

RADICAL SEMISTAR OPERATIONS

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ABSTRACT. We introduce and study the set of radical stable operations of an integral domain D . We show that their set is a complete lattice that is the join-completion of the set of spectral semistar operations, and we characterize when every radical operation is spectral (under the hypothesis that D is rad-colon coherent). When D is a Prüfer domain such that every set of minimal prime ideals is scattered, we completely classify stable semistar operations.

1. INTRODUCTION

Let D be an integral domain. Semistar operations on D are a class of closure operations on the set of D -submodules of the quotient field K of D , defined by Okabe and Matsuda [13] as a generalization of the concept of star operation, originally introduced by Krull [11] and Gilmer [10, Chapter 32]. Semistar operations enjoy greater flexibility than star operations, making them a good tool to use in order to study several topics relative to the properties of ideals of D , as well as the properties of overrings of D . There are several subclasses of semistar operations that are particularly of interest, among which we cite *finite type* operations, *gab* operations (that are related to the valuation overrings of D , cf. [9] and [5, Section 4]) and *spectral* operations (related to the spectrum of D , cf. [3, 2, 8]). See Section 2 for a precise definition.

A semistar operation \star is *stable* if it distributes over finite intersections, i.e., if $(I \cap J)^\star = I^\star \cap J^\star$ for all D -submodules I, J ; in particular, every spectral semistar operation is stable. Stable operations are naturally connected to *localizing systems* [8] and *singular length functions* [17, Theorem 6.5 and subsequent discussion], meaning that there are canonical order-preserving bijections between the sets of stable operations, localizing systems and singular length functions on the same domain D (see Section 7); in particular, any classification of stable semistar operations classifies, as well, localizing systems and singular length functions. However, while spectral semistar operations can be easily classified through subsets of the spectrum of D (see [8, Remark 4.5] and [7, Corollary 4.4]), the same does not hold for stable operations; indeed, a standard representations and a classification of stable

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operations have only be obtained in very specific situations, like for one-dimensional domains with scattered maximal space (see [18]) and for Prüfer domains such that every ideal has only finitely many minimal primes (see [14] and [17]).

In this paper, we study *radical* semistar operations, i.e., stable semistar operations such that, for every ideal I , $1 \in I^*$ if and only if $1 \in \text{rad}(I)^*$. This notion arose in the study of almost Dedekind domains that generalize the notion of SP-domains: indeed, it can be proved that, for the class of *SP-scattered domains*, every stable operation is radical [19]. We systematize the study of this class of stable operations, showing that their set $\text{SStar}_{\text{rad}}(D)$ is a complete lattice (Theorem 4.6) that is, furthermore, the join-completion of the set $\text{SStar}_{\text{sp}}(D)$ of spectral operations inside the set $\text{SStar}(D)$ of all semistar operations. For *rad-colon coherent domains* (a large class of domain that includes domains with Noetherian spectrum and Prüfer and coherent domains), it follows that the set $\text{SStar}_{\text{rad}}(D)$ depends uniquely on the spectrum of D (in the sense that any two such domains with homeomorphic spectra have isomorphic set of radical operations; Theorem 5.3).

In Section 6, we connect the study of radical operations with the use of the derived set and of scattered topological spaces (following [18, 19, 20]) to show that (under the hypothesis that D is rad-colon coherent) the two sets $\text{SStar}_{\text{rad}}(D)$ and $\text{SStar}_{\text{sp}}(D)$ coincide if and only if the space $\text{Min}(I)$ of minimal ideals of D is scattered for every ideal I . Specializing further to the case of Prüfer domain, we show that this property is enough to obtain a full classification of all stable operations of D by means of a standard representation (Theorem 6.7), generalizing the results obtained in [14] and [17] for the case where each $\text{Min}(I)$ is finite; in particular, we show that for these Prüfer domains the set $\text{SStar}_{\text{st}}(D)$ depends only on the spectrum of D (as a topological space) and on which prime ideals are locally principal (Theorem 6.9). In particular, these results hold when the spectrum of D is countable.

In Section 7, we define the concepts analogue to radical semistar operations in the context of localizing systems and length functions.

2. PRELIMINARIES

Throughout the paper, D will denote an integral domain with quotient field K , and $\mathbf{F}(D)$ will denote the set of D -submodules of K . An *overring* of D is a ring between D and K .

2.1. Semistar operations. A *semistar operation* on D is a map $\star : \mathbf{F}(D) \rightarrow \mathbf{F}(D)$, $I \mapsto I^*$, such that, for every $I, J \in \mathbf{F}(D)$, $x \in K$:

- $I \subseteq I^*$;
- if $I \subseteq J$, then $I^* \subseteq J^*$;
- $(I^*)^* = I^*$;
- $(xI)^* = x \cdot I^*$.

A submodule I is said to be \star -closed if $I = I^\star$. The set of \star -closed ideals uniquely determines \star .

The set $\text{SStar}(D)$ of the semistar operations on D has a natural partial order, where $\star_1 \leq \star_2$ if and only if $I^{\star_1} \subseteq I^{\star_2}$ for every $I \in \mathbf{F}(D)$, or equivalently if every \star_2 -closed ideal is \star_1 -closed. Under this order, $\text{SStar}(D)$ is a complete lattice: the infimum of a family $\{\star_\alpha\}_{\alpha \in A}$ is the map

$$I \mapsto \bigcap_{\alpha \in A} I^{\star_\alpha},$$

while its supremum is the semistar operation \sharp such that a submodule I is \sharp -closed if and only if it is \star_α closed for every $\alpha \in A$.

An ideal I of D is said to be a *quasi- \star -ideal* if $I = I^\star \cap D$; if I is a prime quasi- \star -ideal, we say that I is a *quasi- \star -prime*. The set of quasi- \star -primes is called the *quasi-spectrum* of \star , and is denoted by $\text{QSpec}^\star(D)$.

A semistar operation \star is said to be *of finite type* if $I^\star = \bigcup\{J^\star \mid J \subseteq I \text{ is finitely generated}\}$, for every $I \in \mathbf{F}(D)$. It is *semi-finite* (or *quasi-spectral*) if every quasi- \star -ideal is contained in a quasi- \star -prime; every semistar operation of finite type is semi-finite.

A very general way to define semistar operations is through overrings: any family Λ of overrings induces the semistar operation

$$\star_\Lambda : I \mapsto \bigcap_{T \in \Lambda} IT.$$

When Λ is a family of localizations of D , we say that \star is a *spectral* semistar operation. A spectral semistar operation can also be defined through a subset of the spectrum $\text{Spec}(D)$ of D : given a family $\Delta \subseteq \text{Spec}(D)$, we denote by s_Δ the semistar operation

$$s_\Delta : I \mapsto \bigcap_{P \in \Delta} ID_P.$$

Setting $\Delta^\downarrow := \{Q \in \text{Spec}(D) \mid Q \subseteq P \text{ for some } P \in \Delta\}$, we have that $\text{QSpec}^{s_\Delta}(D) = \Delta^\downarrow$, and that $s_\Delta = s_{\Delta^\downarrow}$; moreover, $s_\Delta = s_\Lambda$ if and only if $\Delta^\downarrow = \Lambda^\downarrow$ [8, Remark 4.5]. A spectral operation s_Δ is of finite type if and only if Δ is compact, with respect to the Zariski topology [7, Corollary 4.4].

A semistar operation is *stable* if $(I \cap J)^\star = I^\star \cap J^\star$ for every $I, J \in \mathbf{F}(D)$. Every spectral semistar operation is stable, while every semi-finite stable operation is spectral [1, Theorem 4]. There exist stable operations that are not spectral: an example is the *v -operation* $I \mapsto (D : (D : I))$ when D is a valuation domain with non-principal maximal ideal. The *localizing system associated to \star* is [8, Section 2]

$$\mathcal{F}^\star := \{I \text{ ideal of } D \mid I^\star \cap D = D\} = \{I \text{ ideal of } D \mid 1 \in I^\star\};$$

this set uniquely determines \star , in the sense that if \star_1, \star_2 are stable, then $\star_1 = \star_2$ if and only if $\mathcal{F}^{\star_1} = \mathcal{F}^{\star_2}$. More precisely, $\star_1 \leq \star_2$ if and only if $\mathcal{F}^{\star_1} \subseteq \mathcal{F}^{\star_2}$.

