POLYNOMIALLY INDEPENDENT SUBSETS AND GENERALIZED NAGATA RINGS

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ABSTRACT. Let $(D, \mathfrak{m})$ be a local integral domain. We introduce the concept of *polynomially independent set* over D as a subset $\mathbf{r} \subset F$ (where F is a field containing D) as a set such that every polynomial relation $f(\mathbf{r}) = 0$ has coefficients in $\mathfrak{m}$, and we show that, when considered modulo $\mathfrak{m}$, such sets have many properties in common with sets of independent indeterminates. We study this notion, generalizing a result of Seidenberg about singly-generated algebras over integrally closed domains. Next, we introduce the *polynomial dimension* of F over D as the largest cardinality of a polynomially independent set, and show that this concept generalizes the definition of transcendence degree of a field extension and is better-behaved than the usual Krull dimension in dealing with overrings; we also link it with the Zariski-Riemann space of valuation rings of F containing D. Finally, we define the *Nagata ring* of D with respect to a polynomially independent set $\mathbf{r}$ as a generalization of the classical Nagata ring of an integral domain, and we show that its localizations at the maximal ideals containing $\mathfrak{m}$ are an affine set of a space of rings (which we call the *Seidenberg transforms* of D) that is isomorphic to a product of projective lines over the residue field of D.

1. Introduction

Let D be an integral domain with quotient field K, and let T be an *overring* of D, i.e., a ring between D and K. It is in general impossible to gauge the properties of T from those of D: for example, every Noetherian domain of dimension at least 2 has non-Noetherian overrings, or the dimension of T may be greater than the dimension of D; likewise, it is difficult to understand the relation between the spectrum of T and the spectrum of D, even if T is a finitely generated algebra over D.

In this context, a rather general theorem was proved by Seidenberg for singly-generated algebras [8]: he showed that, if $(D, \mathfrak{m})$ is an integrally closed local domain and α is an element such that $\alpha, \alpha^{-1} \notin D$, then the maximal ideal $\mathfrak{m}$ of D extends to a non-maximal prime ideal P,

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and that the quotient $D[\alpha]/P$ is isomorphic to the ring of polynomials in one variable over the residue field $D/\mathfrak{m}$.

In this paper, we extend Seidenberg's result to the case of more than one element by defining the notion of *polynomially independent set* of an integral domain, which is strongly linked with the notion of *analytically independent set* defined in [6]. Given a local domain $(D,\mathfrak{m})$ and a field F containing D, we say that a subset $\mathbf{r}\subset F$ is polynomially independent if every polynomial relation $f(\mathbf{r})=0$ must have coefficients in the maximal ideal $\mathfrak{m}$ of D. Polynomially independent sets work in many ways as independent indeterminates over D, modulo the maximal ideal $\mathfrak{m}$; for example, we extend Seindeberg's result proving that, if $\mathbf{r}$ is polynomially independent, then $\mathfrak{m}D[\mathbf{r}]$ is a prime ideal of $D[\mathbf{r}]$, and $D[\mathbf{r}]/\mathfrak{m}D[\mathbf{r}]$ is isomorphic to the polynomial ring $(D/\mathfrak{m})[\mathbf{X}]$ (where $\mathbf{X}$ is a set of indeterminates with the same cardinality as $\mathbf{r}$).

Furthermore, this notion allows us to define a new kind of dimension of an integral domain, which we call *polynomial dimension*: more precisely, the polynomial dimension $\dim_p(F/D)$ of F over D is the maximal number of elements in a polynomially independent set; if F is the quotient field of D, we call it simply the polynomial dimension of D, and denote it by $\dim_p(D)$. We show that the polynomial dimension is strongly linked to the Zariski-Riemann space of F over D and that it can be thought of as a way to measure how far is D from being a Prüfer domain: indeed, $\dim_p(D)=0$ if and only if the integral closure of D is a Prüfer domain. Polynomial dimension is better behaved than the usual (Krull) dimension when dealing with extension rings: for example, if D' is a domain between D and F then $\dim_p(F/D')\leq\dim_p(D/F)$, and the polynomial dimension of $D[X]$ is always one more than the polynomial dimension of D. We also show that the polynomial dimension is a generalization of the concept of transcendence degree of a field extension, in the sense that if $D=K$ is a field then $\dim_p(F/K)=\mathrm{trdeg}(F/K)$.

We also generalize the concept of Nagata rings: if $\mathbf{r}$ is a polynomially independent set over D, we define the *Nagata ring of D with respect to* $\mathbf{r}$ as the localization $D[\mathbf{r}]_{\mathfrak{m}D[\mathbf{r}]}$, and we denote it by $D(\mathbf{r})$; this notion reduces to the classical one when $\mathbf{r}=\{X\}$ is an indeterminate over D. Suppose now that $\mathbf{r}=\{r_1,\ldots,r_n\}$ is finite of cardinality n: then, through the isomorphism $D[\mathbf{r}]/\mathfrak{m}D[\mathbf{r}]\simeq D[X_1,\ldots,X_n]$, the localizations of $D[\mathbf{r}]$ at the maximal ideals containing $\mathfrak{m}$ form a space that is naturally homeomorphic to the affine space over $D/\mathfrak{m}$ of dimension n. We show that, if we also add the localizations of $D[r_1^{\epsilon_1},\ldots,r_n^{\epsilon_n}]$ (where each ϵ_i is either 1 or -1) we obtain a space that is homeomorphic to the n-fold product of the projective line over $D/\mathfrak{m}$. We call these localizations the *Seidenberg transforms* of D with respect to $\mathbf{r}$.

In the final part of the paper, we study polynomial independent sets over Noetherian domains. We show that, if $\{x_1,\ldots,x_n\}$ is a system of parameters of D, then $\{x_1/x_n,\ldots,x_2/x_n\}$ is a (maximal) polynomially independent set; in particular, the polynomial dimension of a local Noetherian domain of Krull dimension n is $n-1$. We also show, using the connection with analytically independent sets, that all maximal polynomially independent sets have the same cardinality if and only if D is catenarian.

2. Polynomially independent sets

Throughout the paper, D is an integral domain with quotient field K and F is a field containing K.

We denote by $\mathbf{r} := \{r_\alpha \mid \alpha \in A\}$ a subset of F indexed by A, and by $\mathbf{X} := \{X_\alpha \mid \alpha \in A\}$ a set of independent indeterminates indexed by the same set. Analogously, $\mathbf{r}'$ and $\mathbf{X}'$ will be indexed by A'.

Given $\mathbf{r}$, the *valuation homomorphism* relative to $\mathbf{r}$ is the map $\phi_\mathbf{r}$ given by

$$\begin{aligned} \phi_\mathbf{r}\colon D[\mathbf{X}] &\longrightarrow D[\mathbf{r}], \\ f &\longmapsto f(\mathbf{r}). \end{aligned}$$

Definition 2.1. *Let $(D, \mathfrak{m})$ be a local domain, and let $\mathbf{r} \subseteq F$. We say that $\mathbf{r}$ is a* polynomially independent set *over D if the kernel of the valuation homomorphism $\phi_\mathbf{r}$ is contained in $\mathfrak{m}D[\mathbf{X}]$.*

Example 2.2. Let $D = K$ be a field, and let $\mathbf{r} \subseteq F$. Then, $\mathbf{r}$ is polynomially independent if and only if it is algebraically independent: indeed, $\mathfrak{m}D[\mathbf{X}]$ is just the zero ideal, and thus the condition of being polynomially independent reduces to $f(\mathbf{r}) \neq 0$ for all nonzero polynomials, i.e., to the condition of being algebraically independent.

Example 2.3. Let D be a domain and t be an indeterminate over D. Then, t is polynomially independent over D, since the kernel of the valuation homomorphism is (0).

Example 2.4. If $\mathbf{r} = \{r\}$ is a single element and D is integrally closed, then r is polynomially independent over D if and only if $r, r^{-1} \notin D$; this follows essentially from the proof of [8, Theorem 6] (we shall deal with this case in more detail in Proposition 2.15).

These three examples foreshadow several properties and connections of polynomially independent sets. Indeed, our first result is a generalization of [8, Theorems 6 and 7] (in a different terminology).

Proposition 2.5. *Let $(D, \mathfrak{m})$ be a local domain and let $\mathbf{r} \subseteq F$. Then, the following hold.*

- *(a) $\mathbf{r}$ is polynomially independent if and only if $\phi_\mathbf{r}^{-1}(\mathfrak{m}D[\mathbf{r}]) = \mathfrak{m}D[\mathbf{X}]$.*
- *(b) If $\mathbf{r}$ is polynomially independent, then $\mathfrak{m}D[\mathbf{r}]$ is a prime ideal of $D[\mathbf{r}]$ and $D[\mathbf{r}]/\mathfrak{m}D[\mathbf{r}] \simeq (D/\mathfrak{m})[\mathbf{X}]$.*

Proof. (a) We always have $\phi_\mathbf{r}(\mathfrak{m}D[\mathbf{X}]) = \mathfrak{m}D[\mathbf{r}]$, and thus $\mathfrak{m}D[\mathbf{r}] \subseteq \phi_\mathbf{r}^{-1}(\mathfrak{m}D[\mathbf{X}])$.

