as $M \xrightarrow{*} M$), which does not follow from rule $(\approx)$.

(ii) $M \in \Lambda^0_{+\|}$ is a normal form with respect to $\longrightarrow$ if and only if $M \in \mathbf{W}$. Therefore (closed) values in $\mathbf{V} \setminus \mathbf{W}$ are reducible. By inspection of the reduction rules, it is easily seen that the set $\mathbf{V}$ is closed under $\longrightarrow$. For similar notions of reducible values (i.e. which can be further refined) in presence of parallel operators see e.g. [11, 14].

(iii) The relation $\longrightarrow$ is not Church-Rosser. A simple counter-example is

$$\lambda y.(\lambda x.x)(\lambda x.x) \longleftarrow (\lambda xy.x)((\lambda x.x)(\lambda x.x)) \xrightarrow{*} \lambda y.(\lambda x.x),$$

which is due to the presence of rule (μ) and the absence of rule (ξ) of the classical λ-calculus, the last allowing to reduce under abstraction. The problem, however, is not remedied by adding rule (ξ), as it is shown by a second counter-example. Let $\mathbf{K} \equiv \lambda xy.x$, $\mathbf{O} \equiv \lambda xy.y$ and $\mathbf{I} \equiv \lambda x.x$, then

$$\begin{aligned} (\lambda vx.v)(\mathbf{K}\|(\mathbf{I}+\mathbf{O})) &\longrightarrow \lambda x.\mathbf{K}\|(\lambda vx.v)(\mathbf{K}\|\mathbf{I}+\mathbf{K}\|\mathbf{O}) \qquad \text{by } (\beta_v\|) \\ &\stackrel{*}{\longrightarrow} \lambda x.\mathbf{K}\|\lambda x.(\mathbf{K}\|\mathbf{I}) + \lambda x.\mathbf{K}\|\lambda x.(\mathbf{K}\|\mathbf{O}) \\ &\quad +\lambda x.\mathbf{K}\|\lambda x.(\mathbf{K}\|\mathbf{I}\|\mathbf{O}) \end{aligned}$$

but also

$$\begin{aligned} (\lambda vx.v)(\mathbf{K}\|(\mathbf{I}+\mathbf{O})) &\longrightarrow (\lambda vx.v)(\mathbf{K}\|\mathbf{I}+\mathbf{K}\|\mathbf{O}) \qquad \text{by } (\mu) \text{ and } (+\|) \\ &\stackrel{*}{\longrightarrow} \lambda x.(\mathbf{K}\|\mathbf{I}) + \lambda x.(\mathbf{K}\|\mathbf{O}) + \lambda x.(\mathbf{K}\|\mathbf{I}\|\mathbf{O}). \end{aligned}$$

The discussion in the Introduction justifies the following definition.

4.5. Definition (Convergence). Let $M \in \Lambda^0_{+\|}$ and $V \in \mathbf{V}^0$, then:

(i) $M\Downarrow^{must}$ if and only if $\exists V' \in \mathbf{V}^0.\ M \stackrel{*}{\longrightarrow} V'$;
(ii) $M\Downarrow^{may} V$ if and only if $\exists N.\ M \stackrel{*}{\longrightarrow} V + N$;
(iii) $M\Downarrow^{may}$ if and only if $M\Downarrow^{may} V'$ for some value $V' \in \mathbf{V}^0$;
(iv) $M\Uparrow^{must}$ if and only if $\neg M\Downarrow^{may}$;
(v) $M\Uparrow^{may}$ if and only if $\neg M\Downarrow^{must}$.

For the above definition to make sense we have to ensure that the convergence predicates are preserved under reduction (and hence under conversion, since they are trivially preserved under expansion). As observed, the reduction relation is not confluent, so that the following is a non trivial result.

4.1. Proposition. Let $\Downarrow$ be either $\Downarrow^{must}$ or $\Downarrow^{may}$. Then for any closed M and N:

$$M\Downarrow \quad \text{and} \quad M \longrightarrow N \quad \text{imply} \quad N\Downarrow .$$

We do not prove this Proposition here, since it is an immediate consequence of Theorem 5.3 and Corollaries 5.1 and 7.1. Of course, to avoid circularity, Proposition 4.1 will not be used anywhere in the proofs of the subsequent statements.

4.3. Remark.

(i) A typical divergent term, both with respect to $\Downarrow^{must}$ and to $\Downarrow^{may}$ is $\Omega \equiv \Delta\Delta$, where $\Delta \equiv \lambda x.xx$. The same is true for any term of the shape $\Omega M_1 \cdots M_n$ or $(\lambda v.N)\Omega M_1 \cdots M_n$.

(ii) Although $+$ represents non-deterministic choice, a rule like

$$(+) \qquad M + N \longrightarrow M$$

would invalidate Proposition 4.1. As an example consider $\Omega + \mathbf{I}$ and $\Downarrow^{may}$.

(iii) The classical call-by-value β-rule is unsound in the present context with respect to the $\Downarrow^{must}$ predicate. Let $\mathbf{I}, \mathbf{K}, \mathbf{O}$ be as in 4.2 (iii), and Δ, Ω as in (ii). If $M \equiv \lambda v.(v\Delta\mathbf{I}\Delta\|v\mathbf{I}\Delta\Delta)$, then

$$\begin{aligned} M(\Delta\|(\mathbf{K}+\mathbf{O})) &\longrightarrow M(\Delta\|\mathbf{K}+\Delta\|\mathbf{O}) \text{ by } (\mu) \text{ and } (+\|) \\ &\stackrel{*}{\longrightarrow} \Omega\|\Delta \in \mathbf{V}. \end{aligned}$$

By an unconstrained (β_v) rule like (12) we obtain

$$\begin{aligned} M(\Delta\|(\mathbf{K}+\mathbf{O})) &\longrightarrow (\Delta\|(\mathbf{K}+\mathbf{O}))\Delta\mathbf{I}\Delta\|(\Delta\|(\mathbf{K}+\mathbf{O}))\mathbf{I}\Delta\Delta \\ &\stackrel{*}{\longrightarrow} \Omega\mathbf{I}\Delta\|\Omega + \Omega\mathbf{I}\Delta\|\Delta + \Omega\|\Delta + \Omega\|\Omega, \end{aligned}$$

since $\Delta\|(\mathbf{K}+\mathbf{O}) \in \mathbf{V}$. But the resulting term is neither a value nor is it reducible to a value at all.

(iv) We cannot get rid of the map m in rule $(\beta_v\|)$ because of the $\coprod$ operator and of rule (μ). Indeed suppose that we have instead

$$(\beta_v\|)' \ \frac{U \longrightarrow V \quad U \in \mathbf{U}}{(\lambda v.M)U \longrightarrow M[U/v]\|(\lambda v.M)V}$$

as in [14]. Then consider the term $N \equiv \lambda v.((\coprod v)\mathbf{I}\Omega\Omega + (\coprod v)\Omega\mathbf{I}\Omega)$; take further $P \equiv \lambda xyz.x$, $Q \equiv \lambda xyz.y$ and $R \equiv \lambda xyz.z$, which are all in $\mathbf{W}$. Then

$$N(P\|R + Q\|R) \longrightarrow N(P\|R) + N(Q\|R) + N(P\|Q\|R),$$

since $P\|R + Q\|R \in \mathbf{V} \setminus \mathbf{U}$. Now

$$\begin{aligned} N(P\|R) &\longrightarrow (\coprod(P\|R))\mathbf{I}\Omega\Omega + (\coprod(P\|R))\Omega\mathbf{I}\Omega \\ &\stackrel{*}{\longrightarrow} (P\|R)\mathbf{I}\Omega\Omega + (P\|R)\Omega\mathbf{I}\Omega \\ &\stackrel{*}{\longrightarrow} P\mathbf{I}\Omega\Omega\|R\mathbf{I}\Omega\Omega + P\Omega\mathbf{I}\Omega\|R\Omega\mathbf{I}\Omega \\ &\stackrel{*}{\longrightarrow} \mathbf{I}\|\Omega + \Omega\|\Omega \approx \mathbf{I}\|\Omega + \Omega. \end{aligned}$$

Similarly one checks that $N(Q\|R) \stackrel{*}{\longrightarrow} \mathbf{I}\|\Omega + \Omega$. Finally

$$\begin{aligned} N(P\|Q\|R) &\longrightarrow (\coprod(P\|Q\|R))\mathbf{I}\Omega\Omega + (\coprod(P\|Q\|R))\Omega\mathbf{I}\Omega \\ &\stackrel{*}{\longrightarrow} (P\|Q\|R)\mathbf{I}\Omega\Omega + (P\|Q\|R)\Omega\mathbf{I}\Omega, \end{aligned}$$

where

$$(P\|Q\|R)\mathbf{I}\Omega\Omega \stackrel{*}{\longrightarrow} P\mathbf{I}\Omega\Omega\|Q\mathbf{I}\Omega\Omega\|R\mathbf{I}\Omega\Omega \stackrel{*}{\longrightarrow} \mathbf{I}\|\Omega\|\Omega \approx \mathbf{I}\|\Omega$$

and

$$(P\|Q\|R)\Omega\mathbf{I}\Omega \stackrel{*}{\longrightarrow} P\Omega\mathbf{I}\Omega\|Q\Omega\mathbf{I}\Omega\|R\Omega\mathbf{I}\Omega \stackrel{*}{\longrightarrow} \Omega\|\mathbf{I}\|\Omega \approx \mathbf{I}\|\Omega,$$

so that we conclude

$$N(P\|R + Q\|R) \stackrel{*}{\longrightarrow} \mathbf{I}\|\Omega + \Omega + \mathbf{I}\|\Omega + \Omega + \mathbf{I}\|\Omega \approx \mathbf{I}\|\Omega + \Omega.$$

On the other hand, using rule $(\beta_v\|)'$ and observing that $(P + Q)\|R \in \mathbf{U} \setminus \mathbf{W}$, we have

$$\begin{aligned} N((P+Q)\|R) &\longrightarrow ((\coprod((P+Q)\|R))\mathbf{I}\Omega\Omega + (\coprod((P+Q)\|R))\Omega\mathbf{I}\Omega)\|N(P\|R + Q\|R) \\ &\stackrel{*}{\longrightarrow} ((P\|Q\|R)\mathbf{I}\Omega\Omega + (P\|Q\|R)\Omega\mathbf{I}\Omega)\|N(P\|R + Q\|R) \\ &\stackrel{*}{\longrightarrow} (\mathbf{I}\|\Omega + \mathbf{I}\|\Omega)\|(\mathbf{I}\|\Omega + \Omega) \approx \mathbf{I}\|\Omega\|(\mathbf{I}\|\Omega + \Omega) \\ &\longrightarrow \mathbf{I}\|\Omega\|\mathbf{I}\|\Omega + \mathbf{I}\|\Omega\|\Omega \approx \mathbf{I}\|\Omega. \end{aligned}$$

To sum up, we have found that $N(P\|R + Q\|R) \Uparrow^{must}$ but $N((P + Q)\|R) \Downarrow^{must}$, and of course, by rule (μ), $N((P + Q)\|R) \longrightarrow N(P\|R + Q\|R)$.

The next Proposition states some characteristic properties of term constructors with respect to the may and must convergence predicates. Proofs are immediate by definition.

4.2. Proposition. Let $M, N \in \Lambda^0_{+\|}$, then:

(i) $(M + N)\Downarrow^{must}$ if and only if $M\Downarrow^{must}$ and $N\Downarrow^{must}$.
(ii) $(M\|N)\Downarrow^{must}$ if and only if $M\Downarrow^{must}$ or $N\Downarrow^{must}$.
(iii) $(M + N)\Downarrow^{may}$ if and only if $M\Downarrow^{may}$ or $N\Downarrow^{may}$.
(iv) $(M\|N)\Downarrow^{may}$ if and only if $M\Downarrow^{may}$ or $N\Downarrow^{may}$.
(v) $(\coprod M)\Downarrow^{must}$ if and only if $M\Downarrow^{may}$.

Based on the convergence predicate the following definition adapts to the present setting the notion of contextual theories. This notion stems from [30] and is widely used, e.g. in [8] for the classical theory of solvability, in [3], [11], and [31], where it is shown to be equivalent to applicative bisimulation. The idea is that two terms are operationally equivalent if and only if in all contexts they exhibit the same behavior with respect to some observable properties. Here the only observable is convergence. Now Proposition 4.2 shows that "may" convergence does not distinguish between the behavior of "+" and "‖", while "must" convergence does, hence we put:

4.6. Definition (Operational Preorder). Let $M, N \in \Lambda_{+\|}$. Then:

(i) $M \sqsubseteq^{\mathcal{O}} N$ if and only if for all $C[\ ]$ closing both M and N, $C[M]\Downarrow^{must}$ implies $C[N]\Downarrow^{must}$.

(ii) $\simeq^{\mathcal{O}} = \sqsubseteq^{\mathcal{O}} \cap \sqsupseteq^{\mathcal{O}}$.

It should be noted that $M = N$ implies $M \simeq^{\mathcal{O}} N$, but the opposite does not hold.

5. The Type Assignment System

Subsection 5.1 presents the typing rules, which make the relation between type constructors and syntactic operators explicit. The types which can be deduced for a term turn out to be strictly connected to the syntactic structure of the term itself (Theorem 5.1). In Subsection 5.2 we prove some properties of values.

The main results are contained in Subsection 5.3, where we will prove that:

- types are preserved under conversion of terms;
- must convergent terms can be typed by $\omega \to \omega$,
- may convergent terms can be typed by $(\omega \to \omega) \oplus \omega$.

5.1. Typing Rules

As it is clear from the definition of the reduction relation, a call-by-value variable can be substituted only by a (closed) term in $\mathbf{W}$. This explains why we put $v \in \mathbf{W}$ and the choice, made below, to assume that call-by-value variables are assigned, by the bases, to irreducible types different from ω.

5.1. Definition. A *basis* is a mapping $\Gamma : \mathsf{Vn} \cup \mathsf{Vv} \to Type$ such that:

(i) $\Gamma(x) = \omega$ for all x but a finite subset of Vn;

(ii) $\Gamma(\mathsf{Vv}) \subseteq IType$, and $\Gamma(v) = \omega_1$ for all v but a finite subset of Vv.

The notation $\Gamma, \chi : \sigma$ is a shorthand for the function $\Gamma'(\chi') = \sigma$ if $\chi' \equiv \chi$, $\Gamma(\chi')$ otherwise.

5.2. Definition (The Type Assignment System). The axioms and rules of the type assignment system are the following:

$$(\mathrm{Ax})\quad \Gamma \vdash \chi : \Gamma(\chi) \qquad\qquad (\omega)\quad \Gamma \vdash M : \omega$$

$$(\to \mathrm{I}_n)\ \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau} \qquad (\to \mathrm{I}_v)\ \frac{\Gamma, v : \sigma_i \vdash M : \tau_i \quad i \in I}{\Gamma \vdash \lambda v.M : \bigoplus_{i\in I} \sigma_i \to \bigoplus_{i \in I} \tau_i}$$

$$(\to \mathrm{E})\ \frac{\Gamma \vdash M : \bigoplus_{i\in I}(\sigma \to \tau_i) \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \bigoplus_{i\in I} \tau_i} \qquad (\to \mathrm{E}\omega)\frac{\Gamma \vdash M : \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \bigoplus_{i\in I} \tau_i \oplus \omega}$$

$$(\wedge \mathrm{I}_{\coprod})\ \frac{\Gamma \vdash M : \sigma \oplus \tau}{\Gamma \vdash \coprod M : \sigma \wedge \tau} \qquad (\wedge \mathrm{I}_{\|})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M \| N : \sigma \wedge \tau}$$

$$(\oplus \mathrm{I}_{+})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M + N : \sigma \oplus \tau} \qquad (\wedge\, \mathrm{I})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \wedge \tau}$$

$$(\leq)\quad \frac{\Gamma \vdash M : \sigma \qquad \sigma \leq \tau}{\Gamma \vdash M : \tau}.$$

From now on $\Gamma \vdash M : \sigma$ abbreviates "$\Gamma \vdash M : \sigma$ is derivable using previous axioms and rules".

A few comments are in order. Axiom (ω) is standard for the universal type ω. For call-by-name abstraction, we have the usual arrow introduction rule ($\rightarrow \mathrm{I}_n$).

The introduction rule for call-by-value abstractions can be explained by considering the reduction rule (β_v+), which is the syntactical tool making call-by-value abstractions into morphisms of oper-lattices. Indeed, thanks to the equality **(P0)**, we have:

$$\bigoplus_{i\in I}\sigma_i \rightarrow \bigoplus_{i\in I}\tau_i = \bigoplus_{i\in I}\sigma_i \oplus (\bigwedge_{i\in I}\sigma_i) \rightarrow \bigoplus_{i\in I}\tau_i \oplus (\bigwedge_{i\in I}\tau_i).$$

Moreover in our system $\Gamma, v : \sigma_i \vdash M : \tau_i$ for all $i \in I$ implies $\Gamma, v : \bigwedge_{i\in I}\sigma_i \vdash M : \bigwedge_{i\in I}\tau_i$ (see also 5.1). This is why we do not put explicitly the intersections $\bigwedge_{i\in I}\sigma_i$ and $\bigwedge_{i\in I}\tau_i$ in rule ($\rightarrow \mathrm{I}_v$).