The set $\text{SStar}_{\text{sp}}(D)$ of spectral semistar operations of D is closed by infimum, but not by supremum (see [6, Example 4.5] and Example 4.1 below); note, however, that $\text{SStar}_{\text{sp}}(D)$ is a complete lattice (see below the discussion after Corollary 4.3). On the other hand, the set $\text{SStar}_{\text{st}}(D)$ of stable operations is closed by both infima and suprema [15, Proposition 5.3].

2.2. Topologies on the spectrum. Let $\text{Spec}(D)$ denote the spectrum of D , i.e., the set of all prime ideals of D . We denote by $\mathcal{V}(I)$ and $\mathcal{D}(I)$, respectively, the closed and the open sets of the Zariski topology associated to an ideal I ; i.e., $\mathcal{V}(I) := \{P \in \text{Spec}(D) \mid I \subseteq P\}$, while $\mathcal{D}(I) := \text{Spec}(D) \setminus \mathcal{V}(I)$.

The spectrum of a ring can also be endowed with two other topologies. The *inverse* topology is the topology whose subbasic open sets are those in the form $\mathcal{V}(I)$, as I ranges among the finitely generated ideals of D ; the *constructible* topology is the topology whose subbasic open sets are the $\mathcal{V}(I)$ and the $\mathcal{D}(I)$, for I ranging among the finitely generated ideals of D . In particular, the constructible topology is finer than both the Zariski and the inverse topology, and, furthermore, it is Hausdorff.

If I is an ideal of D and $\text{Min}(I)$ denotes the set of minimal primes of D , the Zariski and the constructible topology agree on $\text{Min}(I)$ (by [4, Corollary 4.4.6(i)], applied to the spectral space $\mathcal{V}(I)$).

2.3. Derived sets and scattered spaces. Let X be a topological space. A point $x \in X$ is *isolated* if $\{x\}$ is an open set; the set of non-isolated points of X is called the *derived set* of X , and is denoted by $\mathcal{D}(X)$. Given an ordinal α , we define the α -th *derived set* as

$$\mathcal{D}^\alpha(X) := \begin{cases} \mathcal{D}(\mathcal{D}^\gamma(X)) & \text{if } \alpha = \gamma + 1; \\ \bigcap_{\beta < \alpha} \mathcal{D}^\beta(X) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

If $\mathcal{D}^\alpha(X) = \emptyset$ for some α , the space X is said to be *scattered*. On the other hand, if $\mathcal{D}(X) = X$, then X is said to be *perfect*.

3. RADICAL SEMISTAR OPERATIONS

Definition 3.1. *We say that a semistar operation \star on D is quasi-radical if, whenever $1 \notin I^\star$ for some ideal I of D , then $1 \notin \text{rad}(I)^\star$.*

We collect in the next few propositions the main properties of quasi-radical semistar operations.

Proposition 3.2. *Let D be an integral domain and \star be a semistar operation on D . If $\star|_{\mathbf{F}(D^\star)}$ is quasi-radical as a semistar operation on D^\star , then \star is quasi-radical.*

Proof. Let I be an ideal of D such that $1 \notin I^*$. Then, $1 \notin (ID)^* = (ID^*)^*$, and thus, by hypothesis, $1 \notin \text{rad}(ID^*)^*$. However, $\text{rad}(I) \subseteq \text{rad}(ID^*)$; hence, $1 \notin \text{rad}(I)^*$. It follows that \star is quasi-radical. \square

Proposition 3.3. *Let D be an integral domain and \star be a semistar operation on D .*

- (a) *If \star is semi-finite, then it is quasi-radical.*
- (b) *If \star is of finite type, then it is quasi-radical.*
- (c) *If \star is induced by overrings, then it is quasi-radical.*
- (d) *If \star is spectral, then it is quasi-radical.*

Proof. Suppose \star is semi-finite, and let I be an ideal of D such that $1 \notin I^*$. Then, $J := I^* \cap D$ is a quasi- \star -ideal such that $1 \notin J^*$. Since \star is semi-finite, there is a quasi- \star -prime ideal P containing J ; thus, $1 \notin P^* \supseteq \text{rad}(J)^* \supseteq \text{rad}(I)^*$. Therefore, \star is quasi-radical.

The next three points follows from the fact that every semistar operation of finite type is semi-finite, as well as any semistar operation induced by overrings, and that any spectral semistar operation is induced by overrings. \square

Proposition 3.4. *Let $\{\star_\alpha\}_{\alpha \in A}$ be a set of quasi-radical semistar operations on D . Then, $\inf_{\alpha \in A} \star_\alpha$ is quasi-radical.*

Proof. Let $\star := \inf_{\alpha \in A} \star_\alpha$, and let I be an ideal of D such that $1 \notin I^*$. Since $I^* = \bigcap_{\alpha \in A} I^{\star_\alpha}$, it follows that there is a $\beta \in A$ such that $1 \notin I^{\star_\beta}$. Since \star_β is quasi-radical, then $1 \notin \text{rad}(I)^{\star_\beta}$, and thus also $1 \notin \text{rad}(I)^*$. Hence \star is quasi-radical. \square

The previous proposition does not extend to the supremum of a family of quasi-radical operations, as the next example shows.

Example 3.5. Let D be a Prüfer domain of dimension 1 such that $\text{Max}(D) = \{P, Q_0, Q_1, \dots, Q_n, \dots\}$ is countable and with a single non-isolated point, P ; suppose also that D_P is not discrete. For every $n \in \mathbb{N}$, let $T_n := \bigcap_{i \geq n} D_{Q_i}$; then, T_n is a Prüfer domain whose maximal ideals are the extensions of Q_i (for $i \geq n$) and of P ; in particular, $\bigcup_n T_n = D_P$.

Recall that a *fractional ideal* of a domain T is an $I \in \mathbf{F}(T)$ such that $dI \subseteq T$ for some $d \in K$, $d \neq 0$. For every n , let \sharp_n and \star_n be the semistar operations defined by

$$I^{\sharp_n} := \begin{cases} IT_n & \text{if } IT_n \text{ is a fractional ideal over } T_n \\ K & \text{otherwise,} \end{cases}$$

and

$$I^{\star_n} := I^{\sharp_n} \cap (ID_P)^{v_P},$$

where v_P is the v -operation on D_P . Since $T_n \subseteq D_P$, for every ideal I of D we have $I^{\star_n} = I^{\sharp_n}$: hence, if $1 \notin I^{\star_n}$ then $IT_n \neq T_n$ and so $\text{rad}(I)T_n \neq T_n$. Thus, every \star_n is quasi-radical.

Let now \star be the supremum of all \star_n . Then, $D_P^\star = D_P$ since $D_P^{\star_n} \subseteq (D_P)^{v_P} = D_P$. Moreover, if $t \in D_P$, then $t \in T_n$ for some n , and thus $t \in D^{\star_n} \subseteq D^\star$. Hence, $D^\star = D_P$. For every n , PD_P is not a fractional ideal over T_n , and thus

$$(PD_P)^{\star_n} = K \cap (PD_P)^{v_P} = K \cap D_P = D_P.$$

Hence,

$$P^\star = (PD)^\star = (PD^\star)^\star = (PD_P)^\star = D_P.$$

On the other hand, if $L \neq P$ is a P -primary ideal, then $(LD_P)^{v_P} = LD_P$; hence, LD_P is \star -closed and thus $L^\star \subseteq LD_P \cap D$, so that $1 \notin L^\star$ while $1 \in P^\star = \text{rad}(L)^\star$. Therefore, \star is not quasi-radical.

The main problem of the previous example is that the restriction of a quasi-radical operation on D to an overring of D is not quasi-radical (as it happens for $\star_i|_{\mathbf{F}(D_P)}$); this in turn is due to the fact that the property of being quasi-radical depends only on the ideals of D , rather than on all D -submodules of K . For this reason, we are only interested in the following subclass of semistar operations.

Definition 3.6. *We say that a semistar operation \star on D is radical if it is quasi-radical and stable.*

Lemma 3.7. *Let \star be a radical stable operation, and suppose that T is an overring of D . Then $\star|_{\mathbf{F}(T)}$ is radical.*

Proof. Let I be a T -ideal such that $1 \notin I^\star$. Then, $1 \notin (I \cap D)^\star$, and since \star is radical we have $1 \notin (\text{rad}(I \cap D))^\star$. However, $\text{rad}(I \cap D) = \text{rad}(I) \cap D$; hence

$$1 \notin \text{rad}(I \cap D)^\star = (\text{rad}(I) \cap D)^\star = \text{rad}(I)^\star \cap D^\star.$$

Thus $1 \notin \text{rad}(I)^\star$ and so $\star|_{\mathbf{F}(T)}$ is radical, as claimed. \square

Proposition 3.8. *Let D be an integral domain and let \star be a radical semistar operation on D such that $D = D^\star$. Let J be an ideal of D such that $J = J^\star$. Then, $\text{rad}(J)^\star = \text{rad}(J)$.*

Proof. Let $s \in \text{rad}(J)^\star$, and let $t \in s^{-1} \text{rad}(J) \cap D$. Then, $st \in \text{rad}(J)$, and thus there is an n such that $s^n t^n \in J$, i.e., $t^n \in s^{-n} J \cap D$. Hence $t \in \text{rad}(s^{-n} J \cap D)$ and so $s^{-1} \text{rad}(J) \cap D \subseteq \text{rad}(s^{-n} J \cap D)$.