Suppose $\mathbf{r}$ is polynomially independent. If $f \in \phi_{\mathbf{r}}^{-1}(\mathfrak{m}D[\mathbf{X}])$, then $f(\mathbf{r}) \in \mathfrak{m}$, say $f(\mathbf{r}) = m$; hence, $g(\mathbf{X}) := f(\mathbf{X}) - m \in \ker\phi_{\mathbf{r}}$. Thus, $g \in \mathfrak{m}D[\mathbf{X}]$ and so $f \in \mathfrak{m}D[\mathbf{X}]$; therefore $\phi_{\mathbf{r}}^{-1}(\mathfrak{m}D[\mathbf{X}]) \subseteq \mathfrak{m}D[\mathbf{r}]$ and the two sets are equal.

Conversely, if $\phi_{\mathbf{r}}^{-1}(\mathfrak{m}D[\mathbf{r}]) = \mathfrak{m}D[\mathbf{X}]$ then in particular $\ker\phi_{\mathbf{r}} = \phi_{\mathbf{r}}^{-1}(0) \subseteq \phi_{\mathbf{r}}^{-1}(\mathfrak{m}D[\mathbf{r}]) = \mathfrak{m}D[\mathbf{X}]$ and so $\mathbf{r}$ is polynomially independent.

(b) follows directly from the previous point. □

Proposition 2.6. *Let $(D, \mathfrak{m})$ be a local domain and $\mathbf{r} \subseteq F$. Then, the following are equivalent:*

(i) $\mathbf{r}$ is polynomially independent over D;
(ii) every subset of $\mathbf{r}$ is polynomially independent over D;
(iii) every finite subset of $\mathbf{r}$ is polynomially independent over D.

Proof. (i) $\Longrightarrow$ (ii). Suppose that $\mathbf{r}$ is polynomially independent and let $\mathbf{r}' \subseteq \mathbf{r}$. We have a commutative diagram

$$\begin{array}{ccc} D[\mathbf{X}'] & \xrightarrow{\phi_{\mathbf{r}'}} & D[\mathbf{r}'] \\ \downarrow & & \downarrow \\ D[\mathbf{X}] & \xrightarrow{\phi_{\mathbf{r}}} & D[\mathbf{r}] \end{array}$$

where the vertical arrows are the obvious inclusions and the horizontal arrows are the valuation homomorphisms $\phi_{\mathbf{r}'}$ and $\phi_{\mathbf{r}}$. Since $\mathbf{r}' \subseteq \mathbf{r}$, we have $\ker\phi_{\mathbf{r}'} \subseteq \ker\phi_{\mathbf{r}}$, and since the latter is contained in $\mathfrak{m}D[\mathbf{X}]$ it follows that the former is contained in $\mathfrak{m}D[\mathbf{X}']$. Hence, $\mathbf{r}'$ is polynomially independent over D.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i). Suppose that the finite subsets of $\mathbf{r}$ are polynomially independent. If $f \in \ker\phi_{\mathbf{r}}$, then f contains only finitely many indeterminates, and thus we can consider f as a polynomial over (say) $D[X_1, \dots, X_n]$. Thus, $f(r_1, \dots, r_n) = 0$, and since the finite subset $\{r_1, \dots, r_n\} \subseteq \mathbf{r}$ is polynomially independent, $f \in \mathfrak{m}D[X_1, \dots, X_n] \subseteq D[\mathbf{X}]$. Hence, $\ker\phi_{\mathbf{r}} \subseteq D[\mathbf{X}]$ and $\mathbf{r}$ is polynomially independent. □

Since $\mathfrak{m}D[\mathbf{r}]$ is a prime ideal, we can give the following definition.

Definition 2.7. *Let $(D, \mathfrak{m})$ be a local domain and let $\mathbf{r}$ be a polynomially independent set over F. The* Nagata ring of D with respect to $\mathbf{r}$, *which we denote by $D(\mathbf{r})$, is the localization $D[\mathbf{r}]_{\mathfrak{m}D[\mathbf{r}]}$.*

Remark 2.8.

(1) The Nagata ring $D(\mathbf{r})$ is always local, with maximal ideal $\mathfrak{m}D(\mathbf{r})$, and its residue field is isomorphic to $(D/\mathfrak{m})(\mathbf{X})$.
(2) If $\mathbf{r} = \{t\}$ is an indeterminate over D, then $D(t)$ is exactly the classical Nagata ring of D.

Proposition 2.9. *Let $(D, \mathfrak{m})$ be a local domain and suppose that $\mathbf{r} \subseteq F$ is the disjoint union of $\mathbf{r}'$ and $\mathbf{r}''$. Then, $\mathbf{r}$ is polynomially independent* *over D if and only if* $\mathbf{r}'$ *is polynomially independent over D and* $\mathbf{r}''$ *is polynomially independent over* $D(\mathbf{r}')$*.*

Proof. We denote by $\overline{\phi_{\mathbf{r}''}}\colon D(\mathbf{r}')[\mathbf{X}''] \longrightarrow D(\mathbf{r}')[\mathbf{r}'']$ the valuation homomorphism relative to $\mathbf{r}''$ over $D[\mathbf{r}']_{\mathfrak{m}D[\mathbf{r}']}$.

Suppose that $\mathbf{r}$ is polynomially independent. Then, $\mathbf{r}'$ is polynomially independent by Proposition 2.6.

Let $f \in \ker \overline{\phi_{\mathbf{r}''}}$. Then, we can write

$$f(\mathbf{X}'') = \frac{g_1(\mathbf{r}')}{h_1(\mathbf{r}')}Y_1 + \cdots + \frac{g_a(\mathbf{r}')}{h_a(\mathbf{r}')}Y_a,$$

where each g_i and h_i is a polynomial, the Y_j are distinct monomials in the $\mathbf{X}''$ and $h_i(\mathbf{r}') \notin \mathfrak{m}D[\mathbf{r}']$ for all i. In particular, since $\mathbf{r}'$ is a polynomially independent set, we have $h_i \notin \mathfrak{m}D[\mathbf{X}']$. Let $H(\mathbf{X}') := h_1(\mathbf{X}')\cdots h_a(\mathbf{X}')$. Then, $F(\mathbf{X}) := H(\mathbf{X}')f(\mathbf{X}'')$ is a polynomial in $D[\mathbf{X}]$, and $F(\mathbf{r}) = H(\mathbf{r}')f(\mathbf{r}'') = 0$ since $f(\mathbf{r}'') = 0$. Since $\mathbf{r}$ is polynomially independent, it follows that $F \in \mathfrak{m}D[\mathbf{X}]$; in particular, each $H(\mathbf{X}')\frac{g_i(\mathbf{X}')}{h_i(\mathbf{X}')}$ must be in $\mathfrak{m}D[\mathbf{X}]$. Since $\frac{H(\mathbf{X}')}{h_i(\mathbf{X}')}$ is a polynomial outside $\mathfrak{m}D[\mathbf{X}']$ (being the product of polynomials outside $\mathfrak{m}D[\mathbf{X}']$), it follows that each $g_i(\mathbf{X}')$ is in $\mathfrak{m}D[\mathbf{X}']$. Thus $f \in \mathfrak{m}D[\mathbf{r}']_{\mathfrak{m}D[\mathbf{r}']}[\mathbf{X}''] = D(\mathbf{r}')[\mathbf{X}'']$, and so $\ker \overline{\phi_{\mathbf{r}''}} \subseteq \mathfrak{m}D[\mathbf{r}']_{\mathfrak{m}D[\mathbf{r}']}[\mathbf{X}'']$. Therefore, $\mathbf{r}''$ is polynomially independent over $D(\mathbf{r}')$, as claimed.

Conversely, suppose that $\mathbf{r}'$ and $\mathbf{r}''$ are polynomially independent respectively over D and over $D(\mathbf{r}')$. Let $f \in \ker \phi_{\mathbf{r}}$, and write

$$f(\mathbf{X}) = g_1(\mathbf{X}')Y_1 + \cdots + g_a(\mathbf{X}')Y_a,$$

where each $g_1(\mathbf{X}')$ is a polynomial and the Y_a are distinct monomials in the $\mathbf{X}''$. We distinguish two cases.

If $g_i(\mathbf{X}') \in \mathfrak{m}D[\mathbf{X}']$ for every i, then $f \in \mathfrak{m}D[\mathbf{X}]$.

Suppose that $g_j(\mathbf{X}') \notin \mathfrak{m}D[\mathbf{X}']$ for some j; then, since $\mathbf{r}'$ is polynomially independent over D it follows from Proposition 2.5 that $g_j(\mathbf{r}') \notin \mathfrak{m}D[\mathbf{r}']$. Evaluating the g_i in $\mathbf{r}'$, we obtain a polynomial $F(\mathbf{X}'') := g_1(\mathbf{r}')Y_1 + \cdots + g_a(\mathbf{r}')Y_a$, which is not in $\mathfrak{m}D[\mathbf{r}'][\mathbf{X}'']$ since $g_i(\mathbf{r}') \notin \mathfrak{m}D[\mathbf{r}']$ (and there is no cancellation). However, $F(\mathbf{r}'') = f(\mathbf{r}) = 0$, and so F belongs to the kernel of the valuation homomorphism $\overline{\phi_{\mathbf{r}''}}$; since $\mathbf{r}''$ is polynomially independent, it follows that $F \in \mathfrak{m}D(\mathbf{r}')$, and thus that $f \in \mathfrak{m}D[\mathbf{X}]$.

In both cases, $f \in \mathfrak{m}D[\mathbf{X}]$, and thus $\ker \phi_{\mathbf{r}} \subseteq \mathfrak{m}D[\mathbf{X}]$, i.e., $\mathbf{r}$ is polynomially independent over D. □

We now want to extend the definition of polynomial independence from local to arbitrary domains.

