Spaces of closure operations on rings and numerical semigroups

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Abstract

A *closure operation* is a map c from a partially ordered set $\mathcal{P}$ to itself that verifies three properties: it is *extensive* (that is, $x \leq c(x)$ for every $x \in \mathcal{P}$), *order-preserving* (if $x \leq y$, then $c(x) \leq c(y)$) and *idempotent* ($c(c(x)) = c(x)$ for every $x \in \mathcal{P}$). In this thesis, closure operations are studied from a global point of view, that is, the focus is on whole sets on closure operations defined on sets of ideals or submodules, studying the cardinality of certain sets of closures and the natural order-theoretic and topological structures with which they can be endowed.

The first chapter deals with *star operations* on numerical semigroups: in particular, it is studied the problem of finding, given a positive integer n, the numerical semigroups with exactly n star operations. It is proved that, for $n > 1$, there are only a finite number of such semigroups, and an estimate for such number is given; these results are proved by estimating the number of star operations through the other invariants of the semigroup (like the multiplicity, the degree of singularity or the Frobenius number). It is then determined explicitly the number of star operation on numerical semigroups of multiplicity 3, and these results are partially extended to the set of residually rational Noetherian domains whose integral closure is a (fixed, but arbitrary) discrete valuation domain.

The second chapter is focused on the study of the topological structure of the set of *semistar operation* on an integral domain: in particular, it is proved that some natural subspaces of this set (for example, the space of finite-type semistar operations) are spectral spaces, that is, they are homeomorphic to the prime spectrum of a commutative unitary ring (endowed with the Zariski topology). Further objects of study are the sets of stable, spectral and valutative semistar operations; results about these spaces are then used to study subspaces of the space of overrings of an integral domain, especially regarding the property of being a compact space, a spectral space or a proconstructible space (that is, a closed set of the constructible topology). Particular spaces studied in this way are the space of localizations of a ring and some of its generalizations (the space of prime semigroups, the space of flat overrings and the space of sublocalizations).

The third chapter deals with the study of star operations on integral domains; in particular, the focus is on the possibility of extending such closures on flat overrings of the starting domain. It is proved that, given an integral domain D and a family Θ of overrings of D with some special properties (what is called a *Jaffard familiy* of D), the space of star operations on D can be represented as the product (both in the topological and in the order-theoretic sense) of the spaces of the star operations on T, as T varies in Θ. These results are then applied to the case of Prüfer domains, obtaining (among the other results) a representation on the class group of D relative to a star operation through the class group of a subset (explicitly determined) of valuation overrings of D.