Since $s \in \text{rad}(J)^\star$, we have $1 \in s^{-1} \text{rad}(J)^\star$; hence also $1 \in \text{rad}(s^{-n} J \cap D)^\star$. Since \star is radical, it follows that $1 \in (s^{-n} J \cap D)^\star$; thus $1 \in s^{-n} J^\star$ and $s^n \in J^\star = J$. Therefore, $s \in \text{rad}(J)$, and $\text{rad}(J)^\star = \text{rad}(J)$. \square

Theorem 3.9. *Let D be an integral domain. Then, the set $\text{SStar}_{\text{rad}}(D)$ of radical stable semistar operations is a complete sublattice of $\text{SStar}(D)$.*

Proof. Let $\{\star_\alpha\}_{\alpha \in A}$ be a family of radical semistar operations. Then, its infimum is quasi-radical by Proposition 3.4 and stable since every \star_α is stable, and thus $\text{SStar}_{\text{rad}}(D)$ is closed by infima. Let \star be the supremum of $\{\star_\alpha\}_{\alpha \in A}$.

Let $T := D^*$: then, T is \star_α -closed for every α . By Proposition 3.2, it suffices to show that $\star|_{\mathbf{F}(T)}$ is radical; furthermore, by Lemma 3.7, each $\star_\alpha|_{\mathbf{F}(T)}$ is radical. Therefore, without loss of generality, we can actually suppose that $T = D$, i.e., that D is \star_α -closed for every α .

Let J be an ideal of D such that $1 \notin J^*$. Let $L := J^*$; then, L is an ideal of D that is \star_α -closed for every α , and thus by Proposition 3.8 also $\text{rad}(L)$ is \star_α -closed for every α ; thus, $\text{rad}(L) = \text{rad}(L)^*$. In particular, $1 \notin \text{rad}(L)^*$; the claim now follows from the fact that $\text{rad}(J) \subseteq \text{rad}(L)$. \square

4. RADICAL OPERATIONS AS A COMPLETION

By Proposition 3.3, each spectral semistar operation s_Δ is radical; in this section, we explore the link between these two classes of semistar operations. Following [6, Example 4.5], we first give an example of a radical operation that is not spectral.

Example 4.1. Let \mathbb{A} be the ring of all algebraic integer, i.e., the integral closure of \mathbb{Z} in $\overline{\mathbb{Q}}$. Then, \mathbb{A} is a Bézout domain (every finitely generated ideal is principal) and, for every maximal ideal P , we have that $\mathbb{A} = \bigcap \{\mathbb{A}_Q \mid Q \in \text{Max}(\mathbb{A}) \setminus \{P\}\}$. Hence, for each P the spectral operation $\sharp(P) := s_{\text{Max}(\mathbb{A}) \setminus \{P\}}$ closes \mathbb{A} , and thus the supremum \star of all the $\sharp(P)$ closes \mathbb{A} too, and thus it closes every principal ideal (since $(x\mathbb{A})^\star = x \cdot \mathbb{A}^\star = x \cdot \mathbb{A}$).

As the supremum of a family of radical operations, \star is itself radical. However, for every P -primary ideal Q , we have $Q^{\sharp(P)} = \mathbb{A}$; therefore, $\text{QSpec}^\star(D)$ contains only the zero ideal. In particular, were \star spectral, it should be equal to $s_{(0)}$, and in particular 1 would belong to I^\star for every nonzero ideal I , contradicting the fact that principal ideals are closed. Hence \star is radical, but not spectral.

The following proposition characterizes which radical operations are spectral.

Proposition 4.2. *Let \star be a radical stable operation on D . Then, \star is spectral if and only if, for every radical ideal I ,*

$$I^\star \cap D = \bigcap \{P \mid P \in \mathcal{V}(I) \cap \text{QSpec}^\star(D)\}.$$

Proof. Suppose first that \star is spectral, say $\star = s_\Delta$ with $\Delta = \Delta^\downarrow$. For every $P \in \Delta$, the ideal ID_P is radical, and its minimal primes are the minimal primes of I contained in P ; all of them belong to Δ , and thus they are all in $\mathcal{V}(I) \cap \text{QSpec}^\star(D)$. Hence,

$$I^\star \cap D = \bigcap_{P \in \Delta} \{QD_P \cap D \mid Q \in \text{Min}(I), Q \subseteq P\} = \bigcap_{Q \in \text{Min}(I) \cap \Delta} Q.$$

The claim follows.

Conversely, suppose that the equality holds, and let $\Delta := \text{QSpec}^*(D)$. For every $P \in \Delta$, PD_P is \star -closed, and thus \star is the identity on $\mathbf{F}(D_P)$; it follows that $I^* \subseteq ID_P$ for every $P \in \Delta$, and thus $\star \leq s_\Delta$.

Suppose that $\star < s_\Delta$: then, there is an ideal I of D such that $I^* \subsetneq I^{s_\Delta}$. Let $x \in I^{s_\Delta} \setminus I^*$ and let $J := (I :_D x)$. Since \star is stable, we have $1 \in J^{s_\Delta}$ while $1 \notin J^*$; since both s_Δ and \star are radical, it follows that $1 \in \text{rad}(J)^{s_\Delta}$ while $1 \notin \text{rad}(J)^*$. However, by the hypothesis and the first part of the proof, $\text{rad}(J)^{s_\Delta} \cap D = \text{rad}(J)^* \cap D$; this is a contradiction, and thus \star must be equal to s_Δ . In particular, \star is spectral, as claimed. \square

Corollary 4.3. *Let D be an integral domain such that every ideal has only finitely many minimal primes. Then, every radical stable operation is spectral.*

Proof. Let I be a radical ideal and P_1, \dots, P_n be its minimal primes. Then, $I = P_1 \cap \dots \cap P_n$, and thus $I^* = P_1^* \cap \dots \cap P_n^*$. Since \star is stable, for each i the ideal $P_i^* \cap D$ is either equal to P_i or to D [14, Lemma 3.1] hence, $I^* \cap D$ is equal to the intersection of the minimal primes that are quasi- \star -ideals. By Proposition 4.2, \star is spectral. \square

The following result is a variant of [14, Lemma 3.1].

Proposition 4.4. *Let \star be a stable semistar operation, and let J be a radical ideal of D . Then, $J^* \cap D$ is either D or a radical ideal.*

Proof. Suppose $J^* \cap D \neq D$. Let $s \in D$ be such that $s^n \in J^*$ for some integer n . Let $L := s^{-n}J \cap D$: since \star is stable, $1 \in L^*$. We claim that $s^{-1}J \cap D = \text{rad}(L)$. Indeed, if $x \in s^{-1}J \cap D$ then $sx \in J$ and thus also $s^n x \in J$, i.e., $x \in s^{-n}J \cap D = L \subseteq \text{rad}(L)$. On the other hand, if $x \in \text{rad}(L)$, then $x^k \in s^{-n}J$ for some k , and thus $x^k s^n \in J$. Since $x, s \in D$, we have $x^N s^N \in J$, where $N := \max\{n, k\}$; since J is radical, it follows that $xs \in J$, that is, $x \in s^{-1}J \cap D$. Thus $s^{-1}J \cap D = L = \text{rad}(L)$.

Since $1 \in L^*$, it follows that $1 \in (s^{-1}J)^* = s^{-1}J^*$, that is, $s \in J^*$. Hence J^* is radical, as claimed. \square

Proposition 4.5. *Let D be a domain, let I be a radical ideal of D and $\Delta = \Delta^\downarrow \subseteq \text{Spec}(D)$. Then, the following are equivalent:*

- (i) $I = I^{s_\Delta} \cap D$;
- (ii) $\mathcal{V}(I) \cap \Delta$ is dense in $\mathcal{V}(I)$;
- (iii) $\text{Min}(I) \cap \Delta$ is dense in $\text{Min}(I)$.