Definition 2.10. *Let D be an integral domain, F a field containing D and let* $\mathbf{r} \subseteq F$*. We say that* $\mathbf{r}$ *is* polynomially independent over D *if there is a maximal ideal* $\mathfrak{m}$ *of D such that* $\ker \phi_{\mathbf{r}} \subseteq \mathfrak{m}D[\mathbf{X}]$.

Lemma 2.11. *Let D be an integral domain and $\mathbf{r} \subseteq F$. Then, $\mathbf{r}$ is polynomially independent over D if and only if it is polynomially independent over $D_\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of D.*

Proof. Let $\phi_\mathbf{r}^{(\mathfrak{m})} : D_\mathfrak{m}[\mathbf{X}] \longrightarrow D_\mathfrak{m}[\mathbf{r}]$ be the valuation homomorphism over $D_\mathfrak{m}$. Then, the kernel of $\phi_\mathbf{r}^{(\mathfrak{m})}$ is just the extension of the kernel of $\phi_\mathbf{r}$; hence, $\ker \phi_\mathbf{r}^{(\mathfrak{m})} \subseteq \mathfrak{m}D_\mathfrak{m}[\mathbf{X}]$ if and only if $\ker \phi_\mathbf{r} \subseteq \mathfrak{m}D[\mathbf{X}]$. The claim follows by comparing the definitions. □

Remark 2.12. The reason why we do not require polynomial independence over *all* localizations of D at maximal ideals is that by doing so we would disqualify any subset (or even any element) that is contained in some localization; for example, if the Jacobson radical of D is (0) then there would be no element of the quotient field of D that is polynomially independent over D.

Lemma 2.13. *Let D, D' be integral domains with $D \subseteq D' \subseteq F$, and let $\mathbf{r} \subseteq F$. If $\mathbf{r}$ is polynomially independent over D', then it is polynomially independent over D.*

Proof. Let $f \in D[\mathbf{X}]$ be such that $f(\mathbf{r}) = 0$. Then, f is also a polynomial over D', and thus there is a maximal ideal $\mathfrak{m}'$ of D' such that $f \in \mathfrak{m}'D'[\mathbf{X}]$, i.e., all its coefficients belong to $\mathfrak{m}'$. Hence, its coefficients are in $\mathfrak{m}' \cap D$, which is contained in some maximal ideal $\mathfrak{m}$ of D, and so $f \in \mathfrak{m}D[\mathbf{X}]$. Thus, $\mathbf{r}$ is polynomially independent over D. □

Proposition 2.14. *Let $(D, \mathfrak{m})$ be a local domain, and let $\mathbf{r} \subseteq F$. Then, $\mathbf{r}$ is polynomially independent over D if and only if it is polynomially independent over $\overline{D}$.*

Proof. If $\mathbf{r}$ is polynomially independent over $\overline{D}$ then it is independent over D by Lemma 2.13. Conversely, suppose that $\mathbf{r}$ is polynomially independent over D. Let Q be the kernel of the valuation homomorphism $\phi_{\mathbf{r},\overline{D}} : \overline{D}[\mathbf{X}] \longrightarrow \overline{D}$; then, $Q \cap D[X]$ is the kernel of the valuation homomorphism $\phi_{\mathbf{r},D}$. By hypothesis, $Q \cap D[\mathbf{X}] \subseteq \mathfrak{m}D[\mathbf{X}]$ for some maximal ideal $\mathfrak{m}$ of D; since $D[\mathbf{X}] \subseteq D'[\mathbf{X}]$ is integral, by going-up there is a prime ideal P of $\overline{D}$ over $\mathfrak{m}$ (so, in particular, P is maximal) such that $Q \subseteq P\overline{D}[\mathbf{X}]$. Hence, $\mathbf{r}$ is polynomially independent over $\overline{D}$. □

For singletons, polynomial independence is very close to an integrality condition; the following result ties polynomial independence with the results in [8, Section 3].

Proposition 2.15. *Let $(D, \mathfrak{m})$ be a local integral domain, and let $r \in F$.*

- *(a) If r is polynomially independent over D, then neither r nor $1/r$ are integral over D.*
- *(b) If the integral closure $\overline{D}$ of D is local, then r is polynomially independent over D if and only if $r, 1/r \notin \overline{D}$.*

Proof. (a) If r were integral, then there would be a monic polynomial $f \in D[X]$ such that $f(r) = 0$; in particular, $f \in \ker\phi_{\{r\}} \setminus \mathfrak{m}D[X]$, a contradiction. Likewise, if $1/r$ were integral, then $g(1/r) = 0$ for some monic polynomial g of degree n; then, $g_1(X) := X^n g(1/X)$ is a polynomial over $D[X]$ outside $\mathfrak{m}D[X]$, but $g_1(r) = r^n g(1/r) = 0$, again a contradiction. Hence neither r nor $1/r$ are integral over D.

(b) By Proposition 2.14, r is polynomially independent over D if and only if it is polynomially independent over $\overline{D}$. The claim now follows from [8, Corollary to Theorem 6]. □

Example 2.16. If $\overline{D}$ is not local, part (b) of the previous proposition does not hold.

Indeed, suppose $D = \mathbb{Z}_{(2)} + Y\overline{\mathbb{Q}}[[Y]]$, and let $r := \frac{1+\sqrt{17}}{4}$. Then, r belongs to the quotient field of D and is a root of $f(X) := 2X^2 - X - 2 \notin \mathfrak{m}D[X]$, and thus r is not polynomially independent. However, r is not integral over D since its trace is $T(r) = \frac{1+\sqrt{17}}{4} + \frac{1-\sqrt{17}}{4} = \frac{1}{2} \notin D$; likewise, $1/r$ is not integral, since $1/r = -\frac{1-\sqrt{17}}{4}$ and so its trace is $T(1/r) = -\frac{1-\sqrt{17}}{4} - \frac{1+\sqrt{17}}{4} = -\frac{1}{2} \notin D$.

Remark 2.17. Let $(D, \mathfrak{m})$ be a local domain with quotient field K, and let $c_0, \ldots, c_h$ be elements of D. Then, $c_0, \ldots, c_h$ are said to be *analytically independent in* D if, whenever $F(c_0, \ldots, c_h) = 0$ for some homogeneous polynomial $F \in D[X_0, \ldots, X_h]$, all coefficients of F are in $\mathfrak{m}$ [6, Section 4.4]. It is not hard to see that $c_0, \ldots, c_h$ are analytically independent if and only if $c_1/c_0, \ldots, c_h/c_0$ are polynomially independent; moreover, if $x_1, \ldots, x_k$ are in K, then $x_1, \ldots, x_k$ are polynomially independent if and only if $yx_1, \ldots, yk_k$ are analytically independent for all $y \in (D : (x_1, \ldots, x_k))$. Therefore, these two notions are essentially equivalent when we are dealing with integral domains and with elements in the quotient field. Since we focus in the study of overrings, and since we also want to deal with elements outside the quotient field, we prefer to use the terminology of polynomial independence.

3. Polynomial dimension

In this section, we introduce the notion of *polynomial dimension*, a kind of dimension that is better behaved than the usual Krull dimension when dealing with ring extensions.

Definition 3.1. *Let D be an integral domain and let F be a field containing D. The* polynomial dimension *of F over D, denoted by $\dim_p(F/D)$, is the largest cardinality of a subset of F that is polynomially independent over D.*

Proposition 3.2. *Let $D \subseteq D'$ be integral domains contained in the field F. Then, the following hold.*

(a) $\dim_p(F/D) = \sup\{\dim_p(F/D_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(D)\}$.

(b) $\dim_p(F/D) \geq \dim_p(F/D')$.
(c) If D' is integral over D, then $\dim_p(F/D) = \dim_p(F/D')$.

Proof. (a) follows from Lemma 2.11, (b) from Lemma 2.13 and (c) from Proposition 2.14. □

Corollary 3.3. *Let D be a domain, and let K be its quotient field. Then,* $\dim_p(K/D) = 0$ *if and only if the integral closure $\overline{D}$ of D is a Prüfer domain.*

Proof. By Proposition 3.2(a), we can suppose without loss of generality that D is local. Suppose first that D is integrally closed. By Lemma 2.15, D admits nonempty polynomially independent sets if and only if there is an r such that $r, 1/r \notin D$, that is, if and only if D is not a valuation domain. Hence, $\dim_p(K/D) = 0$ if and only if D is a valuation domain.

If D is not integrally closed, the claim now follows from Proposition 3.2(a). □

The fact that the polynomial dimension does not change under integral extensions suggests a look at valuation domains. Indeed, we can characterize when a sequence is polynomially independent through chains in the Zariski space. We start with a lemma.