For example, assume that W_1, W_2 are in $\mathbf{W}$, of types σ_1, σ_2 respectively, and that $M[W_1/v]$, $M[W_2/v]$ have types τ_1, τ_2, respectively. It follows that $M[W_1\|W_2/v]$ will have type $\tau_1 \wedge \tau_2$. In this case we expect $(\lambda v.M)(W_1 + W_2)$ to have type $\tau_1 \oplus \tau_2$, since

$$\begin{aligned}(\lambda v.M)(W_1 + W_2) &\longrightarrow (\lambda v.M)W_1 + (\lambda v.M)W_2 + (\lambda v.M)(W_1\|W_2)\\ &\longrightarrow M[W_1/v] + M[W_2/v] + M[W_1\|W_2/v].\end{aligned}$$

As $W_1 + W_2$ has type $\sigma_1 \oplus \sigma_2$, a suitable type for $\lambda v.M$ turns out to be $\sigma_1 \oplus \sigma_2 \rightarrow \tau_1 \oplus \tau_2$.

Rules ($\rightarrow$ E) and ($\rightarrow$ Eω) take care of the fact that a term may correspond to a sum of functions, instead of a single function. Indeed, according to ($+_{app}$), application distributes to the left with respect to the sum.

For example if M_1, M_2 have types $\sigma \rightarrow \tau_1$, $\sigma \rightarrow \tau_2$ and N has type σ, we expect $(M_1 + M_2)N$ to have type $\tau_1 \oplus \tau_2$, since $(M_1 + M_2)N \longrightarrow M_1N + M_2N$. On the other hand, if M_1 has type $\sigma \rightarrow \tau$, M_2 has type ω, and N has type σ, using ($\rightarrow$ Eω) we can deduce the type $\tau \oplus \omega$ for $(M_1 + M_2)N$, which reduces to $M_1N + M_2N$.

The typing rule ($\wedge\mathrm{I}_{\coprod}$) reflects the reduction rules ($\coprod_+$) and ($\coprod_{TV}$). For example if $W_1, W_2 \in \mathbf{W}$ have types σ and τ respectively, then $W_1 + W_2$ will have type $\sigma \oplus \tau$ and $\coprod(W_1 + W_2)$ (which reduces to $W_1\|W_2$) will have type $\sigma \wedge \tau$.

Rule ($\wedge$ I) is standard in intersection type systems. Lastly ($\leq$) is the subsumption rule which includes in the system the pre-order relation on types defined in the previous section.

One can easily show that the following rules are admissible, therefore we will freely use them in the subsequent proofs.

$$(+\mathrm{I})\quad \frac{\Gamma \vdash M\colon\sigma \qquad \Gamma \vdash N\colon\sigma}{\Gamma \vdash M + N\colon\sigma}$$

$$(\|\mathrm{I})\quad \frac{\Gamma \vdash M\colon\sigma}{\Gamma \vdash M\|N\colon\sigma} \qquad\qquad \frac{\Gamma \vdash N\colon\sigma}{\Gamma \vdash M\|N\colon\sigma}$$

$$(\wedge\ \mathrm{E})\quad \frac{\Gamma \vdash M\colon\sigma\wedge\tau}{\Gamma \vdash M\colon\sigma} \qquad\qquad \frac{\Gamma \vdash M\colon\sigma\wedge\tau}{\Gamma \vdash M\colon\tau}$$

$$(\leq_L)\quad \frac{\Gamma, x\colon\sigma \vdash M\colon\tau \qquad \sigma' \leq \sigma}{\Gamma, x\colon\sigma' \vdash M\colon\tau}$$

$$(\oplus\wedge\coprod)\frac{\Gamma \vdash M\colon\bigoplus_{i\in I}\sigma_i}{\Gamma \vdash \coprod M\colon\bigwedge_{i\in I}\sigma_i}.$$

In particular ($\oplus \wedge \coprod$) is derivable since, using **(P0)**, we have:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash M\colon\bigoplus_{i\in I}\sigma_i}{\Gamma \vdash M\colon(\bigoplus_{i\in I}\sigma_i)\oplus(\bigwedge_{i\in I}\sigma_i)}\,(\leq)}{\Gamma \vdash \coprod M\colon(\bigoplus_{i\in I}\sigma_i)\wedge(\bigwedge_{i\in I}\sigma_i)}\,(\wedge\mathrm{I}_{\coprod})}{\Gamma \vdash \coprod M\colon\bigwedge_{i\in I}\sigma_i}\,(\leq)$$

Our assignment system enjoys structural properties which show how the types deducible for a term depend on the syntactic structure of the term itself.

5.1. Theorem (Derivability properties).

(i) $\Gamma \vdash \chi : \tau$ if and only if $\Gamma(\chi) \leq \tau$;

(ii) $\Gamma \vdash \lambda\chi.M : \rho$ if and only if there exist $n,\ \sigma_i, \tau_i\ (i \leq n)$ such that

$$\Gamma \vdash \lambda\chi.M : \sigma_i \to \tau_i, \quad \text{and} \quad \bigwedge_{i \leq n} (\sigma_i \to \tau_i) \leq \rho;$$

(iii) $\Gamma \vdash \lambda x.M : \sigma \to \tau$ if and only if $\Gamma, x : \sigma \vdash M : \tau$;

(iv) $\Gamma \vdash \lambda v.M : \sigma \to \tau$, and $\sigma \leq \omega_1$ if and only if there exist $m,\ \mu_l, \nu_l\ (l \leq m)$ such that

$$\Gamma, v : \mu_l \vdash M : \nu_l, \qquad \bigoplus_{l \leq m} \mu_l = \sigma \quad \text{and} \quad \bigoplus_{l \leq m} \nu_l = \tau;$$

(v) $\Gamma \vdash \lambda v.M : \sigma \to \tau$, and $\sigma \not\leq \omega_1$ if and only if $\tau = \omega$;

(vi) $\Gamma \vdash MN : \tau$, and $\tau \leq \omega_1$ if and only if there exist $n, \tau_i\ (i \leq n), \sigma$ such that

$$\Gamma \vdash M : \bigoplus_{i \leq n} (\sigma \to \tau_i), \quad \Gamma \vdash N : \sigma \quad \text{and} \quad \bigoplus_{i \leq n} \tau_i = \tau;$$

(vii) $\Gamma \vdash MN : \tau$, and $\tau = \tau \oplus \omega \neq \omega$ if and only if there exist $n, \tau_i\ (i \leq n), \sigma$ such that

$$\Gamma \vdash M : \bigoplus_{i \leq n} (\sigma \to \tau_i) \oplus \omega, \quad \Gamma \vdash N : \sigma \quad \text{and} \quad \bigoplus_{i \leq n} \tau_i \oplus \omega = \tau;$$

(viii) $\Gamma \vdash M + N : \tau$ if and only if $\Gamma \vdash M : \sigma$, $\Gamma \vdash N : \sigma'$ and $\sigma \oplus \sigma' \leq \tau$, for some σ, σ';

(ix) $\Gamma \vdash M \| N : \tau$ if and only if $\Gamma \vdash M : \sigma$, $\Gamma \vdash N : \sigma'$ and $\sigma \wedge \sigma' \leq \tau$, for some σ, σ';

(x) $\Gamma \vdash \coprod M : \tau$ if and only if $\Gamma \vdash M : \bigoplus_{i \leq n} \tau_i$ and $\bigwedge_{i \leq n} \tau_i \leq \tau$, for some $n,\ \tau_i\ (i \leq n)$.

Proof. All the reverse implications ($\Leftarrow$) follow immediately from the typing rules. For ($\Rightarrow$) we consider only the interesting cases.

(ii) Given a derivation of $\Gamma \vdash \lambda\chi.M{:}\,\rho$, let

$$\Gamma \vdash \lambda\chi.M{:}\,\sigma_1 \to \tau_1, \ldots, \Gamma \vdash \lambda\chi.M{:}\,\sigma_n \to \tau_n$$

be all the statements in this deduction on which $\Gamma \vdash \lambda\chi.M{:}\,\tau$ depends and which are conclusions of rule $(\to \mathrm{I}_n)$ or of rule $(\to \mathrm{I}_v)$. Then, observing that the only rules that leave the subject unchanged are $(\wedge\ \mathrm{I})$ and $(\leq)$, we conclude that

$$\bigwedge_{i \leq n} (\sigma_i \to \tau_i) \leq \rho.$$

(iii) Let $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ be as in the proof of (ii) where ρ is replaced by $\sigma \to \tau$. Then

$$\bigwedge_{i \leq n} (\sigma_i \to \tau_i) \leq \sigma \to \tau$$

which implies, by **(P4)**,

$$\exists J \subseteq \{1, \cdots, n\}.\ \sigma \leq \bigwedge_{j \in J} \sigma_j\ \&\ \bigwedge_{j \in J} \tau_j \leq \tau.$$

Moreover, by looking at the premises of inferences by which the judgments $\Gamma \vdash \lambda\chi.M : \sigma_i \to \tau_i$ are derived, we know that $\Gamma, x{:}\,\sigma_i \vdash M : \tau_i$ for $1 \leq i \leq n$, so that we conclude that $\Gamma, x{:}\,\sigma \vdash M{:}\,\tau$ by $(\leq_L)$, $(\wedge\ \mathrm{I})$ and $(\leq)$.

(iv) Let $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ be as in the proof of (ii) where ρ is replaced by $\sigma \to \tau$. Similarly to case (iii) we have

$$\exists J \subseteq \{1, \cdots, n\}.\ \sigma \leq \bigwedge_{j \in J} \sigma_j \ \&\ \bigwedge_{j \in J} \tau_j \leq \tau.$$

Let $\Theta(\sigma) = \{\xi_r \mid r \leq p\}$.
By construction

$$\Gamma, v{:}\, \alpha_{i,h} \vdash M{:}\, \delta_{i,h} \tag{41}$$

for some $\alpha_{i,h}, \delta_{i,h}$ such that $\sigma_i = \bigoplus_{h \in H_i} \alpha_{i,h}$ and $\tau_i = \bigoplus_{h \in H_i} \delta_{i,h}$. By **(P2)**

$$\sigma \leq \bigwedge_{j \in J} \sigma_j \quad \text{implies} \quad \bigwedge_{r \leq p} \xi_r \leq \bigwedge_{j \in J} (\bigwedge_{h \in H_j} \alpha_{j,h}) \tag{42}$$

$$\begin{array}{rcl} \forall i \in I, h \in H_i.\ \Gamma, v{:}\, \alpha_{i,h} \vdash M{:}\, \delta_{i,h} & \Rightarrow & \forall j \in J, h \in H_j.\ \Gamma, v{:}\, \bigwedge_{r \leq p} \xi_r \vdash M{:}\, \delta_{j,h} \\ & & \text{by (42) and rule } (\leq_L) \\ & \Rightarrow & \Gamma, v{:}\, \bigwedge_{r \leq p} \xi_r \vdash M{:}\, \bigwedge_{j \in J}(\bigwedge_{h \in H_j} \delta_{j,h}) \\ & & \text{by rule } (\wedge\ \mathrm{I}) \\ & \Rightarrow & \Gamma, v{:}\, \bigwedge_{r \leq p} \xi_r \vdash M{:}\, \bigwedge_{j \in J} \tau_j \\ & & \text{by rule } (\leq) \text{ since} \\ & & \bigwedge_{h \in H_j} \delta_{j,h} \leq \bigoplus_{h \in H_j} \delta_{j,h} = \tau_j \\ & \Rightarrow & \Gamma, v{:}\, \bigwedge_{r \leq p} \xi_r \vdash M{:}\, \tau \qquad (43) \\ & & \text{by rule } (\leq). \end{array}$$

If $\sigma \leq \sigma_j$ for all $j \in J$ then, by **(P2)**,

$$\forall j \in J\ \forall r \leq p\ \exists f(j,r) \in H_j.\ \xi_r \leq \alpha_{j,f(j,r)}. \tag{44}$$

Set $G = \{g : J \to \bigcup_{j \in J} H_j \mid \forall j \in J. g(j) \in H_j\}$.

$$\begin{array}{rcll} \bigwedge_{j \in J} \tau_j \leq \tau & \Rightarrow & \bigwedge_{j \in J}(\bigoplus_{h \in H_j} \delta_{j,h}) \leq \tau & \\ & \Rightarrow & \bigoplus_{g \in G}(\bigwedge_{j \in J} \delta_{j,g(j)}) \leq \tau & \text{by distributivity} \\ & \Rightarrow & \forall g \in G\ \exists \nu_g.\ \bigwedge_{j \in J} \delta_{j,g(j)} \leq \nu_g\ \& & \\ & & \tau = \tau \oplus \nu_g & \text{by } \mathbf{(P1)} \\ & \Rightarrow & \forall r \leq p\ \exists \nu_r.\ \bigwedge_{j \in J} \delta_{j,f(j,r)} \leq \nu_r\ \&\ \tau = \tau \oplus \nu_r & (45) \\ & & & \text{choosing } g(j) = f(j,r) \\ & & & \text{in the above statement.} \end{array}$$

From (41) we have $\Gamma, v{:}\, \bigwedge_{j \in J} \alpha_{j,f(j,r)} \vdash M{:}\, \bigwedge_{j \in J} \delta_{j,f(j,r)}$ by rules $(\leq_L)$, and $(\wedge\mathrm{I})$. Therefore by (44) and (45) we deduce $\forall r \leq p.\ \Gamma, v{:}\, \xi_r \vdash M{:}\, \nu_r$. Taking into account (43), we choose $m = p+1$, $\mu_1 = \xi_1, \ldots,\ \mu_p = \xi_p,\ \mu_{p+1} = \bigwedge_{r \leq p} \xi_r$ and $\nu_{p+1} = \tau$.

(v) Let $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n, J$ be as in the proof of (iv). By definition of basis we have $\sigma_i \leq \omega_1$ for all $i \in I$, therefore $\sigma \leq \bigwedge_{i \in I} \sigma_i$ implies $\sigma \leq \omega_1$. This is a contradiction. Therefore $\Gamma \vdash M{:}\, \sigma \to \tau$ can only be obtained by rule $(\leq)$ stating $\tau = \omega$.

(vi) Given a deduction of $\Gamma \vdash MN{:}\, \tau$, let

$$\Gamma \vdash MN{:}\, \rho_1, \ldots, \Gamma \vdash MN{:}\, \rho_m$$

be all the statements in this deduction on which $\Gamma \vdash MN{:}\,\tau$ depends, and which are conclusions of rule $(\rightarrow \mathrm{E})$ or of rule $(\rightarrow \mathrm{E}\omega)$. Let $H = \{h \leq m \mid \Gamma \vdash MN{:}\,\rho_h$ is conclusion of rule $(\rightarrow \mathrm{E})\}$ and $K = \{k \leq m \mid \Gamma \vdash MN{:}\,\rho_k$ is conclusion of rule $(\rightarrow \mathrm{E}\omega)\}$, $(H \cup K = \{1, \cdots, m\})$. Clearly for all $k \in K$ we have $\rho_k = \rho_k \oplus \omega$.
If $\tau \leq \omega_1$, since $\bigwedge_{j \leq m} \rho_j \leq \tau$, we must have $\rho_j \leq \omega_1$ for some $j \leq m$. Thus H is non-empty, while K can be empty.
For all $h \in H$ there exist $L_h, \mu_h, \nu_{h,l}$, where $l \in L_h$, such that:

$$\Gamma \vdash M{:}\, \textstyle\bigoplus_{l \in L_h}(\mu_h \rightarrow \nu_{h,l}), \qquad \Gamma \vdash N{:}\,\mu_h \quad \text{and} \quad \textstyle\bigoplus_{l \in L_h} \nu_{h,l} = \rho_h.$$

Similarly, $\forall k \in K. \exists L_k, \mu_k, \nu_{k,l}$, where $l \in L_k$, such that:

$$\Gamma \vdash M{:}\, \textstyle\bigoplus_{l \in L_k}(\mu_k \rightarrow \nu_{k,l}) \oplus \omega, \qquad \Gamma \vdash N{:}\,\mu_k \quad \text{and} \quad \textstyle\bigoplus_{l \in L_k} \nu_{k,l} \oplus \omega = \rho_k.$$

Let $\sigma = \bigwedge_{j \leq m} \mu_j$. We can deduce $\Gamma \vdash N{:}\,\sigma$ using $(\wedge\ \mathrm{I})$. Since $\sigma \leq \mu_j$ for all $j \in J$, we have

$$\begin{aligned}
&\left[\bigoplus_{l \in L_h}(\mu_h \rightarrow \nu_{h,l})\right] \wedge \left[\bigoplus_{k \in L_k}(\mu_k \rightarrow \nu_{k,l}) \oplus \omega\right] = \\
&= \left[(\bigoplus_{l \in L_h}(\mu_h \rightarrow \nu_{h,l})) \wedge (\bigoplus_{k \in L_k}(\mu_k \rightarrow \nu_{k,l}))\right] \oplus \left[\bigoplus_{l \in L_h}(\mu_h \rightarrow \nu_{h,l})\right] \\
&\leq \left[(\bigoplus_{l \in L_h}(\sigma \rightarrow \nu_{h,l})) \wedge (\bigoplus_{l \in L_k}(\sigma \rightarrow \nu_{k,l}))\right] \oplus \left[\bigwedge_{l \in L_h}(\sigma \rightarrow \nu_{h,l})\right] \\
&= \left[\bigoplus_{l \in L_h, l' \in L_k}(\sigma \rightarrow \nu_{h,l} \wedge \nu_{k,l'})\right] \oplus \left[\bigoplus_{l \in L_h}(\sigma \rightarrow \nu_{h,l})\right].
\end{aligned}$$