Proof. By Proposition 4.4, the ideal $J := I^{s_\Delta} \cap D$ is a radical ideal of D containing I ; therefore, $\mathcal{V}(J) \subseteq \mathcal{V}(I)$ is a closed set, and $\mathcal{V}(J)$ contains $\mathcal{V}(I) \cap \Delta$ since, if $P \in \mathcal{V}(I) \cap \Delta$, then $J = I^* \cap D \subseteq P^* \cap D = P$. In particular, if $\mathcal{V}(I) \cap \Delta$ is dense then it must be $I = J$. On the other hand, if $\mathcal{V}(I) \cap \Delta$ is not dense, then there is an ideal L such that

$\Delta \cap \mathcal{V}(L) \subsetneq \mathcal{V}(I)$; thus, $ID_P = LD_P$ for every $P \in \Delta$, and $J \supseteq L$, so that $I \neq J$. Thus, the first two conditions are equivalent.

If $\text{Min}(I) \cap \Delta$ is dense in $\text{Min}(I)$, then $\text{Min}(I)$ is contained in the closure of $\mathcal{V}(I) \cap \Delta$; then, $\mathcal{V}(I) \cap \Delta$ is dense since $\text{Min}(I)$ is dense in $\mathcal{V}(I)$. Conversely, suppose $\mathcal{V}(I) \cap \Delta$ is dense in $\mathcal{V}(I)$ and take $P \in \text{Min}(I)$. For every open set Ω meeting $\mathcal{V}(I)$, $\Omega \cap \Delta \cap \mathcal{V}(I)$ is nonempty; if Q belongs to the intersection, then $\Omega \cap \Delta \cap \text{Min}(I)$ contains the minimal primes of I contained in Q . Hence, $\Delta \cap \text{Min}(I)$ is dense in $\text{Min}(I)$. Thus also the last two conditions are equivalent. \square

Let D be an integral domain. The space $\text{SStar}_{\text{sp}}(D)$ of spectral semistar operation on D is a complete lattice: indeed, let $X := \{s_{\Delta_\alpha} \mid \alpha \in A\}$ be a subset of $\text{SStar}_{\text{sp}}(D)$ with $\Delta_\alpha = \Delta_\alpha^\downarrow$. Then, setting $\Delta^\cup := \cup_\alpha \Delta_\alpha$ and $\Delta^\cap := \cap_\alpha \Delta_\alpha$, it is easy to see that the infimum of X in $\text{SStar}_{\text{sp}}(D)$ is s_{Δ^\cup} and that its supremum is s_{Δ^\cap} .

However, while s_{Δ^\cup} is also the infimum of X as a subset of $\text{SStar}(D)$, the same does not hold for s_{Δ^\cap} (see Example 4.1). We now want to prove that the set $\text{SStar}_{\text{rad}}(D)$ of radical semistar operations is the join-completion of $\text{SStar}_{\text{sp}}(D)$ in $\text{SStar}(D)$. In particular, the construction of Example 4.1 is the only way to obtain non-spectral radical semistar operations.

Theorem 4.6. *Let D be an integral domain. Then:*

- (a) $\text{SStar}_{\text{sp}}(D)$ is join-dense in $\text{SStar}_{\text{rad}}(D)$;
- (b) $\text{SStar}_{\text{rad}}(D)$ is the completion of $\text{SStar}_{\text{sp}}(D)$ in $\text{SStar}(D)$.

Proof. Since $\text{SStar}_{\text{rad}}(D)$ is a complete sublattice of $\text{SStar}(D)$ (Theorem 3.9), we only need to prove that every radical stable operation is the supremum of a family of spectral operations.

Fix thus $\star \in \text{SStar}_{\text{rad}}(D)$. Let $\Delta \subseteq \text{Spec}(D)$ be such that $\Delta \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ for every radical ideal I such that $I = I^\star \cap D$. Then, $s_\Delta \leq \star$: indeed, if J is an ideal such that $1 \in J^{s_\Delta}$ and $1 \notin J^\star$, then $\Delta \cap \mathcal{V}(J^\star \cap D)$ would be dense in $\mathcal{V}(J^\star \cap D)$, and thus by Proposition 4.5 $J^\star \cap D$ would be quasi- s_Δ -closed, against the fact that $1 \in J^{s_\Delta}$. Hence, $s_\Delta \leq \star$. Let \sharp be the supremum of all such s_Δ : by construction, $\sharp \leq \star$.

We claim that $\star = \sharp$. Let J be a proper radical ideal: if $1 \in J^\sharp$, then $1 \in J^\star$ since $\sharp \leq \star$. Suppose that $1 \in J^\star$. We claim that $\mathcal{D}(J) \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ for every radical ideal I such that $I = I^\star \cap D$. If not, there is a $P \in \mathcal{V}(J)$ that is not in the closure of $\mathcal{D}(I) \cap \mathcal{V}(J)$; hence, there is a radical ideal L such that $P \in \mathcal{D}(L)$ and $\mathcal{D}(L) \cap \mathcal{D}(J) \cap \mathcal{V}(I) = \emptyset$. Since $\mathcal{D}(L) \cap \mathcal{D}(J) = \mathcal{D}(L \cap J)$, it follows that $\mathcal{D}(L \cap J) \cap \mathcal{V}(I) = \emptyset$, and thus $\mathcal{V}(I) \subseteq \mathcal{V}(L \cap J)$. Thus, $L \cap J \subseteq I$, and $L^\star \cap J^\star = (L \cap J)^\star \subseteq I^\star$. Hence

$$(J^\star \cap D) \cap (L^\star \cap D) \subseteq I^\star \cap D = I.$$

By hypothesis, J^\star contains 1; hence, $J^\star \cap D = D$ and $L^\star \cap D \subseteq I$, so that $L \subseteq I$. In particular, $\mathcal{D}(L) \subseteq \mathcal{D}(I)$; it follows that $\mathcal{D}(L) \cap \mathcal{V}(I) = \emptyset$,

against the hypothesis that $P \in \mathcal{D}(L) \cap \mathcal{V}(I)$. Therefore, $\mathcal{D}(J) \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ for all radical ideal I such that $I = I^* \cap D$; thus, $s_{\mathcal{D}(J)}$ is one of the spectral operations used to define \sharp ; hence, $s_{\mathcal{D}(J)} \leq \sharp$. It follows that $1 \in J^{s_{\mathcal{D}(J)}} \subseteq J^\sharp$. Therefore, $1 \in J^*$ if and only if $1 \in J^\sharp$; since \star and \sharp are stable, it follows that $\star = \sharp$, as claimed, and \star is in the completion of $\text{SStar}_{\text{sp}}(D)$. \square

5. ISOMORPHIC SETS OF RADICAL OPERATIONS

Let D_1, D_2 be two integral domains. If $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$ is an order isomorphism, then ϕ induces an order isomorphism $\Phi : \text{SStar}_{\text{sp}}(D_1) \rightarrow \text{SStar}_{\text{sp}}(D_2)$ by setting $\Phi(s_\Delta) = s_{\phi(\Delta)}$ for every $\Delta \subseteq \text{Spec}(D_1)$. However, Φ does not, in general, extend to a similar isomorphism between the set of radical semistar operations, for example because it may be $\text{SStar}_{\text{sp}}(D_1) = \text{SStar}_{\text{rad}}(D_1)$ while $\text{SStar}_{\text{sp}}(D_2) \neq \text{SStar}_{\text{rad}}(D_2)$ (take for example $D_1 := K[X]$ and $D_2 := \mathbb{A}$, where K is a field of the same cardinality of $\text{Max}(\mathbb{A})$).

In this section, we extend this result to radical operations by using the Zariski topology. We work in a particular class of domains: we say that a domain is *rad-colon coherent* if, for every $x \in K$, the radical of the ideal $(D :_D x)$ is the radical of a finitely generated ideal. This property is linked with the relationship between the Zariski, inverse and constructible topology of $\text{Spec}(D)$ and the Zariski, inverse and constructible topology of $\text{Over}(D)$. Every Noetherian domain (or, more generally, every domain with Noetherian spectrum) is rad-colon coherent; likewise, every Prüfer domain and every coherent domain are rad-colon coherent, as well as every polynomial in finitely many variables over a Prüfer domain. See [16] for applications of this property and for an example of a domain that is not rad-colon coherent.

In our context, the reason why we use this notion is essentially the following lemma.