Lemma 3.4. *Let $(D, \mathfrak{m})$ be a local domain, and let $W \subset V$ be two valuation domains containing D with the same quotient field such that $\mathfrak{m}_W \cap D = \mathfrak{m}_V \cap D = \mathfrak{m}$. Take any $r \in \mathfrak{m}_W \setminus \mathfrak{m}_V$. If $f \in D[X] \setminus \mathfrak{m}D[X]$, then $f(r) \notin \mathfrak{m}_V$.*

Proof. We can write f as the sum $f_1 + f_2$ of two polynomials such that each coefficient of f_1 is out of $\mathfrak{m}$ and each coefficient of f_2 is in $\mathfrak{m}$. Then, $f_2(r) \in \mathfrak{m}W \subseteq \mathfrak{m}_V$. Write $f_1(X) = \sum_j a_j X^j$, with each $a_j \neq 0$; note that f_1 is not the zero polynomial since $f \notin \mathfrak{m}D[X]$. Let v be the valuation relative to V. For every j, the coefficient a_j belongs to $D \setminus \mathfrak{m}$ and thus is a unit in V, so that $v(a_j r^j) = v(r^j)$. Since $r \in \mathfrak{m}_W$, it follows that, for $j \neq j'$, we have

$$v(a_j r^j) = v(r^j) \neq v(r^{j'}) = v(a_{j'} r^{j'});$$

therefore, $v(f_1(r)) = v(r^{j_0})$, where j_0 is the minimal index j such that $a_j \neq 0$. In particular, $f_1(r) \notin \mathfrak{m}_V$. Thus, we also have $f(r) \notin \mathfrak{m}_V$, as claimed. □

Theorem 3.5. *Let $(D, \mathfrak{m})$ be a local domain, F a field containing D, and let $\mathbf{r} := \{r_1, \ldots, r_n\} \subseteq F$. Then, $\mathbf{r}$ is polynomially independent over D if and only if there is a chain $V_0 \supset \cdots \supset V_n$ of elements of $\mathrm{Zar}(F|D)$ with center $\mathfrak{m}$ such that $r_i \in \mathfrak{m}_{V_i} \setminus \mathfrak{m}_{V_{i-1}}$ for all i.*

Proof. Suppose $\mathbf{r}$ is polynomially independent: since $D[\mathbf{r}]/\mathfrak{m}D[\mathbf{r}] \simeq (D/\mathfrak{m})[\mathbf{r}]$, we have a chain of prime ideals $\mathfrak{m}D[\mathbf{r}] \subset (\mathfrak{m}, r_1)D[\mathbf{r}] \subset \cdots \subset (\mathfrak{m}, r_1, \ldots, r_n)D[\mathbf{r}]$ inside $D[\mathbf{r}]$. Then, we can find a chain $W_0 \supset \cdots \supset$

W_n of overrings of $D[\mathbf{r}]$ such that $W_0 \cap D[\mathbf{r}] = \mathfrak{m}D[\mathbf{r}]$ and $W_i \cap D[\mathbf{r}] = (\mathfrak{m}, r_1, \ldots, r_i)D[\mathbf{r}]$ for $i > 0$; each of the W_i has center $\mathfrak{m}$ over D. The claim now follows lifting the chain to a chain in $\mathrm{Zar}(F|D)$.

Conversely, suppose that such a chain exists; we proceed by induction, doing the base case and the step at the same time. Let $S := \{f(r_n) \mid f \in D[X] \setminus \mathfrak{m}D[X]\}$: applying Lemma 3.4 to $V_{n-1} \supset V_n$, we see that $S \cap \mathfrak{m}_{V_{n-1}} = \emptyset$, and since $\mathfrak{m} \subseteq \mathfrak{m}_{V_{n-1}}$ we have $S \cap \mathfrak{m} = \emptyset$, that is, r_n is polynomially independent over D. If $n = 1$ we are done. Suppose $n > 1$, and construct the Nagata ring $D(r_n)$: then, by definition, $D(r_n) = S^{-1}D[r_n] \subseteq S^{-1}V_{n-1} = V_{n-1}$, using the fact that $r_n \in V_n \subset V_{n-1}$. Furthermore, the only prime ideal of $D(r_n)$ above $\mathfrak{m}$ is its maximal ideal; it follows that $\mathfrak{m}_{V_{n-1}} \cap D(r_n) = \mathfrak{m}_{D(r_n)}$, and thus $V_0 \supset \cdots \supset V_{n-1}$ is a chain satisfying the same hypothesis of the statement, but with $D(r_n)$ in place of D. By inductive hypothesis, $\{r_1, \ldots, r_{n-1}\}$ is polynomially independent over $D(r_n)$; hence, $\mathbf{r} = \{r_1, \ldots, r_{n-1}\} \cup \{r_n\}$ is polynomially independent over D by Proposition 2.9. The claim is proved. □

This theorem has a strong consequence.

Theorem 3.6. *Let $(D, \mathfrak{m})$ be a local domain, and let F be a field containing D. Let $\gamma_{F,D} : \mathrm{Zar}(F|D) \longrightarrow \mathrm{Spec}(D)$ be the center map. Then, $\dim_p(F/D) = \dim \gamma_{F,D}^{-1}(\mathfrak{m})$.*

Proof. Suppose first that both $\dim_p(F/D)$ and $\dim \gamma_{F,D}^{-1}(\mathfrak{m})$ are finite. By Theorem 3.5, the cardinality of a maximal independent set is equal to the length of a maximal chain of valuation domains in $\gamma_{F,D}^{-1}(\mathfrak{m})$, i.e., to its dimension.

If $\dim_p(F/D)$ is infinite, then we can find polynomially independent sets of arbitrary finite cardinality, and thus by Theorem 3.5 we can find chains in $\gamma_{F,D}^{-1}(\mathfrak{m})$ of arbitrary length, and thus its dimension is finite. Similarly, if $\dim \gamma_{F,D}^{-1}(\mathfrak{m})$ is infinite then we can find polynomially independent sets of arbitrary finite cardinality, and so the polynomial dimension is infinite. □

The following corollary essentially reduces the study of polynomial dimension over D of an arbitrary field F to the polynomial dimension of the quotient field of D.

Corollary 3.7. *Let $(D, \mathfrak{m})$ be a local domain, and let $L \subseteq F$ be fields containing D. Then,*

$$\dim_p(F/D) = \dim_p(L/D) + \mathrm{trdeg}(F/L).$$

Proof. Let $\mathbf{r} \subset F$ be a polynomially independent set over D, and let $\mathbf{X}$ be a transcendence basis of F over L. Then, $\mathbf{X}$ is polynomially independent over $D(\mathbf{r})$, and thus by Proposition 2.9 $\mathbf{r} \cup \mathbf{X}$ is polynomially independent over D. Thus, $\dim_p(F/D) \geq \dim_p(L/D) + \mathrm{trdeg}(F/L)$.

Let V be a minimal element of $\mathrm{Zar}(L|D)$. Any extension V' of V to F has rank at most $\dim(V) + \mathrm{trdeg}(F/L)$ [4, Proposition 20.5]; moreover, if $W \supset V$ then there is a $W' \supset V'$ that extends W. Therefore, $\dim \gamma_{F,D}^{-1}(\mathfrak{m}) \leq \dim \gamma_{L,D}^{-1}(\mathfrak{m}) + \mathrm{trdeg}(F/L)$, which means, by Theorem 3.6, that $\dim_p(F/D) \leq \dim_p(L/D) + \mathrm{trdeg}(F/L)$. Therefore, the two sides must be equal. □

Corollary 3.8. *Let $(D, \mathfrak{m})$ be a local domain, and let K be its quotient field. Then, $\dim_p(K/D) \leq \dim_v(D) - 1$, where $\dim_v(D)$ denotes the valuative dimension of D. Furthermore, if $\dim(D) = 1$ then equality holds.*

Proof. By Theorem 3.6, $\dim_p(K/P) = \dim \gamma_{K,D}^{-1}(\mathfrak{m})$. Every maximal chain in $\mathrm{Zar}(K|D)$ contains K itself, which does not belong to $\gamma_{K,D}^{-1}(\mathfrak{m})$; thus $\dim \gamma_{K,D}^{-1}(\mathfrak{m}) \leq \dim_v(D) - 1$.

If $\dim(D) = 1$, then $\gamma^{-1}(\mathfrak{m}) = \mathrm{Zar}(D) \setminus \{K\}$, and the claim follows. □

Corollary 3.9. *Let $(D, \mathfrak{m})$ be an integrally closed local domain. If D is not a valuation ring, then there is a chain $V_0 \subset V_1$ of valuation overrings of D such that both V_0 and V_1 have center $\mathfrak{m}$.*

Proof. Since D is integrally closed but not a valuation ring, we have $\dim_p(K/D) > 0$. The claim now follows from Theorem 3.5. □

Let D be an integral domain. If D is Noetherian, the Krull dimension of the polynomial ring $D[X]$ is $\dim(D)+1$; however, in general, we can only say that $\dim(D)+1 \leq \dim(D[X]) \leq 2\dim(D)+1$ [4, Section 30]. In this context, polynomial dimension is much more well-behaved.

Proposition 3.10. *Let D be an integral domain with quotient field K, and let X be an indeterminate over D. Then,*

$$\dim_p(K(X)/D[X]) = \dim_p(K/D) + 1.$$

Proof. Suppose first that D is local with maximal ideal $\mathfrak{m}$. By Corollary 3.7, $\dim_p(K(X)/D) = \dim_p(K/D)+1$; since $D \subseteq D[X]$, by Proposition 3.2(b) we have $\dim_p(K(X)/D) \leq \dim_p(K/D) + 1$.

Let $\mathbf{r} \subseteq K$ be a polynomially independent set over D of maximal cardinality. Then, X is polynomially independent over $D(\mathbf{r})$, and thus by Proposition 2.9 $\mathbf{r} \cup \{X\}$ is polynomially independent over D; hence, $\mathbf{r}$ is polynomially independent over the Nagata ring $D(X)$. By Theorem 3.6 there is a chain $\mathcal{C}$ of valuation domains of cardinality $|\mathbf{r}|$ above the maximal ideal $\mathfrak{m}D(X)$ of $D(X)$.