Now let $G = \{g : J \rightarrow \bigcup_{j \in J} L_j \mid \forall j \in J.\ g(j) \in L_j\}$. Because of

$$\begin{aligned}
\textstyle\bigwedge_{j \leq m}\left[\bigoplus_{l \in L_j}(\sigma \rightarrow \nu_{j,l})\right] &= \textstyle\bigoplus_{g \in G}\left[\bigwedge_{j \leq m}(\sigma \rightarrow \nu_{j,g(j)})\right] \\
&\hspace{10em} \text{by distributivity} \\
&= \textstyle\bigoplus_{g \in G}\left[\bigwedge_{j \leq m}(\sigma \rightarrow \nu_{j,g(j)})\right] \oplus \left[\bigwedge_{g \in G}\bigwedge_{i \leq m}(\sigma \rightarrow \nu_{j,g(j)})\right] \\
&\hspace{10em} \text{by } \mathbf{(P0)} \\
&= \textstyle\bigoplus_{g \in G}(\sigma \rightarrow \bigwedge_{j \leq m} \nu_{j,g(j)}) \oplus (\sigma \rightarrow \bigwedge_{j \leq m, g \in G} \nu_{j,g(j)}) \\
&= \textstyle\bigoplus_{g \in G}(\sigma \rightarrow \bigwedge_{j \leq m} \nu_{j,g(j)}) \oplus (\sigma \rightarrow \bigwedge_{j \leq m, l \in L_j} \nu_{j,l})
\end{aligned}$$

we deduce the last type for M using $(\wedge\ \mathrm{I})$ and $(\leq)$. Therefore we deduce $\Gamma \vdash MN : \bigoplus_{g \in G}(\bigwedge_{j \leq m} \nu_{j,g(j)}) \oplus (\bigwedge_{j \leq m, l \in L_j} \nu_{j,l})$ by rule $(\rightarrow \mathrm{E})$.

$$\begin{aligned}
\textstyle\bigwedge_{j \leq m} \rho_j \leq \tau &\Rightarrow \textstyle\bigwedge_{j \leq m}(\bigoplus_{l \in L_j} \nu_{j,l}) \leq \tau \\
&\Rightarrow \textstyle\bigoplus_{g \in G}(\bigwedge_{j \leq m} \nu_{j,g(j)}) \leq \tau \\
&\hspace{10em} \text{by distributivity} \\
&\Rightarrow \forall g \in G\ \exists \tau_g.\ \textstyle\bigwedge_{j \leq m} \nu_{j,g(j)} \leq \tau_g \ \&\ \tau = \tau_g \oplus \tau \\
&\hspace{10em} \text{by } \mathbf{(P1)} \\
&\Rightarrow \textstyle\bigoplus_{g \in G}(\bigwedge_{j \leq m} \nu_{j,g(j)}) \oplus (\bigwedge_{j \leq m, l \in L_j} \nu_{j,l}) \leq \bigoplus_{g \in G} \tau_g \oplus \tau = \tau.
\end{aligned}$$

So we conclude $\Gamma \vdash MN : \tau$ by rule $(\leq)$.
(vii) Given a deduction of $\Gamma \vdash MN{:}\,\tau = \tau \oplus \omega$, let

$$\Gamma \vdash MN{:}\,\rho_1, \ldots, \Gamma \vdash MN{:}\,\rho_m,$$

$H, L_h, \mu_h, \nu_h, K, \mu_k, \nu_k, \sigma, G$ be as in the proof of (vi). Now H or K can be empty, but not both. Since $\bigoplus_{l\in L_h}(\mu_h \to \nu_{h,l}) \le \bigoplus_{l\in L_h}(\mu_h \to \nu_{h,l}) \oplus \omega$, we have $\bigwedge_{h\in H}[\bigoplus_{l\in L_h}(\mu_h \to \nu_{h,l})] \wedge \bigwedge_{k\in K}[\bigoplus_{l\in L_k}(\mu_k \to \nu_{k,l}) \oplus \omega] \le \bigwedge_{j\le m}[\bigoplus_{l\in L_j}(\mu_j \to \nu_{j,l}) \oplus \omega]$. As in the proof of previous point we have:

$$\bigwedge_{j\le m}\left[\bigoplus_{l\in L_j}(\mu_j \to \nu_{j,l}) \oplus \omega\right] \le \bigoplus_{g\in G}\left(\sigma \to \bigwedge_{j\le m}\nu_{j,g(j)}\right) \oplus \left(\sigma \to \bigwedge_{j\le m, l\in L_j}\nu_{j,l}\right) \oplus \omega,$$

$$\text{and} \qquad \bigoplus_{g\in G}\left(\bigwedge_{j\le m}\nu_{j,g(j)}\right) \oplus \left(\bigwedge_{j\le m, l\in L_j}\nu_{j,l}\right) \oplus \omega \le \tau.$$

So we can deduce the type $\bigoplus_{g\in G}(\sigma \to \bigwedge_{j\le m}\nu_{j,g(j)}) \oplus (\sigma \to \bigwedge_{j\le m,l\in L_j}\nu_{j,l}) \oplus \omega$ for M and by $(\to \mathrm{E}\omega)$ the type $\bigoplus_{g\in G}(\bigwedge_{j\le m}\nu_{j,g(j)}) \oplus (\bigwedge_{j\le m,l\in L_j}\nu_{j,l}) \oplus \omega$ for MN. We conclude using $(\le)$.

(viii) Again, given a deduction of $\Gamma \vdash M + N{:}\,\tau$, let

$$\Gamma \vdash M + N{:}\,\tau_1, \ldots, \Gamma \vdash M + N{:}\,\tau_n$$

be all the statements in this deduction on which $\Gamma \vdash M + N{:}\,\tau$ depends and which are conclusions of rule $(\oplus \mathrm{I}_+)$. Then $\bigwedge_{i\le n}\tau_i \le \tau$ and there are σ_i, σ'_i such that $\Gamma \vdash M{:}\,\sigma_i, \Gamma \vdash N{:}\,\sigma'_i$ and $\tau_i = \sigma_i \oplus \sigma'_i$ for $1 \le i \le n$. So we can choose $\sigma \equiv \bigwedge_{i\le n}\sigma_i, \sigma' \equiv \bigwedge_{i\le n}\sigma'_i$ since $\sigma \oplus \sigma' \le \tau_i$ for all $i \le n$ and we can deduce $\Gamma \vdash M{:}\,\sigma$, $\Gamma \vdash N{:}\,\sigma'$ using $(\wedge\ \mathrm{I})$.

(ix) Given a deduction of $\Gamma \vdash M\|N{:}\,\tau$, let

$$\Gamma \vdash M\|N{:}\,\tau_1, \ldots, \Gamma \vdash M\|N{:}\,\tau_n$$

be all the statements in this deduction on which $\Gamma \vdash M\|N{:}\,\tau$ depends and which are conclusions of rule $(\wedge \mathrm{I}_{\|})$. Then $\bigwedge_{i\le n}\tau_i \le \tau$ and there are σ_i, σ'_i such that $\Gamma \vdash M{:}\,\sigma_i, \Gamma \vdash N{:}\,\sigma'_i$ and $\tau_i = \sigma_i \wedge \sigma'_i$ for $1 \le i \le n$. So we can proceed as in previous case.

(x) Finally, given a deduction of $\Gamma \vdash \coprod M : \tau$, let

$$\Gamma \vdash \coprod M{:}\,\sigma_1, \ldots, \Gamma \vdash \coprod M{:}\,\sigma_m$$

be all the statements in this deduction on which $\Gamma \vdash \coprod M{:}\,\tau$ depends and which are conclusions of rule $(\wedge \mathrm{I}_{\coprod})$. Then $\bigwedge_{i\le m}\sigma_i \le \tau$ and there are ρ_i, ρ_{i+m} such that $\Gamma \vdash M{:}\,\rho_i \oplus \rho_{i+m}$ and $\sigma_i = \rho_i \wedge \rho_{i+m}$ for $1 \le i \le m$. So we can choose $n = 2m$. We have $\Gamma \vdash M{:}\,\bigoplus_{i\le 2m}\rho_i$. We are done since $\bigwedge_{i\le 2m}\rho_i \le \tau$.

□

The previous Lemma shows that the types which can be deduced for a given term respect its formation. In other words, all rules obtainable by reversing the structural rules of our type assignment system are admissible.

5.2. Properties of Values

Inside the set of terms, values play surely a special role, thank to their syntactic form. We prove in this section some properties of the types which can be deduced for values and for terms containing them.

5.2. Theorem.

(i) $V \in \mathbf{V}$ implies $\vdash V : \omega_1$;
(ii) $W \in \mathbf{W}$ and $\Gamma \vdash W : \bigoplus_{i\in I}\sigma_i$ imply $\Gamma \vdash W : \bigwedge_{i\in I}\sigma_i$;
(iii) $\Gamma \vdash WM : \tau$ and $W \in \mathbf{W}$ imply $\Gamma \vdash W : \sigma \to \tau$ and $\Gamma \vdash M : \sigma$, for some σ;

(iv) $\Gamma, \chi : \sigma \vdash W\chi : \tau$ and $W \in \mathbf{W}^0$ imply $\vdash W : \sigma \to \tau$;
(v) $\Gamma, v : \sigma \vdash vM : \tau$ implies $\sigma \leq \rho \to \tau$ and $\Gamma \vdash M : \rho$, for some ρ;
(vi) $U \in \mathbf{U}$ and $\Gamma \vdash m(U) : \tau$ imply $\Gamma \vdash U : \tau$.

Proof. (i). By induction on the definition of values. If $V \equiv v \in \mathsf{Vv}$ then $\Gamma(v) = \omega_1$, since $Dom(\Gamma) = \emptyset$, and the statement follows by (Ax). If $V \equiv \lambda x.M$ then $\Gamma, x : \omega \vdash M : \omega$ is derivable, by rule (ω); hence the statement follows using $(\to \mathrm{I}_n)$. If $V \equiv \lambda v.M$ we do the same as before but assuming $v : \omega_1$. The statement follows using $(\to \mathrm{I}_v)$ and $(\leq)$. If $V \equiv U \| M$ for $U \in \mathbf{U}$, the statement follows by the induction hypothesis using $(\wedge\, \mathrm{I}_{\|})$. Finally if $V \equiv V' + V''$ the statement follows by the induction hypothesis and (+I).

(ii). By induction on W. If $W \equiv v$ then by definition of basis and by 5.1 (i) W is typeable by an irreducible type σ less than $\bigoplus_{i \in I} \sigma_i$. If $W \equiv \lambda\chi.M$ then by 5.1 (ii) W is typeable by an irreducible type σ less than $\bigoplus_{i \in I} \sigma_i$, since an intersection of arrows is in $IType$ by 3.3 (v). In both cases we have that $\sigma \leq \sigma_i$ for all $i \in I$ by **(P2)**. If $W \equiv W' \| W''$, by 5.1 (ix) there exist ρ' and ρ'' such that $\Gamma \vdash W' : \rho'$, $\Gamma \vdash W'' : \rho''$ and $\rho' \wedge \rho'' \leq \bigoplus_{i \in I} \sigma_i$. By the induction hypothesis it is not restrictive to assume that ρ' and ρ'' are themselves irreducible, so that $\rho' \wedge \rho'' \in IType$ by 3.3 (iv) and the thesis follows.

(iii). The case $\tau = \omega$ is trivial. Let $\tau \neq \tau \oplus \omega$.

$$\begin{array}{lll}\Gamma \vdash WM : \tau & \Rightarrow \exists I, \tau_i, \sigma.\ \Gamma \vdash W : \bigoplus_{i \in I}(\sigma \to \tau_i)\ \& & \\ & \quad \Gamma \vdash M : \sigma\ \&\ \bigoplus_{i \in I} \tau_i = \tau & \text{by 5.1 (vi)} \\ & \Rightarrow \exists I, \tau_i, \sigma.\ \Gamma \vdash W : \bigwedge_{i \in I}(\sigma \to \tau_i) & \text{by (ii)} \\ & \Rightarrow \Gamma \vdash W : \sigma \to \tau & \text{by } (\leq) \text{ since } \bigwedge_{i \in I} \tau_i \leq \tau.\end{array}$$

If $\tau = \tau \oplus \omega$ the proof is similar using 5.1 (vii) instead of 5.1 (vi).

(iv) and (v). Immediate from (iii) and 5.1 (i).

(vi). By induction on the definition of $\mathbf{U}$ using rule $(\|\ \mathrm{I})$.

$\square$

From 5.2 (ii) we obtain a restricted form of $\oplus$ elimination. More precisely, the following rule $(\oplus\,\mathrm{E})$ turns out to be admissible:

$$(\oplus\,\mathrm{E})\ \frac{\Gamma, \chi : \sigma_i \vdash M : \tau_i \quad \forall i \in I \quad \Gamma \vdash W : \bigoplus_{i \in I} \sigma_i \quad W \in \mathbf{W}}{\Gamma \vdash M[W/\chi] : \bigoplus_{i \in I} \tau_i}$$

In fact $\Gamma \vdash W : \bigoplus_{i \in I} \sigma_i$ implies $\Gamma \vdash W : \bigwedge_{i \in I} \sigma_i$. From this and $\Gamma, \chi : \sigma_i \vdash M : \tau_i$ it is easy to find a derivation of $\Gamma \vdash M[W/\chi] : \tau_i$ for all $i \in I$. By the way, notice that one can also deduce $\Gamma \vdash M[W/\chi] : \bigwedge_{i \in I} \tau_i$.

Among values in $\mathbf{W}$, call-by-value abstractions set up a subset whose types satisfy particular properties. More specifically, the following Proposition shows the linearity of call-by-value abstractions with respect to the non-deterministic choice operator.

5.1. Proposition.

(i) $\Gamma \vdash \lambda v.M : \sigma \to \tau$ implies either $\sigma \leq \omega_1$ or $\tau = \omega$;
(ii) If $\Gamma \vdash \lambda v.M : \sigma \to \tau$, $\sigma \leq \omega_1$ and $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$ then

$$\Gamma, v : \bigwedge_{i \in I} \sigma_i \vdash M : \tau$$

and for all $i \in I$ there is τ_i such that

$$\Gamma, v : \sigma_i \vdash M : \tau_i \quad \text{and} \quad \tau = \bigoplus_{i \in I} \tau_i \oplus \tau;$$

(iii) $\Gamma \vdash \lambda v.M : \sigma \to \tau$, $\sigma \leq \omega_1$, and $\Gamma \vdash \lambda v.M : \mu \to \nu$, $\mu \leq \omega_1$ imply $\Gamma \vdash \lambda v.M : \sigma \oplus \mu \to \tau \oplus \nu$;

(iv) $\Gamma \vdash \lambda v.M : \sigma \oplus \tau \to \rho$ and $\sigma \oplus \tau \leq \omega_1$ imply that there are ρ_1, ρ_2 such that

$$\Gamma \vdash \lambda v.M : \sigma \to \rho_1, \quad \Gamma \vdash \lambda v.M : \tau \to \rho_2 \quad \text{and} \quad \rho = \rho_1 \oplus \rho_2 \oplus \rho.$$

Proof.

(i). Immediate from 5.1 (iv) and 5.1 (v).

(ii). By 5.1 (iv) $\Gamma \vdash \lambda v.M : \sigma \to \tau$ implies

$$\exists J, \mu_j, \nu_j.\ \sigma = \bigoplus_{j \in J} \mu_j \ \&\ \tau = \bigoplus_{j \in J} \nu_j \ \&\ \forall j \in J.\ \Gamma, v : \mu_j \vdash M : \nu_j. \tag{46}$$

Notice that each μ_j is in $IType$ by definition of basis.

By 3.3 (i) $\sigma = \bigoplus_{i \in I} \sigma_i$ and each σ_i is irreducible by 3.3 (vii). Therefore by **(P2)** $\bigwedge_{i \in I} \sigma_i = \bigwedge_{j \in J} \mu_j$, so we can deduce from (46) that $\Gamma, v : \bigwedge_{i \in I} \sigma_i \vdash M : \tau$ using $(\leq_L)$, $(\wedge\ \mathrm{I})$ and $(\leq)$. Again by **(P2)** we have $\forall i \in I\ \exists f(i) \in J.\ \sigma_i \leq \mu_{f(i)}$. So we can choose $\tau_i \equiv \nu_{f(i)}$.

(iii). By 5.1 (iv) $\Gamma \vdash \lambda v.M : \sigma \to \tau$ implies

$$\exists I, \sigma_i, \tau_i.\ \sigma = \bigoplus_{i \in I} \sigma_i \ \&\ \tau = \bigoplus_{i \in I} \tau_i \ \&\ \forall i \in I.\ \Gamma, v : \sigma_i \vdash M : \tau_i. \tag{47}$$

In the same way $\Gamma \vdash \lambda v.M : \mu \to \nu$ implies

$$\exists J, \mu_j, \nu_j.\ \mu = \bigoplus_{j \in J} \mu_j \ \&\ \nu = \bigoplus_{j \in J} \nu_j \ \&\ \forall j \in J.\ \Gamma, v : \mu_j \vdash M : \nu_j. \tag{48}$$

Applying rule $(\to \mathrm{I}_v)$ to (47) and (48) we can derive the desired statement.