Lemma 5.1. *Let D be a rad-colon coherent domain and let I be a radical ideal. Define $T := \bigcap \{D_P \mid P \in \text{Min}(I)\}$. If \star is a radical semistar operation such that $I = I^* \cap D$, then $T^\star = T$ and $(IT)^\star = IT$.*

Proof. Suppose first that $\star = s_\Delta$ is spectral, with $\Delta = \Delta^\downarrow$. Then, by Proposition 4.5, $\Delta \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$ and $\Delta \cap \text{Min}(I)$ is dense in $\text{Min}(I)$, with respect to the Zariski topology. By [4, Corollary 4.4.6(i)], the Zariski and the constructible topology agree on $\text{Min}(I)$; hence, $\Delta \cap \text{Min}(I)$ is dense in $\text{Min}(I)$ also with respect to the constructible topology.

Let $x \in T^\star$, and let $J := (D :_D x) = x^{-1}D \cap D$. We claim that $\mathcal{V}(J) \cap \text{Min}(I) \cap \Delta = \emptyset$. Indeed, let $P \in \text{Min}(I) \cap \Delta$. Since $x \in T^\star \subseteq D_P^\star$, we have $1 \in (x^{-1}D_P)^\star$, and thus $1 \in (JD_P)^\star$; however, if $P \in \mathcal{V}(J)$ then $(JD_P)^\star \subseteq (PD_P)^\star = PD_P$ since $P \in \Delta$. Therefore, $\mathcal{V}(J) \cap \text{Min}(I) \cap \Delta = \emptyset$, and thus $\text{Min}(I) \cap \Delta \subseteq \mathcal{D}(J)$. Since D is rad-colon

coherent, $\text{rad}(J)$ is the radical of a finitely generated ideal, and thus $\mathcal{D}(J)$ is a closed subset, with respect to the constructible topology; thus $\mathcal{D}(J) \cap \text{Min}(I)$ is closed in $\text{Min}(I)$. Since $\text{Min}(I) \cap \Delta$ is dense in $\text{Min}(I)$, it follows that $\mathcal{D}(J) \cap \text{Min}(I)$ must be equal to the whole of $\text{Min}(I)$, that is, $\mathcal{V}(J) \cap \text{Min}(I) = \emptyset$. Thus, $JD_P = D_P$ for every $P \in \text{Min}(I)$, and $x \in T$. Hence, $T^\star = T$.

This also implies that $(IT)^\star$ is a radical ideal of T contained in PT for every $P \in \text{Min}(I)$. Hence $(IT)^\star = IT$, as claimed.

Suppose now that \star is any radical operation. By Theorem 4.6, \star is the supremum of a family Y of spectral semistar operation. For each $\sharp \in Y$, we have $\sharp \leq \star$, and thus $I = I^\sharp \cap D$; by the previous part of the proof, $T^\sharp = T$ and $(IT)^\sharp = IT$. Hence, also $T^\star = T$ and $(IT)^\star = IT$, as claimed. \square

Proposition 5.2. *Let D be a rad-colon coherent domain and let I be a radical ideal. Let Y be a family of radical semistar operations and let $\sharp := \sup Y$. If $I = I^\star \cap D$ for every $\star \in Y$, then $I = I^\sharp \cap D$.*

Proof. Let $T := \bigcap \{D_P \mid P \in \text{Min}(I)\}$. By Lemma 5.1, $(IT)^\star$ is closed by every $\star \in Y$, and thus also $(IT)^\sharp$ is closed. Then, $I^\sharp \cap D \subseteq (IT)^\sharp \cap D = I$, and thus $I = I^\sharp \cap D$. \square

We are ready to prove the main result of this section.

Theorem 5.3. *Let D_1, D_2 be rad-colon coherent integral domains, and suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \longrightarrow \text{Spec}(D_2)$. Then, there is an order isomorphism*

$$\Phi : \text{SStar}_{\text{rad}}(D_1) \longrightarrow \text{SStar}_{\text{rad}}(D_2)$$

such that $\Phi(s_\Delta) = s_{\phi(\Delta)}$ for every $\Delta \subseteq \text{Spec}(D_1)$.

Proof. Let $X_i := \text{SStar}_{\text{sp}}(D_i)$ and $Y_i := \text{SStar}_{\text{rad}}(D_i)$ for $i = 1, 2$.

By Theorem 3.9, Y_1 is a join-completion of X_1 ; hence, we can consider Y_1 as a sublattice of the set $\mathcal{L}(X_1)$ of lower sets of X_1 by the map ϵ_1 , defined by $\epsilon_1(y) = \{x \in X_1 \mid x \leq y\}$ for every $y \in Y_1$. In particular, $\epsilon_1(x) = \{x\}^\downarrow$ for every $x \in X_1$. Likewise, we can consider Y_2 as a sublattice of $\mathcal{L}(X_2)$ through a map ϵ_2 defined analogously.

The map

$$\begin{aligned} \Phi : X_1 &\longrightarrow X_2, \\ s_\Delta &\longmapsto s_{\phi(\Delta)} \end{aligned}$$

is an order isomorphism; thus, it can be extended to a map $\tilde{\Phi}$ between $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$, that remains an order isomorphism. We claim that $\tilde{\Phi}(\epsilon_1(Y_1)) = \epsilon_2(Y_2)$, and to do so it is enough to prove that, if $A \subseteq X_1$, then the supremum $\sup_{Y_1} A$ in Y_1 (that is, the supremum of A as a semistar operation) is spectral if and only if $\sup_{Y_2} \tilde{\Phi}(A)$ is spectral.

Suppose first that $\star := \sup_{Y_1} A$ is not spectral, and let $\sharp = s_\Delta$ be the supremum of A in X_1 . Let \star' and \sharp' be, respectively, the supremum

of $\Phi(A)$ in Y_2 and X_2 . By construction, $\star < \sharp$, and thus there is a radical ideal I such that $I = I^\star \cap D_1$ while $I^\sharp = D_1^\sharp$. Let now J be the radical ideal such that $\mathcal{V}(J) = \phi(\mathcal{V}(I))$; we claim that $J = J^{\star'} \cap D$ while $J^{\sharp'} = D_2^{\sharp'}$.

Indeed, if $s_\Delta \in \Phi(A)$, then $\phi^{-1}(\Delta) \cap \mathcal{V}(I)$ is dense in $\mathcal{V}(I)$, and thus $\Delta \cap \mathcal{V}(J)$ is dense in $\mathcal{V}(J)$; since D_2 is rad-colon coherent, by Proposition 5.2 $J = J^{\sup \Phi(A)} \cap D_2$, i.e., $J = J^{\star'} \cap D_2$. On the other hand, $\sharp = s_\Delta$ for some Δ such that $\Delta \cap \mathcal{V}(I) = \emptyset$; hence, $\sharp' = \Phi(\sharp) = \Phi(s_\Delta) = s_{\phi(\Delta)}$, where $\phi(\Delta) \cap \mathcal{V}(J)$ is empty. Hence, $J^{\sharp'} = D_2^{\sharp'}$. Thus, $\star' \neq \sharp'$, and $\sup_{Y_2} \Phi(A)$ is not spectral.

The opposite implication follows by applying the same reasoning to the homeomorphism ϕ^{-1} (which induces the map Φ^{-1} on the sets of spectral semistar operations).

Therefore, $\tilde{\Phi}$ restricts to an isomorphism between $\epsilon_1(Y_1)$ and $\epsilon_2(Y_2)$; since $Y_i \simeq \epsilon_i(Y_i)$ for $i = 1, 2$, it follows that $Y_1 = \text{SSStar}_{\text{rad}}(D_1)$ and $Y_2 = \text{SSStar}_{\text{rad}}(D_2)$ are isomorphic, as claimed. \square

6. WHEN EVERY SPECTRAL OPERATION IS RADICAL

We have seen that, in general, not every radical semistar operation is spectral, although the two sets are equal when every ideal has only finitely many minimal primes (Corollary 4.3). In this section, we characterize when the two sets are equal for rad-colon coherent domains; specializing to Prüfer domain, we also show that under this hypothesis we can obtain a standard representation of all stable operations.

We start with two topological lemmas.