The Nagata ring $D(X)$ is the localization of $D[X]$ at the prime ideal $\mathfrak{m}D[X]$; if M is a maximal ideal above $\mathfrak{m}D[X]$, then there is a valuation domain V above M contained in all elements of the chain $\mathcal{C}$. Applying again Theorem 3.6, we obtain that the polynomial dimension of $D[X]$ is at least $|\mathbf{r}|+1$; hence, it must be $\dim_p(K(X)/D[X]) = \dim_p(K/D)+1$, as claimed.

If D is not local, the claim follows from Proposition 3.2(a). □

4. Quotients and automorphisms

A first case where we can calculate polynomial dimension is for pullbacks. We start with a lemma about the behavior of polynomial independence under quotients.

Lemma 4.1. *Let $D \subseteq T \subseteq F$ be domains with $(D, \mathfrak{m})$ local, and suppose that $\mathfrak{m}T \neq T$. Let P be a prime ideal of D, and let $\pi : T \longrightarrow T/PT$ be the quotient map. Let $\mathbf{r} \subseteq T$.*

- *(a) If $\pi(\mathbf{r})$ is polynomially independent over $\pi(D)$, then $\mathbf{r}$ is polynomially independent over D.*
- *(b) If $PT = P$, then $\pi(\mathbf{r})$ is polynomially independent over $\pi(D)$ if and only if $\mathbf{r}$ is polynomially independent over D.*

Proof. (a) Let $f \in D[\mathbf{X}]$ be such that $f(\mathbf{r}) = 0$. Then $\pi(f(\mathbf{r})) = 0$, and thus $\overline{f}(\pi(\mathbf{r})) = 0$, where $\overline{f}$ is the reduction of f modulo PT. Since $\pi(\mathbf{r})$ is polynomially independent over $\pi(D)$, we must have $\overline{f} \in \pi(\mathfrak{m})[\mathbf{X}]$, and thus all coefficients of f are in $\pi^{-1}(\pi(\mathfrak{m})) = \mathfrak{m}$. Hence, $\mathbf{r}$ is polynomially independent.

(b) By the previous point, we need only to show that if $\mathbf{r}$ is polynomially independent over D then $\pi(\mathbf{r})$ is polynomially independent over $\pi(D)$.

Let $\overline{f} \in \pi(D)[\mathbf{X}]$ be such that $\overline{f}(\pi(\mathbf{r})) = 0$. Let $f \in D[\mathbf{X}]$ be a polynomial whose reduction modulo P is $\overline{f}$: then, $\pi(f(\mathbf{r})) = 0$, so $f(\mathbf{r}) \in PT = P \subseteq \mathfrak{m}$, and thus $f \in \mathfrak{m}D[\mathbf{X}]$. Thus $\overline{f} \in \pi(\mathfrak{m})\pi(D)[\mathbf{X}]$ and so $\pi(\mathbf{r})$ is polynomially independent. □

Remark 4.2. Note that the converse of part (a) of the lemma does *not* hold without the hypothesis $PT = P$: for example, suppose $\{r_1, r_2\}$ is a polynomially independent set over D, and suppose there is a valuation overring V with center $\mathfrak{m}$ such that $r_1^2 - r_2 \in \mathfrak{m}_V$. (For example, this can happen if D is a three-dimensional regular local ring, $r_1 = x/z$ and $r_2 = y^2/z^2$, where x, y, z generate $\mathfrak{m}$.) Then, $\pi(r_1)^2 = \pi(r_2)$, so $\{\pi(r_1), \pi(r_2)\}$ is not algebraically independent. The point in which the proof fails is that $f(r_1, r_2)$ will be in $\mathfrak{m}_V$, but not in D (and so not in $\mathfrak{m}$), so that we cannot use the polynomial independence of $\{r_1, r_2\}$.

We also need a result about a "change of basis"-like property.

Proposition 4.3. *Let $(D, \mathfrak{m})$ be a local domain and let F be a field containing D Let $\mathbf{r} := \{r_\alpha \mid \alpha \in A\} \subseteq F$ be polynomially independent over D and, for every $\alpha \in A$, let $\phi_\alpha \in D(t)$, where t is an indeterminate over D, be such that its reduction modulo $\mathfrak{m}D(t)$ is not constant. Let $\boldsymbol{\phi}(\mathbf{r}) := \{\phi_\alpha(r_\alpha) \mid \alpha \in A\}$. Then, $\boldsymbol{\phi}(\mathbf{r})$ is polynomially independent over D.*

Proof. Let $\phi \in D(t)$. Then, $\phi = P/Q$ for some $P, Q \in D[t]$ such that $Q \notin \mathfrak{m}D[t]$; hence, if r is an element that is polynomially independent then $Q(r) \notin \mathfrak{m}D[r]$ and thus $\phi(r) \in D(r) \subseteq D[\mathbf{r}]_{\mathfrak{m}D[\mathbf{r}]}$. Therefore, $\boldsymbol{\phi}(\mathbf{r}) \subseteq D(\mathbf{r})$.

Let π be the quotient of $D(r)$ over its maximal ideal. Then, $\pi(\mathbf{r})$ is a transcendence basis of $(D/\mathfrak{m})(\mathbf{X})$; moreover, $\pi(\phi_\alpha(r_\alpha)) = \overline{\phi_\alpha}(\pi(r_\alpha))$ for every α, where $\overline{\phi_\alpha}$ is the reduction of ϕ_α modulo $\mathfrak{m}D(t)$. Since no $\overline{\phi_\alpha}$ is a constant, the set $\{\overline{\phi_\alpha}(\pi(r_\alpha)) \mid \alpha \in A\} = \pi(\boldsymbol{\phi}(\mathbf{r}))$ is algebraically independent over $D/\mathfrak{m}$, i.e., polynomially independent; by Lemma 4.1(a), $\boldsymbol{\phi}(\mathbf{r})$ is polynomially independent over D. The claim is proved. □

We call an indexed set $\boldsymbol{\epsilon} := \{\epsilon_\alpha \mid \alpha \in A\}$ a ± 1-*sequence* if $\epsilon_\alpha \in \{-1, +1\}$ for all α.

Corollary 4.4. *Let $\mathbf{r} := \{r_\alpha \mid \alpha \in A\} \subseteq K$ be a polynomially independent set, and let $\boldsymbol{\epsilon} := \{\epsilon_\alpha \mid \alpha \in A\}$ be a ± 1-sequence. Then, $\mathbf{r}' := \mathbf{r}^{\boldsymbol{\epsilon}} := \{r_\alpha^{\epsilon_\alpha} \mid \alpha \in A\}$ is polynomially independent.*

Proof. Both $\phi(t) = t$ and $\phi(t) = 1/t$ are elements of $D(t) \setminus \mathfrak{m}D(t)$; the claim now follows from Proposition 4.3. □

Proposition 4.5. *Let $(D, \mathfrak{m})$ be a local domain, let P be a prime ideal of D and let V be a valuation ring containing D such that $PV = P$. Then,*

$$\begin{aligned}\dim_p(K/D) &= \dim_p(Q(V/P)/(D/P)) \\ &= \dim_p(Q(D/P)/(D/P)) + \operatorname{trdeg}(Q(V/P)/Q(D/P)).\end{aligned}$$

Note that, if $PV = V$, then the quotient field of V coincides with the quotient field of D.

Proof. Let $\mathbf{r}$ be a polynomially independent subset of $Q(V)$. For every $r_\alpha \in \mathbf{r}$, either r or r^{-1} is in V; let it be s_α. By Corollary 4.4, $\mathbf{s} := \{s_\alpha \mid \alpha \in A\}$ is polynomially independent, it is contained in V and clearly it has the same cardinality of $\mathbf{r}$. By Lemma 4.1(a), $\pi(\mathbf{s})$ is polynomially independent over D/P, and thus $\dim_p(K/D) \leq \dim_p(Q(V/P)/(D/P))$.

Conversely, if $\mathbf{s} \subseteq Q(V/P)$ is polynomially independent over D/P, then in the same way we can suppose $\mathbf{s} \subseteq V/P$ (since V/P is again a valuation domain); any $\mathbf{r}$ such that $\pi(\mathbf{r}) = \mathbf{s}$ is polynomially independent (by Lemma 4.1(b)) and thus $\dim_p(K/D) \geq \dim_p(Q(V/P)/(D/P))$. Therefore, $\dim_p(K/D) = \dim_p(Q(V/P)/(D/P))$.

The second equality follows from the first part of the proof and Corollary 3.7. □

Example 4.6. Let $D := A + XL[[X]]$, where A is a local integral domain and L a field containing A. Then, we can apply the previous theorem to $P := XL[[X]]$ and $V := L[[X]]$, obtaining

$$\dim_p(K/D) = \dim_p(L/A) = \dim_p(Q(A)/A) + \operatorname{trdeg}(L/Q(A)).$$

For example, if $A = k$ is a field, then $\dim_p(K/D)$ will be exactly the transcendence degree of L over k.

If A is a valuation domain, then $\dim_p(Q(A)/A) = 0$, and so $\dim_p(K/D) = \operatorname{trdeg}(L/Q(A))$, while $\dim(D) = \dim(A) + \operatorname{trdeg}(L/Q(A))$; hence, for any pair (n, m) with $n \leq m$ we can find a domain D with $\dim(D) = n$ and $\dim_p(D) = m$ by choosing A of dimension n and L of transcendence degree $m - n$.