(iv). The case $\rho = \omega$ is trivial. Otherwise let $\Theta(\sigma) = \{\sigma_i \mid i \leq n\}, \Theta(\tau) = \{\tau_j \mid j \leq m\}$. This implies $\Theta(\sigma \oplus \tau) = \Theta(\sigma) \cup \Theta(\tau)$.

$$\begin{aligned} \Gamma \vdash \lambda v.M : \sigma \oplus \tau \to \rho \Rightarrow\ & \exists \mu_1, \ldots, \mu_{n+m}.\ \rho = (\textstyle\bigoplus_{l \leq n+m} \mu_l) \oplus \rho\ \& \\ & \forall i \leq n.\ \Gamma, v : \sigma_i \vdash M : \mu_i\ \& \\ & \forall j \leq m.\ \Gamma, v : \tau_j \vdash \mu_{n+j} \qquad \text{by (ii)} \\ \Rightarrow\ & \Gamma \vdash \lambda v.M : \sigma \to \textstyle\bigoplus_{i \leq n} \mu_i\ \& \\ & \Gamma \vdash \lambda v.M : \tau \to \textstyle\bigoplus_{j \leq m} \mu_{n+j} \ \text{ by } (\to \mathrm{I}_v). \end{aligned}$$

Therefore we can choose $\rho_1 = \bigoplus_{i \leq n} \mu_i$ and $\rho_2 = \bigoplus_{j \leq m} \mu_{n+j}$. □

5.3. Convergence implies Typability

The following statement is the type invariance theorem, which gives a first evidence of the matching between operational and logical semantics. The proof is rather tedious but it is a straightforward consequence of the properties of the pre-order on types and of the structural properties of deductions.

5.3. Theorem (Type Invariance). $\Gamma \vdash M : \tau$ and $M = N$ imply $\Gamma \vdash N : \tau$.

Proof. First we claim that

$$M \approx N \ \&\ \Gamma \vdash M : \tau \Rightarrow \Gamma \vdash N : \tau.$$

The claim is easily established by induction on M using 5.1 and the properties of $\oplus$ and $\wedge$ with respect to the relation $\leq$.

To prove the theorem it suffices to prove the thesis when $M \longrightarrow N$ or $N \longrightarrow M$. We consider only the interesting cases.

$M \longrightarrow N$ by rule $(\beta_v\|)$. In this case we have $M \equiv (\lambda v.L)U$ for some $U \in \mathbf{U}$ and $N \equiv L[m(U)/v]\|(\lambda v.L)V$ under the hypothesis $U \longrightarrow V$. Clearly by induction M and $(\lambda v.L)V$ have the same types, so it suffices to prove that $\Gamma \vdash L[m(U)/v] : \tau$ implies $\Gamma \vdash M : \tau$.

Fix a derivation $\mathcal{D}$ of $\Gamma \vdash L[m(U)/v] : \tau$. Let σ be the intersection of all the types assigned to $m(U)$ in $\mathcal{D}$. The case $\sigma = \omega$ is trivial. Otherwise, let $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$. We have by 5.2 (ii) that $\Gamma \vdash m(U) : \bigwedge_{i\in I}\sigma_i$, being $m(U) \in \mathbf{W}$. By 5.2 (vi) this implies

$$\Gamma \vdash U : \bigwedge_{i\in I}\sigma_i. \tag{49}$$

Moreover from $\mathcal{D}$ we can build a derivation of

$$\Gamma, v : \bigwedge_{i\in I}\sigma_i \vdash L : \tau \tag{50}$$

using $(\leq_L)$. Therefore from (49) and (50) by $(\rightarrow \mathrm{I}_v)$ and $(\rightarrow \mathrm{E})$ we conclude $\Gamma \vdash M : \tau$.

$M \longrightarrow N$ by rule (β_v+). In this case we have $M \approx (\lambda v.L)(\sum_{i\in I} U_i)$ where $U_i \in \mathbf{U}$ for all $i \in I$, and $N \equiv \sum_{i\in I}(\lambda v.L)U_i + (\lambda v.L)(\|_{i\in I}U_i)$.

$$\begin{aligned}\Gamma \vdash M : \tau \Rightarrow \exists\sigma.\ & \Gamma \vdash \lambda v.L : \sigma \rightarrow \tau \\ & \& \ \Gamma \vdash \textstyle\sum_{i\in I} U_i : \sigma \\ & \qquad\qquad \text{by 5.2 (iii)}\end{aligned} \tag{51}$$

We assume that $\sigma \leq \omega_1$, since otherwise we have $\tau = \omega$ by 5.1 (v).

$$\Gamma \vdash \textstyle\sum_{i\in I} U_i : \sigma \Rightarrow \exists\sigma_i.\ \Gamma \vdash U_i : \sigma_i \ \ \& \ \bigoplus_{i\in I}\sigma_i \leq \sigma \quad \text{by 5.1 (viii)} \tag{52}$$

$$\Gamma \vdash \lambda v.L : \sigma \rightarrow \tau \Rightarrow \Gamma \vdash \lambda v.L : \textstyle\bigoplus_{i\in I}\sigma_i \rightarrow \tau \quad \text{by rule } (\leq) \tag{53}$$

$$\Rightarrow \exists\rho_i.\ \Gamma \vdash \lambda v.L : \sigma_i \rightarrow \rho_i \ \ \& \ \textstyle\bigoplus_{i\in I}\rho_i \oplus \tau = \tau \quad \text{by 5.1 (iv).} \tag{54}$$

$$\tag{55}$$

Then we can derive $\Gamma \vdash (\lambda v.L)U_i : \rho_i$, (from (54) and (52)), and $\Gamma \vdash (\lambda v.L)(\|_{i\in I}U_i) : \tau$, from (53), since $\Gamma \vdash \|_{i\in I}U_i : \bigoplus_{i\in I}\sigma_i$. Therefore we have
$\Gamma \vdash \sum_{i\in I}(\lambda v.L)U_i + (\lambda v.L)(\|_{i\in I}U_i) : \bigoplus_{i\in I}\rho_i \oplus \tau$ by $(\oplus\mathrm{I}_+)$, and we are done.

Viceversa suppose that $\Gamma \vdash N : \tau$; then

$$\begin{aligned}\Gamma \vdash N : \tau \Rightarrow \exists\nu_i, \nu.\ & \Gamma \vdash (\lambda v.L)U_i : \nu_i \ \& \\ & \Gamma \vdash (\lambda v.L)(\|_{i\in I}U_i) : \nu \ \& \ \textstyle\bigoplus_{i\in I}\nu_i \oplus \nu \leq \tau \quad \text{by 5.1 (viii)}\end{aligned}$$

$$\Gamma \vdash (\lambda v.L)U_i : \nu_i \Rightarrow \exists\mu_i.\ \Gamma \vdash \lambda v.L : \mu_i \rightarrow \nu_i \ \& \tag{56}$$

$$\Gamma \vdash U_i : \mu_i \quad \text{by 5.2 (iii)} \tag{57}$$

$$\Gamma \vdash (\lambda v.L)(\|_{i\in I}U_i) : \nu \Rightarrow \exists\sigma.\ \Gamma \vdash \lambda v.L : \sigma \rightarrow \nu \ \& \ \Gamma \vdash \|_{i\in I}U_i : \sigma \quad \text{by 5.2 (iii)} \tag{58}$$

$$\Gamma \vdash \|_{i\in I}U_i : \sigma \Rightarrow \exists\sigma_i.\ \Gamma \vdash U_i : \sigma_i \ \ \& \ \textstyle\bigwedge_{i\in I}\sigma_i \leq \sigma \quad \text{by 5.1 (ix).} \tag{59}$$

Let $\rho_i = \mu_i \wedge \sigma_i \wedge \omega_1$ and $\rho = \bigwedge_{i\in I}\rho_i$. From (56), and (58) we deduce, using ($\leq$), that $\Gamma \vdash \lambda v.L : \rho_i \to \nu_i$ and $\Gamma \vdash \lambda v.L : \rho \to \nu$. Notice that $\rho_i \leq \omega_1$, therefore by 5.1 (iii),

$$\Gamma \vdash \lambda v.L : \bigoplus_{i\in I}\rho_i \oplus \rho \to \bigoplus_{i\in I}\nu_i \oplus \nu.$$

(59), (57), and 5.2 (i) imply $\Gamma \vdash U_i : \rho_i$, so by ($\oplus$I$_+$) and ($\leq$) we have

$$\Gamma \vdash \sum_{i\in I} U_i : \bigoplus_{i\in I}\rho_i \oplus \rho,$$

since, by **(P0)**, $\bigoplus_{i\in I}\rho_i \oplus \rho = \bigoplus_{i\in I}\rho_i$. So we conclude

$$\Gamma \vdash (\lambda v.L)(\sum_{i\in I} U_i) : \bigoplus_{i\in I}\nu_i \oplus \nu.$$

$M \longrightarrow N$ by rule ($\|_{app}$). In this case we have $M \equiv (P\|Q)L$ and $N \equiv PL\|QL$ for some P, Q, L. We distinguish two cases, according to whether $\tau \leq \omega_1$ or $\tau = \tau \oplus \omega \neq \omega$ (the case of $\tau = \omega$ being trivial).

Case $\tau \leq \omega_1$: by 5.1 (vi)

$$\exists I, \tau_i, \sigma.\ \Gamma \vdash P\|Q : \bigoplus_{i\in I}(\sigma \to \tau_i)\ \&\ \Gamma \vdash L : \sigma\ \&\ \tau = \bigoplus_{i\in I}\tau_i.$$

By 5.1 (x)

$$\Gamma \vdash P\|Q : \bigoplus_{i\in I}(\sigma \to \tau_i) \Rightarrow \exists \mu, \nu.\ \Gamma \vdash P : \mu\ \&\ \ \Gamma \vdash Q : \nu\ \&\ \mu \wedge \nu \leq \bigoplus_{i\in I}(\sigma \to \tau_i).$$

It follows that $\mu \wedge \nu \leq \omega_1$, so that at least one type among μ and ν is less than ω_1 (notice that $\mu = \mu \oplus \omega$ and $\nu = \nu \oplus \omega$ imply $\mu \wedge \nu = (\mu \wedge \nu) \oplus \mu \oplus \nu \oplus \omega \not\leq \omega_1$). Assume $\mu \leq \omega_1$; then we distinguish among three subcases.

Subcase $\nu \leq \omega_1$:

$$\begin{array}{lll} & \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash P : \bigoplus_{j\in J}(\sigma \to \mu_j)\ \& & \\ & \quad \Gamma \vdash Q : \bigoplus_{l\in L}(\sigma \to \nu_l)\ \& & \\ & \quad (\bigoplus_{j\in J}\mu_j) \wedge (\bigoplus_{l\in L}\nu_l) \leq \bigoplus_{i\in I}\tau_i & \text{by } \textbf{(P5)} \qquad (60) \\ \Rightarrow & \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash PL : \bigoplus_{j\in J}\mu_j\ \& & \\ & \quad \Gamma \vdash QL : \bigoplus_{l\in L}\nu_l & \text{by } (\to \text{E}) \\ \Rightarrow & \Gamma \vdash PL\|QL : (\bigoplus_{j\in J}\mu_j) \wedge (\bigoplus_{l\in L}\nu_l) & \text{by } (\wedge \text{I}_{\|}) \\ \Rightarrow & \Gamma \vdash PL\|QL : \tau & \text{by (60) and } (\leq). \end{array}$$

Subcase $\nu = \nu \oplus \omega \neq \omega$:

$$\begin{array}{lll} & \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash P : \bigoplus_{j\in J}(\sigma \to \mu_j)\ \& & \\ & \quad \Gamma \vdash Q : \bigoplus_{l\in L}(\sigma \to \nu_l) \oplus \omega\ \& & \\ & \quad (\bigoplus_{j\in J}\mu_j) \wedge [(\bigoplus_{l\in L}\nu_l) \oplus \omega] \leq \bigoplus_{i\in I}\tau_i & \text{by } \textbf{(P6)} \qquad (61) \\ \Rightarrow & \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash PL : \bigoplus_{j\in J}\mu_j\ \& & \\ & \quad \Gamma \vdash QL : \bigoplus_{l\in L}\nu_l \oplus \omega & \text{by } (\to \text{E}) \text{ and } (\to \text{E}\omega) \\ \Rightarrow & \Gamma \vdash PL\|QL : (\bigoplus_{j\in J}\mu_j) \wedge [(\bigoplus_{l\in L}\nu_l) \oplus \omega] & \text{by } (\wedge \text{I}_{\|}) \\ \Rightarrow & \Gamma \vdash PL\|QL : \tau & \text{by (61) and } (\leq). \end{array}$$

Subcase $\nu = \omega$: immediate since then $\mu \wedge \nu = \mu$ so that $\Gamma \vdash PL : \tau$ and the thesis follows by $(\wedge\ \mathrm{I}_{||})$, since trivially $\Gamma \vdash QL : \omega$.

Case $\tau = \tau \oplus \omega$:

$$\begin{array}{ll} \Gamma \vdash (P\|Q)L : \tau \oplus \omega \Rightarrow \exists I, \tau_i, \sigma.\ \Gamma \vdash P\|Q : \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega\ \& & \\ \qquad\qquad \Gamma \vdash L : \sigma\ \&\ \tau = \bigoplus_{i\in I} \tau_i \oplus \omega & \text{by 5.1 (vii)} \end{array}$$

$\Gamma \vdash P\|Q : \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega$

$$\begin{array}{lll} \Rightarrow \exists \mu, \nu.\ \Gamma \vdash P : \mu\ \&\ \ \Gamma \vdash Q : \nu\ \& & & \\ \quad \mu \wedge \nu \leq \bigoplus_{i\in I}(\sigma \to \tau_i) \oplus \omega & \text{by 5.1 (x)} & \\ \Rightarrow \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash P : \bigoplus_{j\in J}(\sigma \to \mu_j) \oplus \omega\ \& & & \\ \quad \Gamma \vdash Q : \bigoplus_{l\in L}(\sigma \to \nu_l) \oplus \omega\ \& & & \\ \quad [(\bigoplus_{j\in J} \mu_j) \wedge (\bigoplus_{l\in L} \nu_l)] \oplus \omega \leq \bigoplus_{i\in I} \tau_i \oplus \omega & \text{by } \mathbf{(P7)} & (62) \\ \Rightarrow \exists J, L, \mu_j, \nu_l.\ \Gamma \vdash PL : \bigoplus_{j\in J} \mu_j \oplus \omega\ \& & & \\ \quad \Gamma \vdash QL : \bigoplus_{l\in L} \nu_l \oplus \omega & \text{by } (\to \mathrm{E}\omega) & \\ \Rightarrow \Gamma \vdash PL\|QL : (\bigoplus_{j\in J} \mu_j \oplus \omega) \wedge (\bigoplus_{l\in L} \nu_l \oplus \omega) & \text{by } (\wedge \mathrm{I}_{||}) & \\ \Rightarrow \Gamma \vdash PL\|QL : \tau & \text{by (62) and } (\leq). & \end{array}$$

□

The main result of the present section is that must convergence implies typability by ω_1 and may convergence implies typability by $\omega_1 \oplus \omega$. We will see in Section 7 that also the converse is true, so that these types will completely characterize convergence properties of terms.

5.1. Corollary.

(i) $M \Downarrow^{must}$ implies $\vdash M : \omega_1$;
(ii) $M \Downarrow^{may} V$ and $\vdash V : \sigma$ imply $\vdash M : \sigma \oplus \omega$;
(iii) $M \Downarrow^{may}$ implies $\vdash M : \omega_1 \oplus \omega$.

Proof.
(i).

$$\begin{array}{ll} M \Downarrow^{must} \Rightarrow \exists V \in \mathbf{V}.\ M \xrightarrow{*} V & \text{by 4.5 (i)} \\ \qquad \Rightarrow \vdash M : \omega_1 & \text{by 5.2 (i) and 5.3.} \end{array}$$

(ii).

$$\begin{array}{ll} M \Downarrow^{may} V\ \&\ \vdash V : \sigma \Rightarrow M = V \text{ or } \exists N.\ M = V + N & \text{by 4.5 (ii)} \\ \qquad \Rightarrow \vdash M : \sigma \oplus \omega & \text{by 5.3 using } (\leq) \text{ if } M = V, \\ & \text{and } (\omega),\ (\oplus \mathrm{I}_+) \text{ if } M = V + N. \end{array}$$

(iii). Immediate from (ii) and 5.2 (i).

□

6. The Model

We define formally how to interpret our language $\Lambda_{+||}$ in D^∞. The correct notion of environment should send the call-by-value variables into $\uplus$-irreducible elements different from $\perp$. Call-by-name λ-abstractions are interpreted as functions in the usual way. For call-by-value λ-abstractions, we must take into account the set of $\uplus$-irreducible elements whose $\uplus$ is less than or equal to the current d. So, recalling the operator $\mathcal{C}$ as defined in 2.3 (ii), we put $\mathcal{C}(d) = \Psi^{-1}(\mathcal{C}(\Psi(d)))$. The interpretation of the syntactic operators is the expected one. This justifies the following definition.