Lemma 6.1. *Let D be an integral domain and I a radical ideal that is not prime. Then, $\text{Min}(I)$ is not perfect if and only if there are a prime ideal Q and a radical ideal $J \neq I$ such that $I = Q \cap J$.*

Proof. If I is not perfect, there is an isolated point Q of $\text{Min}(I)$, and $\text{Min}(I) \setminus \{Q\} = \text{Min}(I) \cap \mathcal{V}(J)$ for some radical ideal J . By construction, $J \supsetneq I$ and $\mathcal{V}(J) \cup \mathcal{V}(Q) = \mathcal{V}(I)$, so that $I = Q \cap J$. Conversely, if $I = Q \cap J$, then $\mathcal{V}(I) = \mathcal{V}(Q) \cup \mathcal{V}(J)$. Since $I \neq J$, $\mathcal{V}(J)$ cannot contain all minimal primes of I ; therefore, Q must be contained in $\text{Min}(I)$. Hence, $\{Q\} = \text{Min}(I) \setminus \mathcal{V}(J)$ is open in $\text{Min}(I)$ and Q is isolated; thus $\text{Min}(I)$ is not perfect. \square

Lemma 6.2. *Let D be an integral domain. Then, the following are equivalent:*

- (i) $\text{Min}(I)$ is scattered for every ideal I ;
- (ii) $\text{Min}(I)$ is not perfect for every ideal I .

Proof. (i) \implies (ii) is obvious. To show (ii) \implies (i), let I be a radical ideal and let $X := \bigcap_\alpha \mathcal{D}^\alpha(\text{Min}(I))$: then, X is perfect. Let $J := \bigcap \{Q \mid Q \in X\}$; then, $I \subseteq J \subseteq P$ for all $P \in X$, and thus $X \subseteq \text{Min}(J)$. We claim

that $\text{Min}(J)$ is perfect. Indeed, suppose not: then, it has an isolated point P , and P cannot belong to X , since X is perfect. Since P is isolated, there is a finitely generated ideal L such that $\mathcal{D}(L) \cap \text{Min}(J) = \{P\}$; therefore, $L \not\subseteq P$ while

$$L \subseteq \bigcap_{Q \in \text{Min}(J) \setminus \{P\}} Q \subseteq \bigcap_{Q \in X} Q = J,$$

a contradiction. Thus $\text{Min}(J)$ is perfect, as claimed. \square

Definition 6.3. *We say that D is min-scattered if $\text{Min}(I)$ is a scattered space for every ideal I .*

Proposition 6.4. *Let D be a domain such that $\text{Spec}(D)$ is countable. Then, D is min-scattered.*

Proof. The space $\text{Spec}(D)$, endowed with the constructible topology, is Hausdorff, compact and countable, and thus scattered [12]. Therefore, for every ideal I , also $\text{Min}(I)$ is scattered, with respect to the constructible topology; however, on each $\text{Min}(I)$ the constructible and the Zariski topology coincide. Hence D is min-scattered. \square

Theorem 6.5. *Let D be a rad-colon coherent domain. Then, the following are equivalent:*

- (i) D is min-scattered;
- (ii) every radical semistar operation is spectral.

Proof. (ii) \implies (i) Suppose that there is a radical ideal I such that $\text{Min}(I)$ is perfect. For every $P \in \text{Min}(I)$, let $\sharp(P) := s_{\text{Min}(I) \setminus \{P\}}$, and let \star be the supremum of all these $\sharp(P)$. Then, \star is a radical semistar operation; we claim that \star is not spectral.

Indeed, since $\text{Min}(I)$ is perfect each $\text{Min}(I) \setminus \{P\}$ is dense, and thus by Proposition 4.2 $I = I^{\sharp(P)} \cap D$ for every P ; since D is rad-colon coherent, by Proposition 5.2 we have $I = I^\sharp \cap D$. However, $P^{\sharp(P)} \ni 1$ for every $P \in \text{Min}(I)$; hence, $1 \in P^\sharp$ for every $P \in \mathcal{V}(I)$. By Proposition 4.2, \sharp cannot be spectral.

(i) \implies (ii) Suppose that there is a radical operation \star that is not spectral. By Proposition 4.2, there is an ideal I such that $I^\star \cap D \subsetneq \bigcap \{P \mid P \in \mathcal{V}(I) \cap \text{QSpec}^\star(D)\}$; without loss of generality we can suppose that $I = I^\star$. Let J be equal to the intersection, and let $\Gamma := \text{Min}(I) \setminus \mathcal{V}(J)$. By construction, Γ is nonempty.

The set Γ does not contain isolated points of $\text{Min}(I)$: if $Q \in \Gamma$ is isolated, then $I = Q \cap I_0$ for some $I_0 \supsetneq I$, and thus $I^\star \cap D = (Q \cap I_0)^\star \cap D = Q^\star \cap I_0^\star \cap D$ can only be equal to I if $Q = Q^\star \cap D$, i.e., $Q \in \text{QSpec}^\star(D)$ and $J \subseteq Q$.

For every $P \in \Gamma$, let $\gamma(P)$ be the minimal ordinal number such that $P \notin \mathcal{D}^\gamma(\Gamma)$. Note that $\gamma(P)$ exists since $\text{Min}(I)$ is scattered and no element of Γ is isolated; furthermore, $\gamma(P)$ is a successor ordinal. Let

γ be the minimal element of the set of all $\gamma(P)$, and let $Q \in \text{Min}(I)$ be such that $\gamma(Q) = \gamma$. Let also β be such that $\gamma = \beta + 1$.

By construction, Q is a limit point of $\text{Min}(I)$, while Q is isolated in $\mathcal{D}^\beta(\text{Min}(I))$. Hence, Q is a limit point of $\text{Min}(I) \setminus \mathcal{D}^\beta(\text{Min}(I))$. The latter set is contained in $\mathcal{V}(J)$, by definition of Q ; since $\mathcal{V}(J)$ is closed, it follows that also $Q \in \mathcal{V}(J)$. This is a contradiction, and thus Γ must be empty, i.e., there cannot be a radical non-spectral semistar operation. The claim is proved. \square

We now restrict to the case of Prüfer domains, extending results proved in [14] and mostly following the general method of that paper. Given a semistar operation \star on the Prüfer domain D , we define the *pseudo-spectrum* $\text{PsSpec}^\star(D)$ as the set of those prime ideals Q such that $1 \in Q^\star$, but there is a Q -primary ideal L such that $L = L^\star \cap D$. Using the quasi-spectrum and the pseudo-spectrum, we can define from \star a new semistar operation $\widehat{\star}$, called the *normalized stable version* of D , as

$$\widehat{\star} : I \mapsto \bigcap_{P \in \text{QSpec}^\star(D)} ID_P \cap \bigcap_{Q \in \text{PsSpec}^\star(D)} (ID_Q)^{v_Q},$$

where v_Q is the v -operation on the valuation domain D_Q . Note that v_P is different from the identity on D_P if and only if P is idempotent.

Lemma 6.6. *Let \star be a radical semistar operation. Then, $\star = \widehat{\star}$ if and only if \star is spectral.*

Proof. If \star is spectral, then $\star = s_{\text{QSpec}^\star(D)} = \widehat{\star}$. Conversely, if $\star = \widehat{\star}$ but \star is not spectral, there is a $Q \in \text{PsSpec}^\star(D)$. By definition, $1 \in Q^\star$, while $1 \notin L^\star$ for some Q -primary ideal L ; since $\text{rad}(L) = Q$, this contradicts the fact that \star is radical. Hence \star must be spectral. \square

Theorem 6.7. *Let D be a Prüfer domain. Then, the following are equivalent:*

- (i) D is min-scattered;
- (ii) every radical semistar operation is spectral;
- (iii) $\star = \widehat{\star}$ for every stable semistar operation \star .

Proof. (i) \iff (ii) follows from Theorem 6.5, since a Prüfer domain is rad-colon coherent, while (iii) \implies (ii) follows from Lemma 6.6.

To prove (i) \implies (iii), fix a stable semistar operation \star . By [14, Theorem 3.9], $\star \leq \widehat{\star}$, and thus if $1 \in I^\star$ then also $1 \in \widehat{I}^\star$. Suppose that $1 \in \widehat{I}^\star$ while $1 \notin I^\star$. Then, $J := I^\star \cap D$ is a proper ideal of D that is quasi- \star -closed. Changing notation from I to J , we can suppose without loss of generality that $I = I^\star \cap D$.