5. Seidenberg transforms

Definition 5.1. *Let $(D, \mathfrak{m})$ be a local domain, and let $\mathbf{r}$ be a polynomially independent set over D. An overring T of D is a* Seidenberg transform of D with respect to $\mathbf{r}$ *if there is a ± 1 sequence $\boldsymbol{\epsilon}$ such that T is the localization of $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$ at a maximal ideal containing $\mathfrak{m}$. We denote by $\mathcal{S}_{\mathbf{r}}(D)$ (or simply $\mathcal{S}_{\mathbf{r}}$ if there is no danger of confusion) the set of Seidenberg transforms of D with respect to $\mathbf{r}$.*

Example 5.2. Let $(D, \mathfrak{m})$ be a regular local ring of dimension 2 with $\mathfrak{m} = (x, y)$. Then, the Seidenberg transforms with respect to $\mathbf{r} = \{x/y\}$ are exactly the local quadratic transforms of D.

Lemma 5.3. *Let $\mathbf{r}$ be a polynomially independent set over D, let $\boldsymbol{\epsilon}$ be a ± 1-sequence, and let $T \in \mathcal{S}_{\mathbf{r}}(D)$. If $D[\mathbf{r}^{\boldsymbol{\epsilon}}] \subseteq T$, then T is a localization of $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$.*

Proof. By definition, there is a ± 1-sequence $\boldsymbol{\epsilon}'$ such that T is a localization of $D[\mathbf{r}^{\boldsymbol{\epsilon}'}]$; without loss of generality, all members of $\boldsymbol{\epsilon}'$ are equal to 1, i.e., $D[\mathbf{r}^{\boldsymbol{\epsilon}'}] = D[\mathbf{r}]$. Let S be the set of r_α such that $\epsilon_\alpha = -1$; then, r_α^{-1} belongs to T for all such S, and thus $S^{-1}D[\mathbf{r}] \subseteq T$; in particular, T is a localization of $S^{-1}D[\mathbf{r}]$. By construction, $S^{-1}D[\mathbf{r}]$ contains $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$, and furthermore $S^{-1}D[\mathbf{r}]$ is a localization of $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$; hence, T is a localization of $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$. ☐

Proposition 5.4. *Let $\mathbf{r}$ be a polynomially independent set over D and, for every $T \in \mathcal{S}_{\mathbf{r}}(D)$, let $\Delta_T := \{V \in \operatorname{Zar}_{\min}(D) \mid T \subseteq V\}$. Then, every Δ_T is closed in the constructible topology, and $\{\Delta_T \mid T \in \mathcal{S}_{\mathbf{r}}(D)\}$ is a partition of $\operatorname{Zar}_{\min}(D)$.*

Proof. Each Δ_T is the counterimage of the maximal ideal of T under the center map; since the center map is continuous, it follows that Δ_T is closed in the Zariski and thus in the constructible topology.

Suppose $V \in \operatorname{Zar}_{\min}(D)$. Then, for each $r_\alpha \in \mathbf{r}$ at least one of r_α and r_α^{-1} is in V, say $r_\alpha^{\epsilon_\alpha} \in V$; hence, if $\boldsymbol{\epsilon} := \{\epsilon_\alpha \mid \alpha \in A\}$, then $D' := D[\mathbf{r}^{\boldsymbol{\epsilon}}] \subseteq V$. Since V is minimal over D, its center on D' is a maximal ideal M containing $\mathfrak{m}$; thus, $T := D'_M \subseteq V$. By definition, T is a Seidenberg transform of D with respect to $\mathfrak{m}$, and so $V \in \Delta_T$; hence, $\operatorname{Zar}_{\min}(D)$ is contained in the union of all the Δ_T.

Suppose that $\Delta_T \cap \Delta_{T'} \neq \emptyset$, say V is in the intersection. If T and T' are both contained in the same $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$, then by Lemma 5.3 they are both localization of $D[\mathbf{r}^\epsilon]$ at a maximal ideal; however, the center of V on $D[\mathbf{r}^\epsilon]$ contains both maximal ideals,and thus it must be $T = T'$. □

Lemma 5.5. *Let $\mathbf{r} \subseteq F$ be a polynomially independent set over the local ring $(D, \mathfrak{m})$, and let k be the residue field of D. Then, the quotient map $\pi : D(\mathbf{r}) \longrightarrow D(\mathbf{r})/\mathfrak{m}D(\mathbf{r}) \simeq k(\mathbf{X})$ induces a homeomorphism*

$$\begin{aligned} \pi_0 \colon \mathcal{S}_{\mathbf{r}}(D) &\longrightarrow \mathcal{S}_{\pi(\mathbf{r})}(k), \\ T &\longmapsto \pi(T), \end{aligned}$$

with respect to the Zariski topology.

Proof. By construction, all Seidenberg transforms with respect to $\mathbf{r}$ are contained in the Nagata ring $D(\mathbf{r})$. By Remark 2.8, the image of $\mathbf{r}$ in $k(\mathbf{X})$ is just the set $\mathbf{X}$ of indeterminates, and thus $\pi(\mathbf{r})$ is polynomially independent and it makes sense to consider $\mathcal{S}_{\pi(\mathbf{r})}(k)$. If $T \in \mathcal{S}_{\mathbf{r}}(D)$, then without loss of generality $T = D[\mathbf{r}]_P$ for some maximal ideal P containing $\mathfrak{m}$; thus, $\pi_0(T) = T/(\mathfrak{m}D(\mathbf{r}) \cap T) = T/\mathfrak{m}_T$ is just the localization of $k[\mathbf{X}]$ at the image of P under π. In particular, π_0 is well-defined; it is surjective since $\pi^{-1}(k[\mathbf{X}]_Q)$ is just the localization of $D[\mathbf{r}]$ at $\pi^{-1}(Q)$, and it is clearly injective. To see that it is a homeomoprhism, let $\mathcal{B}(z) := \{T \in \mathcal{S}_{\mathbf{r}}(D) \mid z \in T\}$, for $z \in F$ be a subbasic open set. Then, $\pi_0(\mathcal{B}(z)) = \mathcal{B}(\pi(z))$ (thus π_0 is open), and if $w \in k(\mathbf{X})$ then $\pi_0^{-1}(\mathcal{B}(w)) = \mathcal{B}(w')$, where w' is any element such that $\pi(w') = w$ (thus π_0 is continuous). The claim is proved. □

Theorem 5.6. *Let $\mathbf{r} := \{r_1, \ldots, r_n\}$ be a polynomially independent set over the local ring $(D, \mathfrak{m})$, and let k be the residue field of D. Then, there is a homeomorphism between $\mathcal{S}_{\mathbf{r}}(D)$ and the product $(\mathbb{P}^1_k)^n$ (when both are endowed with the respective Zariski topology).*

Note that the Zariski topology on $(\mathbb{P}^1_k)^n$ is *not* the product topology of the Zariski topologies on the $\mathbb{P}^1_k$s.

Proof. By Lemma 5.5, $\mathcal{S}_{\mathbf{r}}(D)$ is homeomorphic with $\mathcal{S}_{\mathbf{r}'}(k)$, where $\mathbf{r}'$ is polynomially independent over k; therefore, without loss of generality we can suppose that $D = k$ is a field and that $\mathbf{r}$ is algebraically independent over D.

Fix a ± 1-sequence $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, and consider the open subset $\mathcal{B}(\mathbf{r}^\epsilon)$ of $\mathcal{S}_{\mathbf{r}}(k)$. By Lemma 5.3, the elements of $\mathcal{B}(\mathbf{r}^\epsilon)$ are the localizations of $k[\mathbf{r}^\epsilon]$ at the maximal ideals, and thus, since $k[\mathbf{r}^\epsilon] \simeq k[\mathbf{X}]$, the localization map $\lambda : \mathrm{Max}(k[\mathbf{r}^\epsilon]) \longrightarrow \mathcal{S}_{\mathbf{r}}(k)$, $M \mapsto k[\mathbf{r}^\epsilon]_M$, is a homeomorphism onto its image. By definition, $\mathbb{A}^n_k$ is the set of maximal ideals of $k[\mathbf{X}]$, and thus we have a homeomorphism $\psi_{\boldsymbol{\epsilon}} : \mathcal{B}(\mathbf{r}^\epsilon) \longrightarrow \mathbb{A}^n_k$.

The points of $\mathbb{A}^n_k$ can also be expressed as as n-uple $(\alpha_1, \ldots, \alpha_n)$ of elements of the algebraic closure $\overline{k}$, where α_i is the image of X_i under the quotient map of $k[\mathbf{X}]$ onto the residue field of the maximal ideal M corresponding to the point; in this interpretation, $(\alpha_1, \ldots, \alpha_n)$

and $(\alpha'_1,\ldots,\alpha'_n)$ represent the same point if and only if there is a k-automorphism ν such that $\nu(\alpha_i)=\alpha'_i$ for every i. Moreover, the same holds for $\mathbb{P}^1_k$. Hence, we can define a topological embedding

$$\begin{aligned}\sigma_{\boldsymbol{\epsilon}}\colon \mathbb{A}^n_k &\longrightarrow (\mathbb{P}^1_k)^n,\\ (\alpha_1,\ldots,\alpha_n) &\longmapsto (\beta_1,\ldots,\beta_n)\end{aligned}$$

such that

$$\beta_i := \begin{cases}[1:\alpha_i] & \text{if } \epsilon_i=+1,\\ [\alpha_i:1] & \text{if } \epsilon_i=-1.\end{cases}$$

Therefore, the composition $\Psi_{\boldsymbol{\epsilon}} := \sigma_{\boldsymbol{\epsilon}}\circ\psi_{\boldsymbol{\epsilon}} : \mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}}) \longrightarrow (\mathbb{P}^1_k)^n$ is a topological embedding for every $\boldsymbol{\epsilon}$.