6.1. Definition. An *environment* is a map $\eta : \mathsf{Vn} \cup \mathsf{Vv} \to D^\infty$ such that $\eta(\mathsf{Vv}) \subseteq \mathcal{C}(D^\infty) - \{\perp\}$. Call Env^{D^∞} the set of environments over D^∞. Then the interpretation map $[\![\cdot]\!]^{D^\infty}$: $Env^{D^\infty} \times \Lambda_{+||} \to D^\infty$ is defined as follows (writing $[\![\cdot]\!]$ for $[\![\cdot]\!]^{D^\infty}$ and omitting Ψ):

$$\begin{array}{ll}
[\![x]\!]_\eta & = \eta(x) \\
[\![v]\!]_\eta & = \eta(v) \\
[\![\lambda x.M]\!]_\eta & = \{\!|\lambda d \in D^\infty.[\![M]\!]_{\eta[d/x]}|\!\} \\
[\![\lambda v.M]\!]_\eta & = \{\!|\lambda d \in D^\infty.\bigsqcup_{C\in\mathcal{C}(d)}\langle M,\eta,C\rangle|\!\} \\
[\![MN]\!]_\eta & = [\![M]\!]_\eta \cdot [\![N]\!]_\eta \\
[\![M\|N]\!]_\eta & = [\![M]\!]_\eta \sqcup [\![N]\!]_\eta \\
[\![M+N]\!]_\eta & = [\![M]\!]_\eta * [\![N]\!]_\eta \\
[\![\coprod M]\!]_\eta & = \coprod [\![M]\!]_\eta,
\end{array}$$

where $\langle M,\eta,C\rangle =$ if $\perp \notin C$ then $*_{c\in C}\ [\![M]\!]_{\eta[c/v]}$ else $\perp$.

This is a good definition, since the two functions

$$\begin{array}{l}
\lambda d \in D^\infty.[\![M]\!]_{\eta[d/x]}, \\
\lambda d \in D^\infty.\bigsqcup_{C\in\mathcal{C}(d)}\langle M,\eta,C\rangle
\end{array}$$

are both in $[D^\infty \to D^\infty]$, being defined in terms of continuous operators. This can be shown by an easy induction on M, which we omit.

We show how the interpretation $[\![\cdot]\!]^{D^\infty}$ agrees with the type assignment system. It is clear that both definitions of environment and of the interpretation map apply to $\mathcal{F}$ via Φ. Then $Env^{\mathcal{F}}$ will be the set of mappings sending variables into filters such that Vv is sent to $\mathcal{F}_C \setminus \{\uparrow\omega\}$. In fact we have

$$\eta \in Env^{\mathcal{F}} \ \Leftrightarrow\ \Phi \circ \eta \in Env^{D^\infty}.$$

If the previous definition is rephrased using the corresponding operators over $\mathcal{F}$, we get a notion of interpretation $[\![\cdot]\!]^{\mathcal{F}}_\eta$ such that for all $M \in \Lambda_{+||}$

$$[\![M]\!]^{\mathcal{F}}_\eta \in \mathcal{F} \qquad \text{and} \qquad \Phi([\![M]\!]^{\mathcal{F}}_\eta) = [\![M]\!]^{D^\infty}_{\Phi\circ\eta}.$$

6.1. Theorem. For $\eta \in Env^{\mathcal{F}}$ and any basis Γ, define:

$$\Gamma \models \eta \Leftrightarrow \forall\chi \in \mathsf{Vn} \cup \mathsf{Vv}.\ \Gamma(\chi) \in \eta(\chi).$$

Then for all $M \in \Lambda_{+||}$:

$$[\![M]\!]^{\mathcal{F}}_\eta = \{\sigma \mid \exists\Gamma.\ \Gamma \models \eta\ \&\ \Gamma \vdash M : \sigma\}.$$

Proof. By induction on M. The case in which M is a variable is trivial.

$M \equiv \lambda x.L$: let $F = [\![\lambda x.L]\!]^{\mathcal{F}}_{\eta}$ and $f = \lambda d \in D^{\infty}.\Phi([\![L]\!]^{\mathcal{F}}_{\eta[\Phi^{-1}(d)/x]})$. Then

$$\Phi(F) = [\![\lambda x.L]\!]^{D^{\infty}}_{\Phi\circ\eta} = \{\!|\lambda d \in D^{\infty}.[\![L]\!]^{D^{\infty}}_{(\Phi\circ\eta)[d/x]}|\!\} = \{\!|f|\!\}.$$

Now $\sigma \in F$ if and only if $\uparrow\sigma \subseteq F$, i.e. by the isomorphism between $\mathcal{F}$ and D^{∞}, if and only if $(\sigma)^{+} \in K(\{\!|f|\!\})$. If $(\sigma)^{+} \in K(\{\!|f|\!\})$, then $(\sigma)^{+} = \bigsqcup_{i\in I}(e_i \Rightarrow d_i)$ for I finite and $e_i, d_i \in K(D^{\infty})$. By 3.1, there are σ_i, τ_i such that $e_i = (\sigma_i)^{+}$ and $d_i = (\tau_i)^{+}$, so that we deduce that, for all σ, $\Phi^{-1}(g) = \uparrow\sigma \subseteq F$ if and only if $\sigma = \bigwedge_{i\in I}(\sigma_i \to \tau_i)$ for some I and σ_i, τ_i such that $f((\sigma_i)^{+}) \sqsupseteq (\tau_i)^{+}$.

Since $f((\sigma_i)^{+}) = \Phi([\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\sigma_i/x]})$, we have $f((\sigma_i)^{+}) \sqsupseteq (\tau_i)^{+}$ if and only if $\uparrow\tau_i \subseteq [\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\sigma_i/x]}$, which implies $\tau_i \in [\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\sigma_i/x]}$.

By induction hypothesis $\tau_i \in [\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\sigma_i/x]}$ if and only if there is Γ_i such that $\Gamma_i \models \eta[\uparrow\sigma_i/x]$ and $\Gamma_i \vdash L : \tau_i$. Consider the equivalences:

$$\begin{aligned} \exists\Gamma_i.\ \Gamma_i \models \eta[\uparrow\sigma_i/x] \ \&\ \Gamma_i \vdash L : \tau_i \ &\Leftrightarrow\ \exists\Gamma'_i.\ \Gamma'_i \models \eta \ \&\ \Gamma'_i, x : \sigma_i \vdash L : \tau_i \\ &\Leftrightarrow\ \exists\Gamma'_i.\ \Gamma'_i \models \eta \ \&\ \Gamma'_i \vdash \lambda x.L : \sigma_i \to \tau_i \end{aligned}$$

which hold by 5.1 (iii). Then, defining Γ as $\Gamma(\chi) = \bigwedge_{i\in I}\Gamma'_i(\chi)$, we have $\Gamma \models \eta$ and $\Gamma \vdash \lambda x.L : \bigwedge_{i\in I}(\sigma_i \to \tau_i) = \sigma$, so that we conclude, as desired,

$$\sigma \in F \ \Leftrightarrow\ \exists\Gamma.\ \Gamma \models \eta \ \&\ \Gamma \vdash \lambda x.L : \sigma.$$

$M \equiv \lambda v.L$: let $F = [\![\lambda v.L]\!]^{\mathcal{F}}_{\eta}$, $g = \lambda c \in \mathcal{C}(D^{\infty}).\Phi([\![L]\!]^{\mathcal{F}}_{\eta[\Phi^{-1}(c)/v]})$ and $f = \lambda d \in D^{\infty}.\bigsqcup_{C\in\mathcal{C}(d)}$ (if $\perp \notin C$ then $*_{c\in C}\, g(c)$ else $\perp$). Then $\Phi(F) = \{\!|f|\!\}$.

Let $\sigma, e_i, d_i, \sigma_i, \tau_i$ be as in previous case. For all $i \in I$ we have again the condition $d_i \sqsubseteq f(e_i) = \bigsqcup_{C\in\mathcal{C}(e_i)}($ if $\perp \notin C$ then $*_{c\in C}\ g(c)$ else $\perp)$. As this is a directed join and d_i is compact, we know that for all $i \in I$ there exists $C^{(i)} \in \mathcal{C}(e_i)$ such that $d_i \sqsubseteq *_{c\in C^{(i)}}\, g(c)$ or $d_i \sqsubseteq \perp$.

Let $\perp \notin C^{(i)}$. Suppose that $C^{(i)} = \{c^{(i)}_1, \ldots, c^{(i)}_{k_i}\}$. Then for each $j \leq k_i$ there exists $\mu^{(i)}_j \in IType$ such that $c^{(i)}_j = (\mu^{(i)}_j)^{+}$. This implies that $*_{j\leq k_i} c^{(i)}_j \sqsubseteq e_i$ if and only if $\sigma_i \leq \bigoplus_{j\leq k_i} \mu^{(i)}_j$. It follows that

$$\begin{aligned} d_i \sqsubseteq *_{c\in C^{(i)}}\, g(c) \ &\Leftrightarrow\ \tau_i \in *_{j\leq k_i}[\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\mu^{(i)}_j/v]} \\ &\Leftrightarrow\ \exists\nu^{(i)}_1, \ldots, \nu^{(i)}_{k_i}.\forall j \leq k_i.\ \nu^{(i)}_j \in [\![L]\!]^{\mathcal{F}}_{\eta[\uparrow\mu^{(i)}_j/v]} \ \&\ \textstyle\bigoplus_{j\leq k_i} \nu^{(i)}_j \leq \tau_i. \end{aligned}$$

By induction hypothesis, for all $j \leq k_i$ we have

$$\exists\Gamma^{(i)}_j.\ \Gamma^{(i)}_j \models \eta[\uparrow\mu^{(i)}_j/v] \ \&\ \Gamma^{(i)}_j \vdash L : \nu^{(i)}_j,$$

which is equivalent to

$$\exists\bar{\Gamma}^{(i)}_j.\ \bar{\Gamma}^{(i)}_j \models \eta \ \&\ \bar{\Gamma}^{(i)}_j, v : \mu^{(i)}_j \vdash L : \nu^{(i)}_j.$$

This implies that, for all $i \in I$,

$$\exists\Gamma^{(i)}.\ \Gamma^{(i)} \models \eta \ \&\ \forall j \leq k_i.\ \Gamma^{(i)}, v : \mu^{(i)}_j \vdash L : \nu^{(i)}_j.$$

which, since $\forall j \leq k_j.\ \mu^{(i)}_j \leq \omega_1$, is equivalent by 5.1 (iv) to

$$\exists\Gamma^{(i)}.\ \Gamma^{(i)} \models \eta \ \&\ \Gamma^{(i)} \vdash \lambda v.L : \bigoplus_{j\leq k_i} \mu^{(i)}_j \to \bigoplus_{j\leq k_i} \nu^{(i)}_j.$$

The case $\perp \in C^{(i)}$ is easier. In fact we have $f = \perp \Rightarrow \perp$, i.e. $F = \uparrow\{\omega \to \omega\}$, and $\vdash \lambda v.L : \omega \to \omega$.

$M \equiv LN$: then, using the fact that Φ is a morphism of applicative structures, we have

$$\begin{aligned}[\![LN]\!]^{\mathcal{F}}_{\eta} &= \Phi^{-1}([\![LN]\!]^{D^\infty}_{\Phi\circ\eta}) \\ &= \Phi^{-1}([\![L]\!]^{D^\infty}_{\Phi\circ\eta} \cdot [\![N]\!]^{D^\infty}_{\Phi\circ\eta}) \\ &= [\![L]\!]^{\mathcal{F}}_{\eta} \cdot [\![N]\!]^{\mathcal{F}}_{\eta}\end{aligned}$$

where the last $\_ \cdot \_$ is defined in 3.4 (i). The thesis follows by the induction hypothesis, 5.1 (vi) and 5.1 (vii).

$M \equiv L + N$: reasoning as in the previous cases, we have that

$$[\![L+N]\!]^{\mathcal{F}}_{\eta} = [\![L]\!]^{\mathcal{F}}_{\eta} * [\![N]\!]^{\mathcal{F}}_{\eta},$$

where $*$ is defined in 3.2. The thesis follows by the induction hypothesis and 5.1 (viii).

$M \equiv L \| N$:

$$\begin{aligned}[\![L\|N]\!]^{\mathcal{F}}_{\eta} &= \Phi^{-1}([\![L\|N]\!]^{D^\infty}_{\Phi\circ\eta}) \\ &= \Phi^{-1}([\![L]\!]^{D^\infty}_{\Phi\circ\eta} \sqcup [\![N]\!]^{D^\infty}_{\Phi\circ\eta}) \\ &= [\![L]\!]^{\mathcal{F}}_{\eta} \mathbin{\bar{\cup}} [\![N]\!]^{\mathcal{F}}_{\eta}\end{aligned}$$

where $\bar{\cup}$ is defined in 3.4. The thesis follows by the induction hypothesis and 5.1 (ix).

$M \equiv \coprod L$: again we have

$$[\![\textstyle\coprod L]\!]^{\mathcal{F}}_{\eta} = \coprod [\![L]\!]^{\mathcal{F}}_{\eta},$$

where the last $\coprod$ is defined in 3.4 (ii). The thesis follows by the induction hypothesis, and 5.1 (x).

□

The interpretation of the parallel and non-deterministic features of our language is clarified by the following equalities, which are shown in the proof of previous Theorem.

$$[\![L+N]\!]^{\mathcal{F}}_{\eta} = [\![L]\!]^{\mathcal{F}}_{\eta} * [\![N]\!]^{\mathcal{F}}_{\eta}$$

$$[\![L\|N]\!]^{\mathcal{F}}_{\eta} = [\![L]\!]^{\mathcal{F}}_{\eta} \mathbin{\bar{\cup}} [\![N]\!]^{\mathcal{F}}_{\eta}.$$

As an immediate consequence of Theorem 6.1 we have the completeness of our type assignment system, i.e. that

$$\Gamma \vdash M : \sigma \text{ if and only if } \Gamma \models M : \sigma$$

where $\Gamma \models M : \sigma$ is defined in the standard way.

6.1. Corollary. The structure $\mathcal{F} \simeq D^\infty$ is a model of the equivalence defined in 4.4 over $\Lambda^0_{+\|}$ via the interpretations $[\![\cdot]\!]^{\mathcal{F}}$ (or $[\![\cdot]\!]^{D^\infty}$), i.e.

$$\forall M, N \in \Lambda^0_{+\|}.\ M = N \Rightarrow [\![M]\!]^{\mathcal{F}} = [\![N]\!]^{\mathcal{F}},$$

where the environments are omitted since they do not affect the meaning of closed terms.

Proof. Immediate by 5.3 and 6.1.

□

Being $\mathcal{F}$ ordered by subset inclusion, a partial order $\sqsubseteq^{\mathcal{F}}$ (hence an equivalence $\simeq^{\mathcal{F}}$) is induced over $\Lambda_{+\|}$, namely

$$M \sqsubseteq^{\mathcal{F}} N \;\Leftrightarrow\; \forall \eta \in Env^{\mathcal{F}}.\ [\![M]\!]^{\mathcal{F}}_{\eta} \subseteq [\![N]\!]^{\mathcal{F}}_{\eta}.$$

In the following section we will prove that $\sqsubseteq^{\mathcal{F}}$ coincides with the operational pre-order.

7. Full Abstraction

In this section we will prove that the filter model exactly mirrors the operational semantics, i.e. that it is fully abstract. This means that:

- the filter model is adequate, that is it does not equate operationally distinct programs:

$$M \sqsubseteq^{\mathcal{F}} N \text{ implies } M \sqsubseteq^{\mathcal{O}} N \,;$$

- the filter model reflects the operational distinctions:

$$M \sqsubseteq^{\mathcal{O}} N \text{ implies } M \sqsubseteq^{\mathcal{F}} N.$$

Key to this result is the converse of 5.1 (i), i.e.

$$\vdash M : \omega_1 \text{ implies } M \Downarrow^{must} . \tag{63}$$

7.1. Characteristic Terms

The key property on which the proof of full abstraction relies is that any compact element of $\mathcal{F}$, which is of the shape $\uparrow \sigma$, is λ-definable, since for all types σ there exists a characteristic (closed) term R_σ such that

$$\vdash \mathsf{R}_\sigma : \tau \quad \text{if and only if} \quad \sigma \leq \tau, \tag{64}$$

that is $[\![\mathsf{R}_\sigma]\!]^{\mathcal{F}} = \uparrow \sigma$. Such terms are constructed inductively together with test terms: to each type σ we associate a test term T_σ such that for all closed terms M:

$$\mathsf{T}_\sigma M \Downarrow^{must} \quad \text{if and only if} \quad \vdash M : \sigma. \tag{65}$$

Let Ω be any unsolvable of degree 0 (typically $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$) and $\mathbf{I_v} \equiv \lambda v.v$ the call-by-value identity combinator. Then we define two families of terms $\{\mathsf{R}_\sigma\}_{\sigma \in Type}$ and $\{\mathsf{T}_\sigma\}_{\sigma \in Type}$ as follows:

7.1. Definition. The *characteristic terms* R_σ and the *test terms* T_σ are defined by induction on the number of symbols of σ.

$$\mathsf{R}_\sigma \equiv \begin{cases} \Omega & \text{if } \Theta(\sigma) = \{\omega\} \\ \lambda x.(\mathsf{T}_\mu x)\, \mathsf{R}_\nu & \text{if } \Theta(\sigma) = \{\mu \to \nu\} \\ \mathsf{R}_\mu \| \, \mathsf{R}_\nu & \text{if } \Theta(\sigma) = \{\mu \wedge \nu\} \\ \sum_{\sigma' \in \Theta(\sigma)} \mathsf{R}_{\sigma'} & \text{otherwise.} \end{cases}$$

$$\mathsf{T}_\sigma \equiv \begin{cases} \lambda x.\mathbf{I} & \text{if } \Theta(\sigma) = \{\omega\} \\ \lambda x.\mathbf{I_v}((\lambda v.\mathsf{T}_\nu(v\, \mathsf{R}_\mu))x) & \text{if } \Theta(\sigma) = \{\mu \to \nu\} \\ \lambda x.\mathbf{I_v}(\mathsf{T}_\mu\, x + \mathsf{T}_\nu\, x) & \text{if } \Theta(\sigma) = \{\mu \wedge \nu\} \\ \lambda x.\mathbf{I_v}(\sum_{\sigma' \in \Theta(\sigma)} \mathsf{T}_{\sigma'}(\coprod x)) & \text{if } \omega \in \Theta(\sigma) \neq \{\omega\} \\ \lambda x.\mathbf{I_v}((\lambda v.\|_{\sigma' \in \Theta(\sigma)} \mathsf{T}_{\sigma'} v)x + \sum_{\sigma' \in \Theta(\sigma)} \mathsf{T}_{\sigma'}(\coprod x)) & \text{otherwise.} \end{cases}$$

To see that this inductive definition is correct, let $|\sigma|$ be the number of symbols occurring in σ. A simple induction on σ shows that, if $\sigma' \in \Theta(\sigma)$ then $|\sigma'| \leq |\sigma|$, and that, if the cardinality of $\Theta(\sigma)$ is greater than 1, then this inequality is strict.