Since $\text{Min}(I)$ is not perfect, there is an isolated point Q . Since $\text{Min}(I) \setminus \{Q\}$ is closed, it is equal to $\mathcal{V}(I_1) \cap \text{Min}(I_1)$ for some radical ideal I_1 . Let $T := \bigcap \{D_P \mid P \in \mathcal{V}(Q)\}$ and $S := \bigcap \{D_P \mid P \in \mathcal{V}(I_1)\}$:

Then, $I = IS \cap IT$, and in particular

$$I^* \cap D = (IS)^* \cap D \cap (IT)^* \cap D.$$

The radical of $IT \cap D$ is Q , which is a prime ideal. By the proof of [14, Theorem 4.5], since $1 \in (IT \cap D)^{\widehat{\star}}$, we also have $1 \in (IT \cap D)^*$, and thus $(IT)^* \cap D = D$. On the other hand, $IS \cap D$ is not contained in Q ; hence, neither does $(IS)^* \cap D$. By construction, $I = I^* \cap D$; this is a contradiction, and thus \star and $\widehat{\star}$ must be equal, as claimed. \square

Corollary 6.8. *Let D be a domain such that $\text{Spec}(D)$ is countable. Then, every radical semistar operation is spectral. If D is Prüfer, moreover, $\star = \widehat{\star}$ for every stable semistar operation \star .*

Proof. If $\text{Spec}(D)$ is countable, then D is min-scattered by Proposition 6.4. The claims now follow from Theorems 6.5 and 6.7. \square

The following is a version of Theorem 5.3 for stable operations on a Prüfer domain; it can also be seen as a variant of [17, Theorem 5.12] (in view of [17, Section 6]).

Theorem 6.9. *Let D_1, D_2 be Prüfer domains. Suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$ such that a prime ideal P is idempotent if and only if $\phi(P)$ is idempotent. If D_1 is min-scattered, then there is an isomorphism $\Phi : \text{SStar}_{\text{st}}(D_1) \rightarrow \text{SStar}_{\text{st}}(D_2)$.*

Proof. We first note that if J is an ideal of D_2 , then $\text{Min}(J)$ is the set of minimal elements of the closed set $\mathcal{V}(J)$; then, $\phi^{-1}(\mathcal{V}(J))$ is a closed set of $\text{Spec}(D_1)$, and thus it is equal to $\mathcal{V}(J')$ for some ideal J' of D_1 . By hypothesis, $\text{Min}(J')$ is scattered, and thus also $\phi(\text{Min}(J')) = \text{Min}(J)$ is scattered. Hence also D_2 is min-scattered.

Given a stable semistar operation \star on D_1 , we define $\Phi(\star)$ as the map

$$\Phi(\star) : I \mapsto \bigcap_{P \in \phi(\text{QSpec}^{\star}(D_1))} I(D_2)_P \cap \bigcap_{Q \in \phi(\text{PsSpec}^{\star}(D_1))} (I(D_2)_Q)^{v_Q}.$$

We claim that $\text{QSpec}^{\Phi(\star)}(D_2) = \phi(\text{QSpec}^{\star}(D_1))$ and $\text{PsSpec}^{\Phi(\star)}(D_2) = \phi(\text{PsSpec}^{\star}(D_1))$.

Indeed, let $P \in \text{Spec}(D_1)$ and let $Q := \phi(P)$. If $P \in \text{QSpec}^{\star}(D_1)$, then

$$Q^{\Phi(\star)} \cap D \subseteq Q(D_2)_Q \cap D_2 = Q,$$

and thus $Q \in \text{QSpec}^{\Phi(\star)}(D_2)$; conversely, if $Q \in \text{QSpec}^{\Phi(\star)}(D_2)$, then either $Q(D_2)_A \neq (D_2)_A$ for some $A \in \phi(\text{QSpec}^{\star}(D_1))$ or $(Q(D_2)_B)^{v_B} \neq Q(D_2)_B$ for some $B \in \phi(\text{PsSpec}^{\star}(D_1))$. In the former case, $P \subseteq A$; since the quasi-spectrum is closed by generizations [14, Proposition 3.4(a)], $P \in \text{QSpec}^{\star}(D_1)$ and thus $Q \in \phi(\text{QSpec}^{\star}(D_1))$. In the latter case it must be $Q \subsetneq B$, and thus $Q \in \phi(\text{QSpec}^{\star}(D_1))$ since every $B' \subsetneq B$ is in the quasi-spectrum [14, Proposition 3.4(b)]. Therefore, $\text{QSpec}^{\Phi(\star)}(D_2) = \phi(\text{QSpec}^{\star}(D_1))$.

Suppose now that $P \in \text{PsSpec}^*(D_1)$. By the previous paragraph, $Q \notin \text{QSpec}^{\Phi(\star)}(D_1)$. There is a P -primary ideal $L \subsetneq P$ such that $L = L^* \cap D$; in particular, P is not branched in the valuation domain $(D_1)_P$. The map ϕ induced a homeomorphism between $\text{Spec}((D_1)_P)$ and $\text{Spec}((D_2)_Q)$; therefore, neither Q is branched, and thus there exist a Q -primary ideal $L' \subsetneq Q$. By definition,

$$(L')^{\Phi(\star)} \cap D_2 \subseteq L'(D_2)_Q \cap D_2 = L',$$

and thus $Q \in \text{PsSpec}^{\Phi(\star)}(D_2)$. Conversely, if $Q \in \text{PsSpec}^{\Phi(\star)}(D_2)$, then there is a Q -primary ideal $L \subsetneq Q$ such that $L^{\Phi(\star)} \cap D_2 = L$. By the previous part of the proof, $P \notin \text{QSpec}^*(D_1)$; if P is not even in $\text{PsSpec}^*(D_1)$, then $L^{\Phi(\star)}$ would just be equal to $D^{\Phi(\star)}$, a contradiction. Hence, $Q = \phi(P) \in \phi(\text{PsSpec}^*(D_1))$. Therefore, $\text{PsSpec}^{\Phi(\star)}(D_2) = \phi(\text{PsSpec}^*(D_1))$.

Consider now the map $\Psi : \text{SStar}_{\text{st}}(D_2) \longrightarrow \text{SStar}_{\text{st}}(D_1)$ defined by

$$\Psi(\sharp) : I \mapsto \bigcap_{P \in \phi^{-1}(\text{QSpec}^\sharp(D_2))} I(D_1)_P \cap \bigcap_{Q \in \phi^{-1}(\text{PsSpec}^\sharp(D_2))} (I(D_1)_Q)^{v_Q}$$

for every ideal I of D_1 and every $\sharp \in \text{SStar}_{\text{st}}(D_2)$. Then, Ψ is the map associated to the homeomorphism ϕ^{-1} by the previous construction; hence,

$$\begin{aligned} \Psi \circ \Phi(\star) : I &\mapsto \bigcap_{P \in \phi^{-1}(\text{QSpec}^{\Phi(\star)}(D_2))} I(D_1)_P \cap \bigcap_{Q \in \phi^{-1}(\text{PsSpec}^{\Phi(\star)}(D_2))} (I(D_1)_Q)^{v_Q} = \\ &= \bigcap_{P \in \text{QSpec}^*(D_1)} I(D_1)_P \cap \bigcap_{Q \in \text{PsSpec}^*(D_1)} (I(D_1)_Q)^{v_Q} = \widehat{I} \end{aligned}$$

since ϕ is a homeomorphism. By Theorem 6.7, $\widehat{\star} = \star$, and thus $\Psi \circ \Phi(\star) = \star$, i.e., $\Psi \circ \Phi$ is the identity on $\text{SStar}_{\text{st}}(D_1)$. By symmetry, also $\Phi \circ \Psi$ is the identity on $\text{SStar}_{\text{st}}(D_2)$; hence, Φ and Ψ are inverses one of each other. It is straightforward to see that they are also order-preserving; thus, they establish an isomorphism between $\text{SStar}_{\text{st}}(D_1)$ and $\text{SStar}_{\text{st}}(D_2)$, as claimed. \square

Corollary 6.10. *Let D_1, D_2 be Prüfer domains with countable spectrum. Suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \longrightarrow \text{Spec}(D_2)$ such that a prime ideal P is idempotent if and only if $\phi(P)$ is idempotent. Then, there is an isomorphism $\Phi : \text{SStar}_{\text{st}}(D_1) \longrightarrow \text{SStar}_{\text{st}}(D_2)$.*

Proof. If $\text{Spec}(D_1), \text{Spec}(D_2)$ are countable, then D_1, D_2 are min-scattered by Proposition 6.4. The claim now follows from Theorem 6.9. \square

7. OTHER VERSIONS

Stable semistar operations are linked to two other structures on a ring: localizing systems and length functions.

A *localizing system* on a domain D is a set of ideals \mathcal{F} such that:

- if $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$;

- if $I \in \mathcal{F}$ and $(J :_D iD) \in \mathcal{F}$ for all $i \in I$, then $J \in \mathcal{F}$.