Suppose now that $T\in\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}})\cap\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}'})$; we claim that $\Psi_{\boldsymbol{\epsilon}}(T)=\Psi_{\boldsymbol{\epsilon}'}(T)$. Let $\psi_{\boldsymbol{\epsilon}}(T):=(\alpha_1,\ldots,\alpha_n)$ and $\psi_{\boldsymbol{\epsilon}'}(T):=(\alpha'_1,\ldots,\alpha'_n)$, while set $\Psi_{\boldsymbol{\epsilon}}(T):=(\beta_1,\ldots,\beta_n)$ and $\Psi_{\boldsymbol{\epsilon}'}(T):=(\beta'_1,\ldots,\beta'_n)$.

By construction, α_i is equal to the image of $r_i^{\epsilon_i}$ under the quotient $T\longrightarrow T/\mathfrak{m}_T$. If $\epsilon_i=\epsilon'_i$, we have $\alpha_i=\alpha'_i$ and, by definition of the maps $\sigma_{\boldsymbol{\epsilon}}$ and $\sigma_{\boldsymbol{\epsilon}'}$, also $\beta_i=\beta'_i$. If $\epsilon_i\neq\epsilon'_i$, then T must contain both r_i and r_i^{-1}, which thus do not belong to its maximal ideal and are mapped to units of $\overline{k}$ under the quotients. Hence, $\alpha'_i=\alpha_i^{-1}$; if, without loss of generality, $\epsilon_i=+1$ and $\epsilon'_i=-1$, we have

$$\beta'_i=[\alpha'_i:1]=[\alpha_i^{-1}:1]=[1:\alpha_i]=\beta_i.$$

Therefore, $\beta_i=\beta'_i$ for all i, and so $\Psi_{\boldsymbol{\epsilon}}(T)=\Psi_{\boldsymbol{\epsilon}'}(T)$.

Since each element of $\mathcal{S}_{\mathbf{r}}(k)$ is the localization of some $\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}})$, these sets are a cover of $\mathcal{S}_{\mathbf{r}}(k)$, and so the maps $\Psi_{\boldsymbol{\epsilon}}$ can be glued into a single map $\Psi:\mathcal{S}_{\mathbf{r}}(k)\longrightarrow(\mathbb{P}^1_k)^n$, that is continuous and open since every $\Psi_{\boldsymbol{\epsilon}}$ is a homeomorphism; to prove that Ψ itself is a homeomorphism it is enough to show that it is bijective.

We first show that Ψ is surjective. Let $(\beta_1,\ldots,\beta_n)\in(\mathbb{P}^1_k)^n$, and suppose without loss of generality that $\beta_1,\ldots,\beta_s$ are equal to $[0:1]$ and that $\beta_{s+1},\ldots,\beta_n$ are different from $[0:1]$, say $\beta_t:=[1:\alpha_t]$. Let $\boldsymbol{\epsilon}$ be the ±1-sequence having -1 as the first s elements and $+1$ as the other elements; then, $(\beta_1,\ldots,\beta_n)=\sigma_{\boldsymbol{\epsilon}}(0,\ldots,0,\alpha_{s+1},\ldots,\alpha_n)$. Since $\psi_{\boldsymbol{\epsilon}}$ is a homeomorphism, there is a $T\in\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}})$ such that $\psi_{\boldsymbol{\epsilon}}(T)=(0,\ldots,0,\alpha_{s+1},\ldots,\alpha_n)$, and so $\Psi(T)=\Psi_{\boldsymbol{\epsilon}}(T)=(\beta_1,\ldots,\beta_n)$. Thus Ψ is surjective.

To prove that Ψ is injective,, suppose that $\Psi(T)=\Psi(T')=(\beta_1,\ldots,\beta_n)$. As above, without loss of generality $\beta_1,\ldots,\beta_s$ are equal to $[0:1]$, while $\beta_{s+1},\ldots,\beta_n$ are different from $[0:1]$, say $\beta_t:=[1,\alpha_t]$; define $\boldsymbol{\epsilon}$ as above. Then, both T and T' belong to $\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}})$, and since $\Psi_{\boldsymbol{\epsilon}}$ is injective we have $\Psi(T)=\Psi_{\boldsymbol{\epsilon}}(T)\neq\Psi_{\boldsymbol{\epsilon}}(T')=\Psi(T')$, a contradiction. Hence Ψ is injective.

Therefore, Ψ is a bijective map such that all its restrictions to $\mathcal{B}(\mathbf{r}^{\boldsymbol{\epsilon}})$ are homeomorphisms; it follows that Ψ itself is a homeomorphism, as claimed. □

Remark 5.7. Despite the apparent symmetry between the rings $D[\mathbf{r}]$ and $D[\mathbf{r}^{\boldsymbol{\epsilon}}]$ (where $\mathbf{r}$ is polynomially independent and $\boldsymbol{\epsilon}$ is a ± 1-sequence), in general the Seidenberg transforms with respect to $\mathbf{r}$ are not isomorphic to each other.

For example, let k be a field and $A = k[X, XY, XY^2, \ldots, XY^n, \ldots]$; let P be the maximal ideal generated by the monomials and set $D := A_P$. Then, D is a local domain with maximal ideal $\mathfrak{m} = PD$ and Y is polynomially independent over D. Consider the Seidenberg transforms of D with respect to $\{Y\}$.

The ring $D[Y]$ is a localization of the polynomial ring $A[Y] = k[X, Y]$; in particular, all its localizations are regular local rings. On the other hand, in the ring $D[Y^{-1}]$, for every n the element Y^{-n} divides X; it follows that, in the localization $D[Y^{-1}]_{(\mathfrak{m},Y^{-1})}$, X belongs to the intersection of the ideals generated by the powers Y^{-n}. Therefore, $D[Y^{-1}]_{(\mathfrak{m},Y^{-1})}$ cannot be a Noetherian ring (in fact, it is a two-dimensional valuation ring), and thus it cannot be isomorphic to the other Seidenberg transforms.

6. Maximal polynomially independent subsets

We say that $\mathbf{r}$ is a *maximal polynomially independent set* (of F over D) if $\mathbf{r}$ cannot be enlarged to a bigger polynomially independent set, i.e., if no $\mathbf{r}' \supsetneq \mathbf{r}$ is polynomially independent. It's easy to see that such sets exist.

Proposition 6.1. *Let $\mathbf{r}$ be a polynomially independent set over D. Then, there is a maximal polynomially independent set $\mathbf{r}'$ containing $\mathbf{r}$.*

Proof. It is enough to apply Zorn's lemma to the set of polynomially independent subsets of F, keeping in mind Proposition 2.6. □

A natural question is whether all maximal polynomially independent set of the same field have the same cardinality, like it happens for algebraically independent elements in a field extension; this is in general not true. We shall characterize when this happens in the Noetherian case and then give two sufficient conditions in the general case.

Proposition 6.2. *Let D be a Noetherian local integrally closed domain with quotient field K. Then, all maximal polynomially independent set of K over D have the same cardinality if and only if D is catenarian.*

Proof. By [7, Theorem 4.18] (applied with $R = S = D$), D is catenary if and only if every maximal set of analytically independent elements have the same cardinality. Since $x_0, \ldots, x_k$ are analytically independent if and only if $x_1/x_0, \ldots, x_k/x_0$ are polynomially independent (see Remark 2.17), the claim follows. □

Proposition 6.3. *Let D be an integral domain. Suppose there is a maximal polynomially independent set $\mathbf{r}$ such that $D(\mathbf{r})$ is integrally closed. Then, all maximal polynomially independent subsets over D have the same cardinality.*

Proof. By Proposition 2.9 and Corollary 3.3, since $\mathbf{r}$ is maximal the ring $D(\mathbf{r})$ is a Prüfer domain and thus a valuation domain.

Let $\mathbf{s}$ be a maximal polynomially independent set. For every $\alpha \in A$, define:

$$\phi_\alpha(t) := \begin{cases} t & \text{if } s_\alpha \in D(\mathbf{s}) \setminus \mathfrak{m}D(\mathbf{s}), \\ t+1 & \text{if } s_\alpha \in \mathfrak{m}D(\mathbf{s}), \\ t^{-1}+1 & \text{if } s_\alpha \notin D(\mathbf{s}). \end{cases}$$

Then, each ϕ_α is an invertible element of $D(t)$ whose reduction modulo $\mathfrak{m}D(t)$ is not constant; by Proposition 4.3, $\boldsymbol{\phi}(\mathbf{s}) := \{\phi_\alpha(s_\alpha) \mid \alpha \in A\}$ is polynomially independent over D. Furthermore, since $D(\mathbf{r})$ is a valuation domain $\phi_\alpha(s_\alpha) \in D(\mathbf{r}) \setminus \mathfrak{m}D(\mathbf{r})$ for every α.

Let π be the quotient of $D(\mathbf{r})$ onto its residue field. By Proposition 4.3, $\pi(\boldsymbol{\phi}(\mathbf{s}))$ is algebraically independent over $D/\mathfrak{m}$; hence, $|\mathbf{s}| \leq |\mathbf{r}|$, since the transcendence degree of the residue field of $D(\mathbf{r})$ over $D/\mathfrak{m}$ is $|\mathbf{r}|$.