In all cases except $\sigma \equiv \omega$, T_σ has the shape $\lambda x.\mathbf{I_v} M$, for some M: so, for any N, either $\mathsf{T}_\sigma N$ must converge or it must diverge (i.e. it cannot be both may divergent and may convergent).

$\mathsf{T}_{\mu\to\nu}$ has an internal call-by-value abstraction besides that one in $\mathbf{I_V}$: indeed, in view of (63) and (65), if the argument of $\mathsf{T}_{\mu\to\nu}$ has an arrow type, then it has also type ω_1, so that it is expected to be must convergent.

Concerning the definition of T_σ when $\Theta(\sigma)$ has more than one element, observe that, if x has type σ, then $\coprod x$ has type $\bigwedge_{\sigma'\in\Theta(\sigma)}\sigma'$. So for example, if $\sigma\equiv\mu\oplus\nu\oplus\omega$ and μ,ν are arrow types, we have:

$$\begin{aligned}\mathsf{T}_{\mu\oplus\nu\oplus\omega}(\mathsf{R}_\mu+\mathsf{R}_\nu+\mathsf{R}_\omega)\overset{*}{\longrightarrow}\mathbf{I_V}\ (\ &\mathsf{T}_\mu(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\omega)+\\ &\mathsf{T}_\nu(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\omega)+\\ &\mathsf{T}_\omega(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\omega)\ \ ),\end{aligned}$$

which is a convergent term assuming (64) and (65). Now let us consider $\sigma\equiv(\mu\wedge\nu)\oplus\tau$, where μ,ν,τ are arrow types. In this case we have:

$$\begin{aligned}\mathsf{T}_{(\mu\wedge\nu)\oplus\tau}((\mathsf{R}_\mu\|\mathsf{R}_\nu)+\mathsf{R}_\tau)\overset{*}{\longrightarrow}\mathbf{I_V}\ (\ &\mathsf{T}_{\mu\wedge\nu}(\mathsf{R}_\mu\|\mathsf{R}_\nu)\|\mathsf{T}_\tau(\mathsf{R}_\mu\|\mathsf{R}_\nu)+\\ &\mathsf{T}_{\mu\wedge\nu}\mathsf{R}_\tau\|\mathsf{T}_\tau\mathsf{R}_\tau+\\ &\mathsf{T}_{\mu\wedge\nu}(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)\|\mathsf{T}_\tau(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)+\\ &\mathsf{T}_{\mu\wedge\nu}(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)+\\ &\mathsf{T}_\tau(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)\ \ ),\end{aligned}$$

which also converges. To understand the use of the subterm $(\lambda v.\|_{\sigma'\in\Theta(\sigma)}\mathsf{T}_{\sigma'}v)x$, consider that by dropping it we would obtain:

$$\mathsf{T}_{(\mu\wedge\nu)\oplus\tau}(\mathsf{R}_\mu+\mathsf{R}_\nu+\mathsf{R}_\tau)\overset{*}{\longrightarrow}\mathbf{I_V}(\mathsf{T}_{\mu\wedge\nu}(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)+\mathsf{T}_\tau(\mathsf{R}_\mu\|\mathsf{R}_\nu\|\mathsf{R}_\tau)),$$

which converges, in spite of the fact that $\mathsf{R}_\mu+\mathsf{R}_\nu+\mathsf{R}_\tau$ does not have in general the type $(\mu\wedge\nu)\oplus\tau$.
Lastly, to justify the subterms with the may-must operator $\coprod$, notice that

$$(\lambda v.(\mathsf{T}_\mu v\|\mathsf{T}_\nu v))(\mathsf{R}_\mu+\mathsf{R}_\mu)\overset{*}{\longrightarrow}\mathsf{T}_\mu\mathsf{R}_\mu\|\mathsf{T}_\nu\mathsf{R}_\mu,$$

which converges, while $\mathsf{R}_\mu+\mathsf{R}_\mu$ cannot be typed by $\mu\oplus\nu$ whenever $\mu\not\leq\nu$.

For the following proof it is useful to consider the sets of types of $\mathbf{I}$ and $\mathbf{I_V}$. Let us define $\mathcal{I}$ as the least filter including all types of the shape $\sigma\to\sigma$.

7.1. Proposition.
(i) $\vdash\mathbf{I}:\tau$ if and only if $\tau\in\mathcal{I}$:
(ii) $\sigma,\tau\in\mathcal{I}$ if and only if $\sigma\oplus\tau\in\mathcal{I}$;
(iii) $\sigma\to\tau\in\mathcal{I}$ if and only if $\sigma\leq\tau$;
(iv) $\vdash\mathbf{I_V}:\sigma\to\tau$ and $\tau\neq\omega$ imply $\sigma\leq\tau\wedge\omega_1$;
(v) $\Gamma\vdash\mathbf{I_V}M:\tau$ and $\tau\neq\omega$ imply $\Gamma\vdash M:\tau\wedge\omega_1$.

Proof.

(i) follows from 5.1 (i).
(ii). ($\Rightarrow$). Notice that $\sigma\wedge\tau\leq\sigma\oplus\tau$. ($\Leftarrow$) follows from (i) and 5.2 (ii).
(iii). Easy.
(iv) follows from 5.1 (i) and 5.1 (i).
(v). By 5.2 (iii) $\vdash\mathbf{I_V}:\sigma\to\tau$ and $\vdash M:\sigma$ for some σ. So we are done by (iv) and rule ($\leq$).

□

7.1. Lemma.
(i) $\vdash\mathsf{T}_\sigma:\tau\to\rho\oplus\omega$, and $\rho\leq\omega_1$ imply $\vdash\mathsf{T}_\sigma:\tau\to\rho\oplus\omega_1$;
(ii) $\vdash\mathsf{R}_\sigma:\tau$ if and only if $\sigma\leq\tau$;
(iii) for all $\tau\in\mathit{Type}$, $\vdash\mathsf{T}_\sigma:\sigma\to\tau\to\tau$;

(iv) $\vdash \mathsf{T}_\sigma : \tau \to \rho$, and $\rho \le \omega_1$ imply $\tau \le \sigma$, and $\rho \in \mathcal{I}$.

Proof. (i) If $\sigma \equiv \omega$ then $\mathsf{T}_\omega \equiv \lambda x.\mathbf{I}$, which has type $\tau \to \rho \oplus \omega$ if and only if $\vdash \mathbf{I} : \rho \oplus \omega$ by 5.1 (iii). This implies that $\vdash \mathbf{I} : \rho$ by 5.2 (ii). By hypothesis $\rho \le \omega_1$, so that $\rho \le \rho \oplus \omega_1$, and the thesis follows.

In all other cases, using the fact that T_σ has the shape $\lambda x.\mathbf{I_v} M$, we have

$$\begin{array}{lll} \vdash \mathsf{T}_\sigma : \tau \to \rho \oplus \omega & \Rightarrow x : \tau \vdash \mathbf{I_v} M : \rho \oplus \omega & \text{by 5.1 (iii)} \\ & \Rightarrow x : \tau \vdash M : (\rho \oplus \omega) \wedge \omega_1 & \text{by 7.1 (v) and } \rho \le \omega_1 \\ & \Rightarrow x : \tau \vdash M : \rho \oplus \omega_1 & \text{by } (\le) \\ & \Rightarrow \vdash \mathsf{T}_\sigma : \tau \to \rho \oplus \omega_1 & \text{by } (\to \mathrm{E}) \text{ and } (\to \mathrm{I}_n), \\ & & \text{since } \vdash \mathbf{I_v} : \rho \oplus \omega_1 \to \rho \oplus \omega_1. \end{array}$$

We prove (ii)-(iv) by a simultaneous induction on σ. Notice that all test terms are terms in $\mathbf{W}^0$. Therefore we will freely use the properties stated in 5.2. We consider only the most interesting cases.

$\Theta(\sigma) = \mu \to \nu$.

- (ii) $\mathsf{R}_{\mu\to\nu} \equiv \lambda x.(\mathsf{T}_\mu x)\,\mathsf{R}_\nu$ and suppose $\vdash \mathsf{R}_{\mu\to\nu} : \tau$. Then we proceed by a subordinate induction on the structure of τ, the base case being that $\tau \equiv \alpha \to \beta$, since $\mathsf{R}_{\mu\to\nu}$ is an abstraction (see 5.1 (ii)).
  We consider two cases, $\beta \neq \beta \oplus \omega$ and $\beta = \beta \oplus \omega$.
  If $\beta \neq \beta \oplus \omega$ then

  $$\begin{array}{lll} \vdash \lambda x.(\mathsf{T}_\mu x)\,\mathsf{R}_\nu : \alpha \to \beta & & \\ & \Rightarrow x : \alpha \vdash (\mathsf{T}_\mu x)\,\mathsf{R}_\nu : \beta & \text{by 5.1 (iii)} \\ & \Rightarrow \exists I, \beta_i, \delta.\ x : \alpha \vdash \mathsf{T}_\mu x : \bigoplus_{i\in I}(\delta \to \beta_i) & \\ & \quad \&\ \vdash \mathsf{R}_\nu : \delta\ \&\ \beta = \bigoplus_{i\in I}\beta_i & \text{by 5.1 (vi)} \\ & \Rightarrow \vdash \mathsf{T}_\mu : \alpha \to \bigoplus_{i\in I}(\delta \to \beta_i)\ \&\ \nu \le \delta & \text{by induction and 5.2 (iv)} \\ & \Rightarrow \vdash \mathsf{T}_\mu : \alpha \to \bigoplus_{i\in I}(\nu \to \beta_i) & \text{by } (\le) \\ & \Rightarrow \alpha \le \mu\ \&\ \bigoplus_{i\in I}(\nu \to \beta_i) \in \mathcal{I} & \text{by induction} \\ & \Rightarrow \alpha \le \mu\ \&\ \forall i \in I. \nu \to \beta_i \in \mathcal{I} & \text{by 7.1 (ii)} \\ & \Rightarrow \alpha \le \mu\ \&\ \forall i \in I.\ \nu \le \beta_i & \text{by 7.1 (iii)} \\ & \Rightarrow \alpha \le \mu\ \&\ \nu \le \beta & \text{since } \beta = \bigoplus_{i\in I}\beta_i \\ & \Rightarrow \mu \to \nu \le \alpha \to \beta. & \end{array}$$

  When $\beta = \beta \oplus \omega$ we have

  $$\begin{array}{l} \exists I, \beta_i, \delta.\ x : \alpha \vdash \mathsf{T}_\mu x : \bigoplus_{i\in I}(\delta \to \beta_i) \oplus \omega\ \&\ \vdash \mathsf{R}_\nu : \delta\ \&\ \beta = \bigoplus_{i\in I}\beta_i \oplus \omega \text{ by 5.1 (vii)} \\ \Rightarrow \vdash \mathsf{T}_\mu : \alpha \to \bigoplus_{i\in I}(\delta \to \beta_i) \oplus \omega\ \&\ \nu \le \delta \text{ as above} \\ \Rightarrow \vdash \mathsf{T}_\mu : \alpha \to \bigoplus_{i\in I}(\nu \to \beta_i) \oplus (\nu \to \omega) \text{ by (i) and } (\le), \end{array}$$

  and then the proof goes on as in the previous case.
  When $\tau \equiv \alpha \wedge \beta$ or $\alpha \oplus \beta$ the thesis follows using the subordinate induction hypothesis and, in the last case, using 5.2 (ii) which implies both $\vdash \mathsf{R}_{\mu\to\nu} : \alpha$ and $\vdash \mathsf{R}_{\mu\to\nu} : \beta$, since $\mathsf{R}_{\mu\to\nu} \in \mathbf{W}$.
- (iii) Since $\tau \to \tau \in IType$, we know that for all τ

  $$\vdash \mathbf{I_v} : (\tau \to \tau) \to (\tau \to \tau). \tag{66}$$

  By induction hypothesis, $\vdash \mathsf{T}_\nu : \nu \to \tau \to \tau$ and $\vdash \mathsf{R}_\mu : \mu$; therefore, because $\mu \to \nu \in IType$,

  $$\dfrac{\dfrac{v : \mu \to \nu \vdash \mathsf{T}_\nu : \nu \to \tau \to \tau \quad \dfrac{v : \mu \to \nu \vdash v : \mu \to \nu \quad v : \mu \to \nu \vdash \mathsf{R}_\mu : \mu}{v : \mu \to \nu \vdash v\mathsf{R}_\mu : \nu}\,(\to \mathrm{E})}{v : \mu \to \nu \vdash \mathsf{T}_\nu(v\mathsf{R}_\mu) : \tau \to \tau}\,(\to \mathrm{E})}{\vdash \lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu) : (\mu \to \nu) \to \tau \to \tau}\,(\to \mathrm{I}_v)$$

By this and rule $(\to\ \mathrm{E})$ we have $x:\mu\to\nu\vdash(\lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu))x:\tau\to\tau$, and the thesis follows by (66), rule $(\to\ \mathrm{E})$ and rule $(\to\ \mathrm{I}_n)$.

- (iv) $\mathsf{T}_{\mu\to\nu}\equiv\lambda x.\mathbf{I_v}((\lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu))x)$ and suppose that $\vdash\mathsf{T}_{\mu\to\nu}:\tau\to\rho,\ \rho\leq\omega_1$. Then we have

$$\begin{array}{ll}
\quad x:\tau\vdash\mathbf{I_v}((\lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu))x):\rho & \\
\Rightarrow x:\tau\vdash(\lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu))x:\rho & \text{by 7.1 (v)}\\
\Rightarrow\ \vdash\lambda v.\mathsf{T}_\nu(v\mathsf{R}_\mu):\tau\to\rho & \text{by 5.2 (iv)}\\
\Rightarrow\exists I,\tau_i,\rho_i.\ \forall i\in I.\ v:\tau_i\vdash\mathsf{T}_\nu(v\mathsf{R}_\mu):\rho_i\ \& & \\
\quad\tau=\bigoplus_{i\in I}\tau_i\ \&\ \rho=\bigoplus_{i\in I}\rho_i & \text{by 5.1 (iv)}\\
\Rightarrow\forall i\in I.\exists\alpha_i.\ \vdash\mathsf{T}_\nu:\alpha_i\to\rho_i\ \&\ v:\tau_i\vdash v\mathsf{R}_\mu:\alpha_i & \text{by 5.2 (iii)}\\
\Rightarrow\forall i\in I.\exists\alpha_i.\ \alpha_i\leq\nu\ \&\ \tau_i\leq\mu\to\alpha_i\ \&\ \rho_i\in\mathcal{I} & \text{by induction and 5.2 (v)}\\
\Rightarrow\forall i\in I.\ \tau_i\leq\mu\to\nu\ \&\ \rho\in\mathcal{I} & \text{by 7.1 (ii) as }\rho=\bigoplus_{i\in I}\rho_i\\
\Rightarrow\tau\leq\mu\to\nu\ \&\ \rho\in\mathcal{I} & \text{since }\tau=\bigoplus_{i\in I}\tau_i.
\end{array}$$

$\Theta(\sigma)=\{\mu\wedge\nu\}$.

- (iv) $\mathsf{T}_{\mu\wedge\nu}\equiv\lambda x.\mathbf{I_v}(\mathsf{T}_\mu x+\mathsf{T}_\nu x)$ and suppose that $\vdash\mathsf{T}_{\mu\wedge\nu}:\tau\to\rho,\ \rho\leq\omega_1$. Then we have

$$\begin{array}{ll}
\quad x:\tau\vdash\mathbf{I_v}(\mathsf{T}_\mu x+\mathsf{T}_\nu x):\rho & \text{by 5.1 (iii)}\\
\Rightarrow x:\tau\vdash\mathsf{T}_\mu x+\mathsf{T}_\nu x:\rho & \text{by 7.1 (v)}\\
\Rightarrow\exists\alpha,\beta.\ x:\tau\vdash\mathsf{T}_\mu x:\alpha\ \&\ x:\tau\vdash\mathsf{T}_\nu x:\beta\ \&\ \alpha\oplus\beta\leq\rho & \text{by 5.1 (viii)}\\
\Rightarrow\exists\alpha,\beta.\ \vdash\mathsf{T}_\mu:\tau\to\alpha\ \&\ \vdash\mathsf{T}_\nu:\tau\to\beta & \text{by 5.2 (iv)}\\
\Rightarrow\exists\alpha,\beta.\ \tau\leq\mu\ \&\ \tau\leq\nu\ \&\ \alpha,\beta\in\mathcal{I} & \text{by induction}\\
\Rightarrow\tau\leq\mu\wedge\nu\ \&\ \rho\in\mathcal{I} & \text{by (67) and 7.1 (ii).}
\end{array}\tag{67}$$

$\omega\in\Theta(\sigma)\neq\{\omega\}$.