The map $\star \mapsto \mathcal{F}^\star := \{I \mid 1 \in I^\star\}$ establishes a bijective correspondence between the set of stable semistar operations and the set of all localizing systems, whose inverse is given by the map associating to \mathcal{F} the semistar operation [8, Section 2]

$$\star_{\mathcal{F}} : I \mapsto \bigcup_{J \in \mathcal{F}} (I :_D J).$$

We say that a localizing system is *radical* if, for every ideal I such that $\text{rad}(I) \in \mathcal{F}$, we have $I \in \mathcal{F}$. This notion corresponds exactly to radical semistar operations.

Proposition 7.1. *Let \star be a stable semistar operation. Then, \star is radical if and only if \mathcal{F}^\star is a radical localizing system.*

Proof. If \star is radical and $\text{rad}(I) \in \mathcal{F}^\star$, then $1 \in \text{rad}(I)^\star$ and thus $1 \in I^\star$ by definition, so that $I \in \mathcal{F}$ and \mathcal{F}^\star is radical. Conversely, if \star is not radical there is an ideal I such that $1 \notin I^\star$ while $1 \in \text{rad}(I)^\star$: then, $\text{rad}(I) \in \mathcal{F}^\star$ while $I \notin \mathcal{F}^\star$, so that \mathcal{F}^\star is not radical. \square

Therefore, the bijection between stable operations and localizing systems restricts to a bijection between $\text{SStar}_{\text{rad}}(D)$ and the set $\text{LS}_{\text{rad}}(D)$ of radical localizing systems; it follows that Theorems 5.3 and 6.5 can be expressed also in the terminology of localizing systems.

A *singular length function* on D is a map ℓ from the set of all D -modules to $\{0, \infty\}$ such that:

- ℓ is additive on exact sequences;
- for each module M , $\ell(M)$ is the supremum of $\ell(N)$, as N ranges among the finitely generated submodules of M .

A singular length function is uniquely determined by its *ideal colength* τ , where τ is the map associated to each ideal I the length $\ell(D/I)$ [21, Proposition 3.3]. We denote by $\mathcal{L}_{\text{sing}}(D)$ the set of singular length functions on D .

There is a bijective correspondence between localizing systems and singular length functions, where we associate to a localizing system \mathcal{F} the colength [17, Section 6]

$$\tau_{\mathcal{F}}(I) = \begin{cases} 0 & \text{if } I \in \mathcal{F}, \\ \infty & \text{if } I \notin \mathcal{F}. \end{cases}$$

We say that a length function ℓ with associated colength τ is *radical* if $\tau(I) = \tau(\text{rad}(I))$ for every ideal I . This definition corresponds to radical semistar operations and radical localizing systems.

Proposition 7.2. *Let \mathcal{F} be a localizing system. Then, \mathcal{F} is radical if and only if the associated length function $\ell_{\mathcal{F}}$ is radical.*

Proof. If I is an ideal, by definition $\tau(I) = 0$ if and only if $I \in \mathcal{F}$. Therefore, $\tau(I) = \tau(\text{rad}(I)) = 0$ if and only if $I, \text{rad}(I) \in \mathcal{F}$, while $\tau(I) = \tau(\text{rad}(I)) = \infty$ if and only if $I, \text{rad}(I) \notin \mathcal{F}$. The claim follows. \square

Let D be a Prüfer domain. To every singular length function ℓ with associated colength τ we can associate the space

$$\Sigma(\ell) := \{P \in \text{Spec}(D) \mid \tau(Q) > 0 \text{ for some } P\text{-primary ideal } Q\},$$

and the length function

$$\ell^\# := \sum_{P \in \Sigma(\ell)} \ell \otimes D_P$$

(where $(\ell \otimes D_P)(M) := \ell(M \otimes D_P)$). Then, we get an analogue of Theorem 6.7.

Proposition 7.3. *Let D be a Prüfer domain. The following are equivalent:*

- (i) D is min-scattered;
- (ii) $\ell = \ell^\#$ for every singular length function ℓ .

Proof. Let Φ be the isomorphism between $\text{SStar}_{\text{st}}(D)$ and $\mathcal{L}_{\text{sing}}(D)$ obtained composing the bijections of the two with the set of localizing systems. The claim follows from Theorem 6.7 and the fact that $\Phi(\widehat{\star}) = \Phi(\star)^\#$ [17, Proposition 6.8]. \square

As a consequence, we also obtain a version of Theorem 6.9 (compare with [17, Theorem 5.12]).

Theorem 7.4. *Let D_1, D_2 be Prüfer domain. Suppose that there is a homeomorphism $\phi : \text{Spec}(D_1) \rightarrow \text{Spec}(D_2)$ such that a prime ideal P is idempotent if and only if $\phi(P)$ is idempotent. If D_1 is min-scattered, then there is an isomorphism $\Phi : \mathcal{L}_{\text{sing}}(D_1) \rightarrow \mathcal{L}_{\text{sing}}(D_2)$.*

To conclude, we express [19, Corollary 7.5] in the terminology of this paper; see [19] for the definition of SP-scattered domain.

Proposition 7.5. *Let D be an SP-scattered domain. Then, every stable semistar operation on D is radical.*

REFERENCES

- [1] D. D. Anderson. Star-operations induced by overrings. *Comm. Algebra*, 16(12):2535–2553, 1988.
- [2] D. D. Anderson and Sharon M. Clarke. Star-operations that distribute over finite intersections. *Comm. Algebra*, 33(7):2263–2274, 2005.
- [3] D. D. Anderson and Sylvia J. Cook. Two star-operations and their induced lattices. *Comm. Algebra*, 28(5):2461–2475, 2000.
- [4] Max Dickmann, Niels Schwartz, and Marcus Tressl. *Spectral spaces*, volume 35 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2019.

- [5] Carmelo A. Finocchiaro, Marco Fontana, and K. Alan Loper. The constructible topology on spaces of valuation domains. *Trans. Amer. Math. Soc.*, 365(12):6199–6216, 2013.
- [6] Carmelo A. Finocchiaro, Marco Fontana, and Dario Spirito. Spectral spaces of semistar operations. *J. Pure Appl. Algebra*, 220(8):2897–2913, 2016.
- [7] Carmelo A. Finocchiaro and Dario Spirito. Some topological considerations on semistar operations. *J. Algebra*, 409:199–218, 2014.
- [8] Marco Fontana and James A. Huckaba. Localizing systems and semistar operations. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 169–197. Kluwer Acad. Publ., Dordrecht, 2000.
- [9] Marco Fontana and K. Alan Loper. Cancellation properties in ideal systems: a classification of e.a.b. semistar operations. *J. Pure Appl. Algebra*, 213(11):2095–2103, 2009.
- [10] Robert Gilmer. *Multiplicative Ideal Theory*. Marcel Dekker Inc., New York, 1972. Pure and Applied Mathematics, No. 12.
- [11] Wolfgang Krull. Beiträge zur Arithmetik kommutativer Integritätsbereiche i-ii. *Math. Z.*, 41(1):545–577; 665–679, 1936.
- [12] Stefan Mazurkiewicz and Waclaw Sierpiński. Contribution à la topologie des ensembles dénombrables. *Fundam. Math.*, 1:17–27, 1920.
- [13] Akira Okabe and Ryūki Matsuda. Semistar-operations on integral domains. *Math. J. Toyama Univ.*, 17:1–21, 1994.
- [14] Dario Spirito. Towards a classification of stable semistar operations on a Prüfer domain. *Comm. Algebra*, 46(4):1831–1842, 2018.
- [15] Dario Spirito. The Zariski topology on sets of semistar operations without finite-type assumptions. *J. Algebra*, 513:27–49, 2018.
- [16] Dario Spirito. Topological properties of localizations, flat overrings and sublocalizations. *J. Pure Appl. Algebra*, 223(3):1322–1336, 2019.
- [17] Dario Spirito. Decomposition and classification of length functions. *Forum Math.*, 32(5):1109–1129, 2020.
- [18] Dario Spirito. The derived sequence of a pre-Jaffard sequence. *Mediterr. J. Math.*, 19(4):146, 2022.
- [19] Dario Spirito. Almost Dedekind domains without radical factorization. submitted.
- [20] Dario Spirito. Localizations of integer-valued polynomials and of their Picard group. submitted.
- [21] Paolo Zanardo. Multiplicative invariants and length functions over valuation domains. *J. Commut. Algebra*, 3(4):561–587, 2011.

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