Suppose that $|\mathbf{s}| < |\mathbf{r}|$: then, we can find $s_0 \in D(\mathbf{r}) \setminus \mathfrak{m}D(\mathbf{r})$ such that $\pi(\boldsymbol{\phi}(\mathbf{s})) \cup \pi(s_0)$ is algebraically independent over $D/\mathfrak{m}$, and thus by Lemma 4.1(a) $\boldsymbol{\phi}(\mathbf{s}) \cup \{s_0\}$ is polynomially independent over D. Applying again Proposition 4.3, we see that $\boldsymbol{\phi}^{-1}(\boldsymbol{\phi}(\mathbf{s})) \cup \{s_0\} = \mathbf{s} \cup \{s_0\}$ is polynomially independent over D. This contradicts the maximality of $\mathbf{s}$, and thus it cannot be that $|\mathbf{s}| < |\mathbf{r}|$. Therefore, $|\mathbf{s}| = \mathbf{r}$. Since $\mathbf{s}$ was arbitrary, the claim is proved. □

Corollary 6.4. *Let D be a local Noetherian integral domain that is not catenarian. For every maximal polynomially independent set $\mathbf{r}$, the ring $D[\mathbf{r}]$ is not integrally closed.*

Proof. If it were integrally closed, so would be $D(\mathbf{r})$, and so all maximal polynomially independent sets would have the same cardinality. This contradicts Proposition 6.2. □

When $(D, \mathfrak{m})$ is a regular local ring, and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then it is known [1, (1.4)] that the localization of $D[\mathfrak{m}/x]$ at $(\mathfrak{m}/x)(D[\mathfrak{m}/x])$, and thus $D(\mathfrak{m}/x)$, is integrally closed; hence, Proposition 6.3 gives an alternate proof of the fact that in this case all polynomially independent subsets have the same cardinality. It would be interesting to characterize when the hypothesis of Proposition 6.3 holds.

7. The polynomial dimension of a Noetherian domain

When D is a Noetherian domain, we can obtain a precise calculation of the polynomial dimension.

Lemma 7.1. *Let $(D,\mathfrak{m})$ be a Noetherian local domain, and let $\mathbf{r}\subseteq K$ be a maximal polynomially independent set. Then, $\mathfrak{m}D[\mathbf{r}]$ has height 1 in $D[\mathbf{r}]$.*

Proof. Note first that $\mathbf{r}$ is finite, since by Corollary 3.8 $|\mathbf{r}|\leq\dim_p D\leq\dim_v D-1<\infty$ since D is Noetherian.

The polynomial dimension of $D':=D(\mathbf{r})$ is 0: indeed, if $\mathbf{r}'$ were polynomially independent over D', then by Proposition 2.9 $\mathbf{r}\cup\{\mathbf{r}'\}$ would be polynomially independent over D, against the maximality of $\mathbf{r}$. By Corollary 3.3, the integral closure of D' is a Prüfer domain; however, since $\mathbf{r}$ is finite, D' is Noetherian, and thus it must have dimension 1, i.e., $\mathfrak{m}D[\mathbf{r}]$ has height 1 in $D[\mathbf{r}]$. □

The following is a new way to see a well-known result (see e.g. [3, Lemme 2]).

Corollary 7.2. *Let D be a Noetherian domain and let $\mathfrak{p}$ be a prime ideal of D. Then, there is a discrete valuation overring V of D with center $\mathfrak{p}$.*

Proof. Without loss of generality, we can suppose that D is local with maximal ideal $\mathfrak{p}=\mathfrak{m}$. We can find a maximal polynomially independent set $\mathbf{r}$: by the previous lemma, $D(\mathbf{r})$ is a Noetherian local domain of dimension 1 whose maximal ideal contracts to $\mathfrak{m}$. Since all valuation overrings of a one-dimensional Noetherian domain are discrete, the claim follows. □

Lemma 7.3. *Let $(D,\mathfrak{m})$ be a local domain, let $P\subsetneq\mathfrak{m}$ be a prime ideal, and let $\mathbf{r}$ be a set that is polynomially independent over D_P. Let $S:=D\setminus P$. Then, $D(\mathbf{r})\subsetneq D_P(\mathbf{r})=S^{-1}D(\mathbf{r})$.*

Proof. It is obvious that $D(\mathbf{r})\subseteq S^{-1}D(\mathbf{r})\subseteq D_P(\mathbf{r})$. By definition, $D_P(\mathbf{r})$ is the quotient of $D_P[\mathbf{X}]$ by the kernel of the valuation homomorphism $\phi_{\mathbf{r}}^{(D_P)}$ over D_P, which is exactly the localization of the kernel of the valuation homomorphism $\phi_{\mathbf{r}}^{(D)}$ relative to D at the multiplicatively closed subset S; thus,

$$D_P(\mathbf{r})=\frac{S^{-1}D[\mathbf{X}]}{S^{-1}\ker\phi_{\mathbf{r}}^{(D)}}=S^{-1}D(\mathbf{r}).$$

To conclude the proof, let $t\in\mathfrak{m}\setminus P$. Then, $t^{-1}\in D_P\subseteq D_P(\mathbf{r})$. However, if $t^{-1}\in D(\mathbf{r})$ then $1=tt^{-1}\in\mathfrak{m}D(\mathbf{r})$, against the fact that $\mathfrak{m}D(\mathbf{r})$ is the maximal ideal of $D(\mathbf{r})$. Then, $t^{-1}\in D_P(\mathbf{r})\setminus D(\mathbf{r})$, and the claim is proved. □

The next theorem is a direct consequence of [6, Chapter 4, Theorem 3] and of the correspondence between analytically and polynomially independent sets (see Remark 2.17). We give an alternate proof with a more overring-centric approach; note that the correspondence also implies that this also gives a proof of [6, Chapter 4, Theorem 3].

Theorem 7.4. *Let $(D, \mathfrak{m})$ be a Noetherian local domain of dimension $n \geq 2$. Let $x_1, \ldots, x_n$ be a system of parameters. Then, $\{x_1/x_n, \ldots, x_{n-1}/x_n\}$ is a polynomially independent set of K over D.*

Proof. We proceed by induction on n: suppose first that $n = 2$. There is a maximal ideal M of the integral closure $\overline{D}$ of D that is minimal over (x_1, x_2); hence, x_1, x_2 is a system of parameters of $T := \overline{D}_M$. In particular, T contains neither x_1/x_2 nor x_2/x_1; since T is integrally closed, by Proposition 2.15(b) x_1/x_2 is polynomially independent over T, and by Lemma 2.13 is polynomially independent over D.

Suppose now that the claim holds up to $n-1$. Let P be a minimal prime over $x_2, \ldots, x_n$: then, $x_2, \ldots, x_n$ is a system of parameters over D_P, and by inductive hypothesis $\mathbf{r} := \{x_2/x_n, \ldots, x_{n-1}/x_n\}$ is a polynomially independent set over D_P and thus over D. We claim that x_1, x_n is a system of parameters over $D(\mathbf{r})$.

Indeed, $(x_1, \ldots, x_n)D(\mathbf{r}) = (x_1, x_n)D(\mathbf{r})$ since, for $2 \leq i \leq n-1$, $x_i = x_n \frac{x_i}{x_n} \in x_n D(\mathbf{r})$ as $\frac{x_i}{x_n} \in D(\mathbf{r})$. Moreover, $\mathfrak{m}D[\mathbf{X}]$ is minimal over $(x_1, \ldots, x_n)D[\mathbf{X}]$, so that since $\mathfrak{m}D[\mathbf{r}]$ is minimal over $(x_1, \ldots, x_n)D[\mathbf{r}]$ and thus (since $D(\mathbf{r})$ is a localization of $D[\mathbf{r}]$) also $\mathfrak{m}D(\mathbf{r})$ is minimal over $(x_1, \ldots, x_n)D(\mathbf{r})$. To conclude, we need to prove that $D(\mathbf{r})$ has dimension 2. If not, then it must have dimension 1 (since $D(\mathbf{r})$ is Noetherian and its maximal ideal is minimal over a 2-generated ideal), and since it is also local its unique localizations are itself and its quotient field. By Lemma 7.3, it would follow that $D_P(\mathbf{r})$ is a field, against the fact that its maximal ideal $PD_P(\mathbf{r})$ is nonzero. Hence x_1, x_n is a system of parameters over $D(\mathbf{r})$.

Applying again the case $n = 2$, we have that x_1/x_n is polynomially independent over $D(\mathbf{r})$; by Proposition 2.9, it follows that $\{x_1/x_n\} \cup \mathbf{r} = \{x_1/x_n, \ldots, x_{n-1}/x_n\}$ is polynomially independent over D. □

Corollary 7.5. *Let D be a Noetherian domain that is not a field. Then,* $\dim_p(K/D) = \dim(D) - 1$.

Proof. Without loss of generality we can suppose that D is local. If $\dim(D) = 1$ then the integral closure $\overline{D}$ of D is a PID, in particular a Prüfer domain, and so $\dim_p(K/D) = 0$ by Corollary 3.3. If $\dim(D) > 1$, then Theorem 7.4 gives a polynomially independent set of cardinality $\dim(D) - 1$, and as in the end of the previous proof $\dim_p(K/D) \leq \dim_v(D) - 1 = \dim(D) - 1$; hence $\dim_p(K/D) = \dim(D) - 1$, as claimed. □

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