- (iii) By derivability of rule $(\oplus\wedge\coprod)$ we know that $x:\sigma\vdash\coprod x:\sigma'$ for all $\sigma'\in\Theta(\sigma)$; therefore

$$\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{x:\sigma\vdash\mathsf{T}_{\sigma'}:\sigma'\to\tau\to\tau}\quad x:\sigma\vdash\coprod x:\sigma'\quad\forall\sigma'\in\Theta(\sigma)}{x:\sigma\vdash\mathsf{T}_{\sigma'}(\coprod x):\tau\to\tau\quad\forall\sigma'\in\Theta(\sigma)}\,(\to\mathrm{E})}{x:\sigma\vdash\sum_{\sigma'\in\Theta(\sigma)}\mathsf{T}_{\sigma'}(\coprod x):\tau\to\tau}\,(\oplus\mathrm{I}_+)$$

So we conclude using (66), $(\to\mathrm{E})$, and $(\to\mathrm{I}_n)$.

- (iv) Let $\Theta(\sigma)=\{\sigma_i\mid i\in I\}$.

$$\begin{array}{lll}
\vdash\mathsf{T}_\sigma:\tau\to\rho & \Rightarrow x:\tau\vdash\mathbf{I_v}(\sum_{i\in I}\mathsf{T}_{\sigma_i}(\coprod x)):\rho & \text{by 5.1 (iii)}\\
& \Rightarrow x:\tau\vdash\sum_{i\in I}\mathsf{T}_{\sigma_i}(\coprod x):\rho & \text{by 7.1 (v) since }\rho\leq\omega_1\\
& \Rightarrow\forall i\in I\ \exists\mu_i.\ x:\tau\vdash\mathsf{T}_{\sigma_i}(\coprod x):\mu_i\ \& & \\
& \qquad\qquad\bigoplus_{i\in I}\mu_i\leq\rho & \text{by 5.1 (viii)}\\
& \Rightarrow\forall i\in I\ \exists\mu_i,\nu_i.\ x:\tau\vdash\mathsf{T}_{\sigma_i}:\nu_i\to\mu_i\ \& & \\
& \qquad\qquad x:\tau\vdash\coprod x:\nu_i & \text{by 5.2 (iii)}\\
& \Rightarrow\forall i\in I\ .\nu_i\leq\sigma_i\ \&\ \mu_i\in\mathcal{I} & \text{by induction}\\
& \Rightarrow\forall i\in I\ .x:\tau\vdash\coprod x:\sigma_i\ \&\ \rho\in\mathcal{I} & \text{by 7.1 (ii).}
\end{array}$$

Now $x:\tau\vdash\coprod x:\sigma_i$ implies by 5.1 (x) that there are $J_i,\tau_j^{(i)}$ such that

$$\tau=\bigoplus_{j\in J_i}\tau_j^{(i)}\quad\text{and}\quad\bigwedge_{j\in J_i}\tau_j^{(i)}\leq\sigma_i.\tag{68}$$

Observe that $\omega \in \Theta(\sigma)$ implies that $\sigma = \sigma \oplus \omega$, so that

$$\begin{array}{ll}\tau = \bigoplus_{i\in I}(\bigoplus_{j\in J_i} \tau_j^{(i)}) & \\ \quad = [\bigwedge_{i\in I}(\bigwedge_{j\in J_i} \tau_j^{(i)})] \oplus \bigoplus_{i\in I} \bigoplus_{j\in J_i} \tau_j^{(i)} \text{ by } \mathbf{(P0)} & \\ \quad \leq [\bigwedge_{i\in I}(\bigwedge_{j\in J_i} \tau_j^{(i)})] \oplus \omega & \\ \quad \leq (\bigwedge_{i\in I} \sigma_i) \oplus \omega & \text{by (68)} \\ \quad = \sigma & \text{by } \mathbf{(P3)}.\end{array}$$

$\omega \notin \Theta(\sigma)$.

- (iii) Because of $\omega \notin \Theta(\sigma)$, for all $\sigma' \in \Theta(\sigma)$, $v : \sigma'$ is a basis. Hence

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{ind. hyp.}}{x:\sigma, v:\sigma' \vdash \mathsf{T}_{\sigma'} : \sigma' \to \tau \to \tau} \quad x:\sigma, v:\sigma' \vdash v:\sigma'}{x:\sigma, v:\sigma' \vdash \mathsf{T}_{\sigma'} v : \tau \to \tau \quad \forall \sigma' \in \Theta(\sigma)}(\to \mathrm{E})}{x:\sigma, v:\sigma' \vdash \|_{\sigma'\in\Theta(\sigma)} \mathsf{T}_{\sigma'} v : \tau \to \tau \quad \forall \sigma' \in \Theta(\sigma)}(\wedge \mathrm{I}_{\|}), (\leq)}{x:\sigma \vdash \lambda v.\|_{\sigma'\in\Theta(\sigma)} \mathsf{T}_{\sigma'} v : \sigma \to \tau \to \tau}(\to \mathrm{I}_v) \qquad x:\sigma \vdash x:\sigma}{x:\sigma \vdash (\lambda v.\|_{\sigma'\in\Theta(\sigma)} \mathsf{T}_{\sigma'} v)x : \tau \to \tau}(\to \mathrm{E})$$

Moreover we can derive $x : \sigma \vdash \sum_{\sigma'\in\Theta(\sigma)} \mathsf{T}_{\sigma'}(\coprod x) : \tau \to \tau$ as in previous case. Now the thesis follows applying rules $(\oplus \mathrm{I}_+)$, $(\to \mathrm{E})$, $(\to \mathrm{I}_n)$, and using (66).
- (iv) Let $\Theta(\sigma) = \{\sigma_i \mid i \in I\}$. As in the proof of previous case we get

$$x : \tau \vdash (\lambda v.\|_{i\in I} \mathsf{T}_{\sigma_i} v)x + \sum_{i\in I} \mathsf{T}_{\sigma_i}(\coprod x)) : \rho.$$

By 5.1 (viii), this judgment implies that there exist μ and μ_i such that

$$x : \tau \vdash (\lambda v.\|_{i\in I} \mathsf{T}_{\sigma_i} v)x : \mu \tag{69}$$

$$\forall i \in I.\ x : \tau \vdash \mathsf{T}_{\sigma_i}(\coprod x) : \mu_i \tag{70}$$

$$\mu \oplus \bigoplus_{i\in I} \mu_i \leq \rho. \tag{71}$$

As above from (70) we have $\mu_i \in \mathcal{I}$ and (68). Moreover we get

$$\begin{array}{rll}(69) & \Rightarrow\ \vdash \lambda v.\|_{i\in I} \mathsf{T}_{\sigma_i} v : \tau \to \mu & \text{by 5.2 (iv)} \\ & \Rightarrow \exists H, \tau_h, \nu_h.\tau = \bigoplus_{h\in H} \tau_h \ \&\ \mu = \bigoplus_{h\in H} \nu_h \ \& & \\ & \qquad \forall h \in H.\ v : \tau_h \vdash \|_{i\in I} \mathsf{T}_{\sigma_i} v : \nu_h & \text{by 5.1 (iv).}\end{array} \tag{72}$$

$$\begin{array}{ll}v : \tau_h \vdash \|_{i\in I} \mathsf{T}_{\sigma_i} v : \nu_h & \\ \qquad \Rightarrow \forall i \in I\ \exists \xi_h^{(i)}.\ v : \tau_h \vdash \mathsf{T}_{\sigma_i} v : \xi_h^{(i)} \ \&\ \bigwedge_{i\in I} \xi_h^{(i)} \leq \nu_h & \text{by 5.1 (ix)} \\ \qquad \Rightarrow \forall i \in I\ \exists \xi_h^{(i)}.\ \vdash \mathsf{T}_{\sigma_i} : \tau_h \to \xi_h^{(i)} & \text{by 5.2 (iv).}\end{array}$$

(71) and $\rho \leq \omega_1$ imply that $\mu \leq \omega_1$; now from $\mu = \bigoplus_{h\in H} \nu_h$ and $\bigwedge_{i\in I} \xi_h^{(i)} \leq \nu_h$ we get $\bigwedge_{i\in I} \xi_h^{(i)} \leq \omega_1$. This implies $\exists i_h \in I.\ \xi_h^{(i_h)} \leq \omega_1$. In fact, if we assume $\forall i \in I.\ \xi_h^{(i)} = \xi_h^{(i)} \oplus \omega$, we have $\bigwedge_{i\in I} \xi_h^{(i)} = (\bigwedge_{i\in I} \xi_h^{(i)}) \oplus \omega$ by **(P3)**. Moreover by (i) we have that $\xi_h^{(i)} = \xi_h^{(i)} \oplus \omega$ implies $\xi_h^{(i)} = \omega$. So by induction

$$\forall h \in H \exists i_h \in I.\ \tau_h \leq \sigma_{i_h} \ \&\ \xi_h^{(i_h)} \in \mathcal{I}. \tag{73}$$

Therefore from $\forall h \in H\ \exists i_h \in I.\ \xi_h^{(i_h)} \in \mathcal{I}$, $\forall h \in H.\ \bigwedge_{i\in I}\xi_h^{(i)} \leq \nu_h$, $\forall h \in H\ \forall i \in I$ either $\xi_h^{(i)} \leq \omega_1$ or $\xi_h^{(i)} = \omega$, $\mu = \bigoplus_{h\in H}\nu_h$, $\forall i \in I.\ \mu_i \in \mathcal{I}$, and (71), we conclude $\rho \in \mathcal{I}$. Lastly

$$\begin{array}{ll}\tau = \bigoplus_{i\in I}(\bigoplus_{j\in J_i}\tau_j^{(i_h)}) & \text{by (68)}\\ \quad = [\bigwedge_{i\in I}(\bigwedge_{j\in J_i}\tau_j^{(i_h)})] \oplus \tau & \text{by } \mathbf{(PO)}\\ \quad = [\bigwedge_{i\in I}(\bigwedge_{j\in J_i}\tau_j^{(i_h)})] \oplus \bigoplus_{h\in H}\tau_h & \text{by (72)}\\ \quad \leq (\bigwedge_{i\in I}\sigma_i) \oplus \bigoplus_{h\in H}\sigma_{i_h} & \text{by (68) and (73)}\\ \quad \leq \sigma & \text{by } \mathbf{(P2)}.\end{array}$$

□

7.2. Typability implies Convergence

The aim of this subsection is to prove that types completely characterize must and may convergence, i.e.

$$\vdash M : \omega_1 \;\Rightarrow\; M\Downarrow^{must} \qquad\qquad \vdash M : \omega_1 \oplus \omega \;\Rightarrow\; M\Downarrow^{may}$$

for all closed terms M. The $\Leftarrow$ were proved in 5.1 (i), 5.1 (iii).

The proof of $\Rightarrow$ requires a double induction on types and deductions. Following a standard methodology, we split this induction by introducing a "realizability interpretation" of types as sets of closed terms.

7.2. Definition (Realizability). We define the mapping $[\![\cdot]\!] : \mathit{Type} \to \mathcal{P}(\Lambda^0_{+\|})$ by induction:

(i) $[\![\omega]\!] = \Lambda^0_{+\|}$;

(ii) $M \in [\![\sigma \to \tau]\!]$ if and only if for some n and $U_1, \ldots, U_n \in \mathbf{U}^0 :\ M \xrightarrow{*} \sum_{i\leq n} U_i$, and for all $N \in [\![\sigma]\!]$ and $i \leq n :\ m(U_i)N \in [\![\tau]\!]$;

(iii) $M \in [\![\sigma \wedge \tau]\!]$ if and only if $M \in [\![\sigma]\!]$ and $M \in [\![\tau]\!]$;

(iv) $M \in [\![\sigma \oplus \tau]\!]$ if and only if for some $P, Q \in \Lambda^0_{+\|}$, $M \xrightarrow{*} P + Q$, $P \in [\![\sigma]\!]$ and $Q \in [\![\tau]\!]$.

Let s be a map from term variables to $\Lambda^0_{+\|}$. If M si any term such that $FV(M) = \{\chi_1, \ldots, \chi_n\}$ then $s(M) \equiv M[s(\chi_1)/\chi_1, \ldots, s(\chi_n)/\chi_n]$. Now we define:

(i) $s \models_r \Gamma$ if and only if for all $\chi : \sigma \in \Gamma :\ s(\chi) \in [\![\sigma]\!]$;

(ii) $\Gamma \models_r M : \sigma$ if and only if $s(M) \in [\![\sigma]\!]$ for all s such that $s \models_r \Gamma$.

The correctness of this definition is due to the fact that all types are inhabited by some closed term; in fact we have $\models_r \mathsf{R}_\sigma : \sigma$ for all types σ.

The following Lemma states some key properties of our realizability interpretation.

7.2. Lemma.

(i) $M \in [\![\sigma]\!]$ and $N \xrightarrow{*} M$ imply $N \in [\![\sigma]\!]$;

(ii) $M, N \in [\![\sigma]\!]$ implies $M + N \in [\![\sigma]\!]$;

(iii) $M + N \in [\![\sigma]\!]$ and σ irreducible imply $M, N \in [\![\sigma]\!]$;

(iv) $M \in [\![\sigma]\!]$ implies that for all closed $N :\ M\|N \in [\![\sigma]\!]$;

(v) if $U \in \mathbf{U}^0$ and $m(U)N \in [\![\sigma]\!]$ for some closed N, then $UN \in [\![\sigma]\!]$;

(vi) $\sigma \leq \tau$ implies $[\![\sigma]\!] \subseteq [\![\tau]\!]$;

(vii) $M \in [\![\sigma]\!]$ implies $\coprod M \in [\![\sigma]\!]$;

(viii) $M \in [\![\sigma]\!]$ and $\sigma \in \mathit{IType}$, imply that for some n, $U_1, \ldots, U_n \in \mathbf{U}^0$, $M \xrightarrow{*} \sum_{i\leq n} U_i$ and $m(U_i) \in [\![\sigma]\!]$ for all $i \leq n$.

Proof. All proofs but that of (vi) are by induction on σ.

(i) is straightforward from Definition 7.2.

(ii) The only interesting case is:

$\sigma \equiv \tau \oplus \rho$. If $M, N \in [\![\tau \oplus \rho]\!]$ then for some $M_0, N_0 \in [\![\tau]\!]$ and $M_1, N_1 \in [\![\rho]\!]$ it is the case that $M \xrightarrow{*} M_0 + M_1$ and $N \xrightarrow{*} N_0 + N_1$. By induction hypothesis $M_0 + N_0 \in [\![\tau]\!]$ and $M_1 + N_1 \in [\![\rho]\!]$; therefore

$$M + N \xrightarrow{*} (M_0 + N_0) + (M_1 + N_1) \in [\![\tau \oplus \rho]\!].$$

(iii) If $\sigma \equiv \tau \to \rho$ then $M + N \xrightarrow{*} \sum_{i \le m+k} U_i$, $M \xrightarrow{*} \sum_{i \le m} U_i$ and $N \xrightarrow{*} \sum_{m < i \le m+k} U_i$, say. Then $M, N \in [\![\sigma]\!]$ follows immediately by definition.

(iv) We consider the cases in which σ is an arrow or a sum.

$M \in [\![\tau \to \rho]\!]$ implies by definition that there are n and $U_1, \cdots, U_n \in \mathbf{U}^0$ such that $M \xrightarrow{*} \sum_{i \le n} U_i$ and $\forall L \in [\![\tau]\!]$ and $i \le n$, $m(U_i)L \in [\![\rho]\!]$. We have $M \| N \xrightarrow{*} \sum_{i \le n}(U_i \| N)$ by rule $(+\|)$ and $m(U_i \| N) \equiv m(U_i) \| m(N)$ or $m(U_i \| N) \equiv m(U_i)$ according to $N \in \mathbf{U}^0$ or not. In the first case $m(U_i)L \in [\![\rho]\!]$ implies $m(U_i)L \| m(N)L \in [\![\rho]\!]$ by induction, and therefore $m(U_i \| N)L \in [\![\rho]\!]$ by (i) since $m(U_i \| N)L \equiv (m(U_i) \| m(N))L \longrightarrow m(U_i)L \| m(N)L$. The second case is immediate.

$M \in [\![\tau \oplus \rho]\!]$ implies by definition that there are $P, Q \in \Lambda^0_{+\|}$ such that $M \xrightarrow{*} P + Q$, $P \in [\![\tau]\!]$ and $Q \in [\![\rho]\!]$. We have $M \| N \xrightarrow{*} P \| N + Q \| N$ by distributivity and $P \| N \in [\![\tau]\!]$, $Q \| N \in [\![\rho]\!]$ by induction.

(v) By Remark 4.1 $m(U)$ is either U itself, when $U \in \mathbf{W}^0$, or there exist $W \in \mathbf{W}^0$ and a closed $M \notin \mathbf{U}$ such that $U \approx W \| M$ and $m(U) \equiv W$. In the first case the thesis is immediate; in the second one, if $WN \in [\![\sigma]\!]$ then $WN \| MN \in [\![\sigma]\!]$ by (iv), so that $UN \in [\![\sigma]\!]$ by (i) because $UN \approx (W \| M)N \longrightarrow WN \| MN$.

(vi) By induction on the axiomatic presentation of $\le$. We use (ii) for the case $\sigma \oplus \sigma \le \sigma$. The most interesting case is $(\nu \oplus \rho) \wedge \mu \le (\nu \wedge \mu) \oplus (\rho \wedge \mu)$. $M \in [\![\sigma]\!]$ implies $M \in [\![\nu \oplus \rho]\!]$ and $M \in [\![\mu]\!]$. From $M \in [\![\nu \oplus \rho]\!]$ we have $M \xrightarrow{*} P + Q$ for some $P \in [\![\nu]\!]$ and $Q \in [\![\rho]\!]$. Therefore by (iv) we have $P \| M \in [\![\nu \wedge \mu]\!]$ and $Q \| M \in [\![\rho \wedge \mu]\!]$. So we can conclude $M \approx M \| M \xrightarrow{*} (P + Q) \| M \longrightarrow P \| M + Q \| M \in [\![(\nu \wedge \mu) \oplus (\rho \wedge \mu)]\!]$.

(vii) We consider the case in which σ is a sum. $M \in [\![\tau \oplus \rho]\!]$ implies by definition that there are $P, Q \in \Lambda^0_{+\|}$ such that $M \xrightarrow{*} P + Q$, $P \in [\![\tau]\!]$ and $Q \in [\![\rho]\!]$. Then $\coprod M \xrightarrow{*} \coprod P \| \coprod Q$ by rule $(\coprod_+)$ and by induction $\coprod P \in [\![\tau]\!]$ and $\coprod Q \in [\![\rho]\!]$. Therefore we have $\coprod M \in [\![\tau \wedge \rho]\!]$ by (iv), so we can conclude using (vi): $\coprod M \in [\![\tau \oplus \rho]\!]$.

(viii) If $\sigma \equiv \tau \wedge \rho$, where $\tau, \rho \in \mathit{IType}$ we have by induction $M \xrightarrow{*} \sum_{i \in I} U_i$ and $M \xrightarrow{*} \sum_{j \in J} U'_j$ for some some finite sets of indexes I, J and $m(U_i) \in [\![\tau]\!]$ and $m(U'_j) \in [\![\rho]\!]$. Now $M \approx M \| M \xrightarrow{*} (\sum_{i \in I} U_i) \| (\sum_{j \in J} U'_j) \xrightarrow{*} \sum_{i \in I} \sum_{j \in J} (U_i \| U'_j)$ and $m(U_i \| U'_j) \equiv m(U_i) \| m(U'_j) \in [\![\tau \wedge \rho]\!]$. The other cases are immediate.

□

As expected realizability coincides with derivability in $\vdash$ and this implies that we can show that all closed terms typeable by ω_1 must converge.

7.1. Theorem. The assignment system is sound and complete with respect to the realizability interpretation, namely

$$\Gamma \vdash M : \sigma \text{ if and only if } \Gamma \models_r M : \sigma.$$

Proof. Soundness (that is $\Rightarrow$) is by induction on derivations. We consider just the more complex cases (the other being standard or similar).

Suppose that the derivation ends by

$\Gamma \vdash \lambda v.M : \bigoplus_{i \le n} \sigma_i \to \bigoplus_{i \le n} \tau_i$ $\Gamma, v : \sigma_i \vdash M : \tau_i \quad i = 1, \ldots, n$

where each $\sigma_i \in IType$. Since $\lambda v.M \in \mathbf{W}$, we have to show that, if $s \models_r \Gamma$, then $s(\lambda v.M)N \in [\![\bigoplus_{i\leq n} \tau_i]\!]$ for all $N \in [\![\bigoplus_{i\leq n} \sigma_i]\!]$. If $N \in [\![\bigoplus_{i\leq n} \sigma_i]\!]$ then by 7.2 $N \xrightarrow{*} \sum_{i\leq n} N_i$ for some $N_1, \ldots, N_n$ such that $N_i \in [\![\sigma_i]\!]$ for each i. By Lemma 7.2 (viii) for each i there exist $n_i \in \mathbb{N}$ and $U_{i,1}, \ldots, U_{i,n_i} \in \mathbf{U}$ such that

$$N_i = \sum_{j\leq n_i} U_{i,j} \quad \text{and} \quad \forall j \leq n_i.\ m(U_{i,j}) \in [\![\sigma_i]\!]. \tag{74}$$

Now:

$$(\lambda v.M)N \xrightarrow{*} (\lambda v.M)(\sum_{i\leq n}\sum_{j\leq n_i} U_{i,j}) \xrightarrow{*} \sum_{i\leq n}\sum_{j\leq n_i}((\lambda v.M)U_{i,j}) + Q \xrightarrow{*} \sum_{i\leq n}\sum_{j\leq n_i} P_{i,j} + Q'$$

where $Q \equiv (\lambda v.M)(\|_{i\leq n, j\leq n_i} U_{i,j})$. The $P_{i,j}$ and Q' may have different forms, according to the following cases. The $P_{i,j}$ may be of the following shapes

(a) $P_{i,j} \equiv M[U_{i,j}/v]$ if $U_{i,j} \in \mathbf{W}^0$ (in this case $m(U_{i,j}) \equiv U_{i,j}$)
(b) $P_{i,j} \equiv M[m(U_{i,j})/v] \| (\lambda v.M)V_{i,j}$ if $U_{i,j} \longrightarrow V_{i,j}$, otherwise.

On the other hand Q' has one of the following shapes:

(c) $Q' \equiv M[\|_{i\leq n, j\leq n_i} U_{i,j}/v]$ when $\|_{i\leq n, j\leq n_i} U_{i,j} \in \mathbf{W}^0$;
(d) $Q' \equiv M[m(\|_{i\leq n, j\leq n_i} U_{i,j})/v] \| (\lambda v.M)V$ for some V such that $\|_{i\leq n, j\leq n_i} U_{i,j} \longrightarrow V$, otherwise.

Since the case (c) of Q' is similar to case (d) but easier, we consider cases (a) and (b) of $P_{i,j}$ and case (d) of Q'.

Let $s_{i,j} = s[m(U_{i,j})/v]$, then by (74) and the induction hypothesis $s_{i,j}(M) \in [\![\tau_i]\!]$. In case (a) $s(P_{i,j}) \equiv s_{i,j}(M)$, so $s(P_{i,j}) \in [\![\tau_i]\!]$. In case (b) $s(P_{i,j}) \equiv s_{i,j}(M) \| s((\lambda v.M)V_{i,j})$, and again $s(P_{i,j}) \in [\![\tau_i]\!]$ by 7.2 (iv). For Q', notice that $m(\|_{i\leq n, j\leq n_i} U_{i,j}) \equiv \|_{i\leq n, j\leq n_i} m(U_{i,j})$, and therefore by (74) and 7.2 (iv) we have $m(\|_{i\leq n, j\leq n_i} U_{i,j}) \in [\![\sigma_i]\!]$ for all $i \leq n$. Let $s' = s[m(\|_{i\leq n, j\leq n_i} U_{i,j})/v]$, then, again by induction hypothesis $s'(M) \in [\![\tau_i]\!]$, so that, by 7.2 (iv) $s(Q') \equiv s'(M) \| s((\lambda v.M)V) \in [\![\tau_i]\!]$ for all $i \leq n$. We finally conclude, by 7.2 (ii) and 7.2 (i) that $(\lambda v.M)N \in [\![\bigoplus_{i\leq n} \tau_i]\!]$ as desired.

If the derivation ends by

$$\frac{\Gamma \vdash M : \bigoplus_{i\leq n}(\sigma \to \tau_i) \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \bigoplus_{i\leq n} \tau_i}$$

choose any s such that $s \models_r \Gamma$. By induction $s(M) \in [\![\bigoplus_{i\leq n}(\sigma \to \tau_i)]\!]$, i.e. $s(M) \xrightarrow{*} \sum_{i\leq n} M_i$ for some $M_1, \ldots, M_n$ such that $M_i \in [\![\sigma \to \tau_i]\!]$. Again by induction $s(N) \in [\![\sigma]\!]$. Now, by definition, each $M_i \xrightarrow{*} \sum_{j_i \in J_i} U_{j_i}$ such that, for all $j_i \in J_i$, $m(U_{j_i})N \in [\![\tau_i]\!]$. By 7.2 (v) this implies that $U_{j_i}N \in [\![\tau_i]\!]$ for all i and $j_i \in J_i$, so that $M_i s(N) \in [\![\tau_i]\!]$, i.e. $s(MN) \in [\![\bigoplus_{i\leq n} \tau_i]\!]$ by 7.2 (i) since $s(MN) \xrightarrow{*} \sum_{i\leq n} M_i s(N)$.

To prove completeness (namely $\Leftarrow$) first we observe that, given Γ, Γ', such that $FV(\Gamma) \cap FV(\Gamma') = \emptyset$, the map $s_{\Gamma,\Gamma'}$ defined by

$$s_{\Gamma,\Gamma'}(\chi) = \begin{cases} \chi & \text{if } \chi \in FV(\Gamma') \\ \mathsf{R}_\tau & \text{if } \chi : \tau \in \Gamma \\ \mathsf{R}_{\omega\to\omega} & \text{if } \chi \notin FV(\Gamma \cup \Gamma') \text{ and } \chi \text{ is call-by-value} \\ \mathsf{R}_\omega & \text{otherwise} \end{cases}$$

is such that $s_{\Gamma,\emptyset} \models_r \Gamma$, hence by hypothesis $s_{\Gamma,\emptyset}(M) \in [\![\sigma]\!]$. It is easy to prove by induction on M that for all Γ, Γ', such that $FV(\Gamma) \cap FV(\Gamma') = \emptyset$:

$$\forall M, \sigma.\ \Gamma' \vdash s_{\Gamma,\Gamma'}(M) : \sigma \Rightarrow \Gamma \cup \Gamma' \vdash M : \sigma.$$

Now we prove:

$$\Gamma \models_r M : \sigma \Rightarrow \Gamma \vdash M : \sigma$$

by induction on σ. All cases are immediate but that of the arrow types. Let $\sigma \equiv \tau \to \rho$.

By the soundness part of this theorem we know that $\mathsf{R}_\tau \in [\![\tau]\!]$. By definition $s_{\Gamma,\emptyset}(M) \equiv M' \xrightarrow{*} \sum_{i\leq n} U_i$ for some n and $U_1, \ldots, U_n \in \mathbf{U}^0$ such that, for all $i \leq n$, $m(U_i)\mathsf{R}_\tau \in [\![\rho]\!]$. By induction hypothesis we have $\Gamma \vdash m(U_i)\mathsf{R}_\tau : \rho$. Since $m(U) \in \mathbf{W}^0$, by 5.2 (iii) there exists τ' such that $\vdash m(U_i) : \tau' \to \rho$ and $\vdash \mathsf{R}_\tau : \tau'$. Now 7.1 (ii) implies that $\tau \leq \tau'$, so that $\tau' \to \rho \leq \tau \to \rho$ and therefore, by rule $(\leq)$, we know that $\vdash m(U_i) : \tau \to \rho$ for all $i \leq n$. Using 5.2 (vi) we also know that $\vdash U_i : \tau \to \rho$ for all i, so that, by rule (+I), $\vdash \sum_{i\leq n} U_i : \tau \to \rho$. Therefore $\vdash M' : \tau \to \rho$ by 5.3, which immediately implies by the above claim that $\Gamma \vdash M : \tau \to \rho$.

□

The main result we obtain is the converse of 5.1, namely:

7.1. Corollary.
(i) $\vdash M : \omega_1$ implies $M \Downarrow^{must}$;
(ii) $\vdash M : \omega_1 \oplus \omega$ implies $M \Downarrow^{may}$.

As a byproduct of these results we get a proof of Proposition 4.1, namely that, $M \Downarrow^{must}$ and $M \longrightarrow N$ imply $N \Downarrow^{must}$ for closed M, N. Indeed, if $M \Downarrow^{must}$ then $\vdash M : \omega_1$ by 5.1 (i). Since types are preserved by reduction (actually by conversion) by Theorem 5.3, we know that $\vdash N : \omega_1$. Therefore, by Corollary 7.1 (i), we conclude that $N \Downarrow^{must}$ as desired. A similar argument, using 5.1 (iii) and 7.1 (ii), shows that if $M \Downarrow^{may}$ and $M \longrightarrow N$ then $N \Downarrow^{may}$.

7.3. Full Abstraction Theorem

To establish the main result of this paper we use the discriminability power of test terms.

7.2. Theorem. Let M be a closed term.

$$\mathsf{T}_\sigma M \Downarrow^{must} \quad \text{if and only if} \quad \vdash M : \sigma.$$

Proof.
$(\Rightarrow)$.

$$\begin{array}{lll} \mathsf{T}_\sigma M \Downarrow^{must} & \Rightarrow \vdash \mathsf{T}_\sigma M : \omega_1 & \text{by 5.1 (i)} \\ & \Rightarrow \exists \rho.\ \vdash \mathsf{T}_\sigma : \rho \to \omega_1 \ \ \& \ \ \vdash M : \rho & \text{by 5.2 (iii) since } \mathsf{T}_\sigma \in \mathbf{W}^0 \\ & \Rightarrow \exists \rho.\ \rho \leq \sigma \ \ \& \ \ \vdash M : \rho & \text{by 7.1 (iv)} \\ & \Rightarrow \vdash M : \sigma & \text{by rule } (\leq). \end{array}$$

$(\Leftarrow)$.

$$\begin{array}{lll} \vdash M : \sigma & \Rightarrow \vdash \mathsf{T}_\sigma M : \omega_1 & \text{by 7.1 (iii)} \\ & \Rightarrow \mathsf{T}_\sigma M \Downarrow^{must} & \text{by 7.1 (i).} \end{array}$$

□

Adequacy is a consequence of 7.1 and of the fact that $\simeq^{\mathcal{F}}$ is a congruence. For the converse it suffices to observe that test terms discriminate internally, that is with respect to the convergence predicate, terms having different interpretations in the filter model.

7.3. Theorem (Full Abstraction Theorem).

$$M \sqsubseteq^{\mathcal{F}} N \ \text{ if and only if } \ M \sqsubseteq^{\mathcal{O}} N.$$

Proof. ($\Rightarrow$) (Adequacy) Since $\sqsubseteq^{\mathcal{F}}$ is a precongruence, we immediately have that

$$M \sqsubseteq^{\mathcal{F}} N \Rightarrow \forall C[\ ].\ \vdash C[M] : \omega_1 \Rightarrow \Gamma \vdash C[N] : \omega_1.$$

It follows that

$$C[M]\Downarrow^{must} \Rightarrow \vdash C[M] : \omega_1 \text{ by 5.1 (i) } \Rightarrow \vdash C[N] : \omega_1 \Rightarrow C[N]\Downarrow^{must} \text{ by 7.1 (i).}$$

($\Leftarrow$) (Completeness) By definition

$$M \not\sqsubseteq^{\mathcal{F}} N \ \Leftrightarrow\ \exists \Gamma, \sigma.\ \Gamma \vdash M : \sigma \ \&\ \Gamma \not\vdash N : \sigma.$$

Let $FV(MN) = \{\chi_i \mid 1 \leq i \leq n\}$, $\Gamma = \{\chi_i : \tau_i \mid 1 \leq i \leq n\}$ and $\tau = \tau_1 \to \cdots \to \tau_n \to \sigma$, then $\vdash \lambda\chi_1 \ldots \chi_n.M : \tau$ and $\not\vdash \lambda\chi_1 \ldots \chi_n.N : \tau$ by 5.1 (iii) and (iv). Therefore choosing $C[\,] = \mathsf{T}_\tau(\lambda\chi_1 \ldots \chi_n.[\,])$ we have that $C[M]\Downarrow^{must}$ and $C[N]\Uparrow^{may}$ which imply $M \not\sqsubseteq^{\mathcal{O}} N$. □

The full abstraction theorem shows that the calculus we have defined is a complete description of the powerdomain structure devised in Section 2, being the match between the two provided by the logical system.

8. Related Work and Conclusions

Most of the related works have been quoted in the Introduction. Here we just recall some topics, summarize our results and give hints for further research.

The subject of the interaction between functionality and various forms of non-determinism and parallelism has been studied both in the case of term rewriting systems [10, 22, 23, 1] and of λ-calculus, typed [6, 7, 38] and type-free [11, 27, 31, 32, 14, 15]. The present paper is meant as an improvement and/or a development of [31, 32] and [15]: with respect to Ong's papers we give a sound denotational semantics even in presence of call-by-value, parallelism and non-determinism. With respect to our own work we do not limit ourself to upper (also called Smyth) powerdomain, while retaining (at least conceptually) the simplicity of the domain description and of the type assignment system.

The theory of powerdomains has been pioneered by Plotkin [35] and is now an established topic in domain theory. In particular the free construction at the basis of our definition of the functor $\mathcal{P}$ is presented e.g. in [20] and, in its most general form, in [24]. It would be interesting to know more about powerdomains which are endofunctors over the category of algebraic lattices, and it is unclear whether there exists a functor yielding the best approximation of the convex powerdomain functor within **ALG**.

The use of strong monads to account for "computational types" has been advocated by Moggi [29]. The functor $T = \mathcal{P} \circ (\_)_\perp$ is the composition of two strong monads and it is a strong monad itself: it is however false, in general, even that the composition of monads is a monad. To know when this is the case seems relevant to Moggi's approach when several features (like non-determinism and partiality) are included in the same language, something which deserves further investigation.

Models are usually build to give a mathematical treatment of the meaning of programs. We use the implicit definition of a model, via a recursive domain equation, to guide the definition of the language (even if we do not start from scratch, building over previous versions of the concurrent λ-calculus). Since our construction follows an implicit pattern, it makes sense to ask whether this pattern can be made explicit and to which extent denotational semantics can be used as a design tool.

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