WILF'S CONJECTURE FOR NUMERICAL SEMIGROUPS WITH LARGE SECOND GENERATOR

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Abstract. We study Wilf's conjecture for numerical semigroups S such that the second least generator a_2 of S satisfies $a_2 > \frac{c(S)+\mu(S)}{3}$, where $c(S)$ is the conductor and $\mu(S)$ the multiplicity of S. In particular, we show that for these semigroups Wilf's conjecture holds when the multiplicity is bounded by a quadratic function of the embedding dimension.

1. Introduction and preliminaries

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ that contains 0, is closed under addition and such that the complement $\mathbb{N} \setminus S$ is finite. In particular, there is a largest integer not contained in S, which is called the *Frobenius number* of S and is denoted by $F(S)$. The *conductor* of S is defined as $c(S) := F(S) + 1$, and it is the minimal integer x such that $x + \mathbb{N} \subseteq S$. Calculating $F(S)$ is a classical problem (called the *Diophantine Frobenius problem*), introduced by Sylvester [11]; see [8] for a general overview.

Given coprime integers $a_1 < \ldots < a_n$, the numerical semigroup *generated* by $a_1, \ldots, a_n$ is the set

$$\langle a_1, \ldots, a_n \rangle := \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in \mathbb{N}\}.$$

Conversely, if S is a numerical semigroup, there are always a finite number of integers $a_1, \ldots, a_n$ such that $S = \langle a_1, \ldots, a_n \rangle$; moreover, there is a unique minimal set of such integers, whose cardinality, called the *embedding dimension* of S, is denoted by $\nu(S)$. The integer a_1, the smallest minimal generator of S, is called the *multiplicity* of S, and is denoted by $\mu(S)$.

In 1978, Wilf [12] suggested a relationship between the conductor and the embedding dimension of S. More precisely, set

$$L(S) := \{x \in S \mid 0 \leq x < c(S)\}.$$

Wilf hypothesized that the inequality

$$\nu(S)|L(S)| \geq c(S)$$

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holds for every numerical semigroup S; this question is known as *Wilf's conjecture*. The conjecture is still unresolved in the general case, although there have been several partial results: for example, it has been proven that Wilf's conjecture holds when $\nu(S) \leq 3$ [11, 3], when $|\mathbb{N} \setminus S| \leq 60$ [4], when $c(S) \leq 3\mu(S)$ [5, 1] and when $\nu(S) \geq \mu(S)/2$ [10].

In this paper, we study Wilf's conjecture when a_2, the second smallest generator of S, is large, in the sense that

$$a_2 > \frac{c(S) + \mu(S)}{3}.$$

This condition is favorable to the study of Wilf's condition because it implies that the conductor $c(S)$ is not too large with respect to the other parameters of S, and that the embedding dimension $\nu(S)$ is not too small with respect to the multiplicity $\mu(S)$ (see Proposition 2.4). From a technical point of view, the main advantage is that we can split the generators of S into three sets (see Section 2) in a way that make easier to estimate the cardinality of $L(S)$.

The main result of this paper is Theorem 5.2, which says that, for every $\epsilon > 0$, Wilf's conjecture holds (among the semigroups with $a_2 > \frac{c(S)+\mu(S)}{3}$) whenever

$$\mu(S) \leq \frac{8}{25}\nu(S)^2 + \frac{1}{5}\nu(S) - \frac{1}{2} - \epsilon$$

and ν is larger than a quantity $\nu_0(\epsilon)$ depending on ϵ; this could be compared to the results in [7], where Wilf's conjecture is proved (without hypothesis on the second generator) for $\mu(S) \leq c'\nu(S)^{4/3}$, where c' is a constant (see Remark 5.4 for details).

Following the same method of proof, we also show a few variants of this result: we show that for $\nu(S) \geq 10$ we can take $\epsilon = 3/4$ (Proposition 5.3), we prove the conjecture under the additional hypothesis that $c(S) \equiv 0 \mod \mu(S)$ (Proposition 5.5) and we improve the previous inequality to $\mu(S) \leq \frac{4}{9}\nu(S)^2$ provided that we allow a finite number of counterexamples for every (large) value of $\nu(S)$ (Proposition 5.6).

The structure of the paper is at follows. Section 2 is focused on the combinatorial features of semigroups satisfying the condition $a_2 > \frac{c+\mu}{3}$, and in particular on what bounds the condition imposes on the parameters of S. Section 3 estimates the cardinality of $L(S)$ in function of the size of generators of S, first in a general way and then distinguishing six different cases according to a parameter $\theta(S)$ (see Definition 3.3). In Section 4 we present two inequalities that are proved through purely analytic methods. In Section 5, we give a criterion summing up the previous results (Proposition 5.1) and then we prove the main Theorem 5.2 and its variants (Propositions 5.3-5.7).

For general information and results about numerical semigroups, the reader may consult [9].

2. Splitting the generators

From now on, S will be a numerical semigroup, $\mu := \mu(S)$ its multiplicity, $\nu := \nu(S)$ its embedding dimension, and $c := c(S)$ its conductor. We denote by $\mathrm{Ap}(S)$ the *Apéry set* of S with respect to its multiplicity, i.e.,

$$\mathrm{Ap}(S) := \{i \in S \mid i - \mu \notin S\}.$$

We recall that, for every $t \in \{0, \dots, \mu-1\}$, there is a unique $x \in \mathrm{Ap}(S)$ such that $x \equiv t \bmod \mu$; in particular, $\mathrm{Ap}(S)$ has cardinality μ. Note also that, since $F(S) = c(S) - 1$ is the maximal integer not belonging to S, the largest element of $\mathrm{Ap}(S)$ is $F(S) + \mu$, and thus every element of $\mathrm{Ap}(S)$ is strictly smaller than $c + \mu$.

Let now $P := \{a_1, \dots, a_\nu\}$ be the set of minimal generators of S, with $\mu = a_1 < a_2 < \cdots < a_\nu$. We shall always suppose that $a_2 > \frac{c+\mu}{3}$. Since each $x \in P \setminus \{\mu\}$ belongs to $\mathrm{Ap}(S)$, we can subdivide $P \setminus \{\mu\}$ into the following three sets:

$$P_1 := \left\{a \in P \setminus \{\mu\} \mid \frac{1}{3}(c+\mu) < a < \frac{1}{2}(c+\mu)\right\},$$
$$P_2 := \left\{a \in P \setminus \{\mu\} \mid \frac{1}{2}(c+\mu) \le a < \frac{2}{3}(c+\mu)\right\},$$
$$P_3 := \left\{a \in P \setminus \{\mu\} \mid \frac{2}{3}(c+\mu) \le a < c+\mu\right\}.$$

We set $q_i := |P_i|$, for $i \in \{1, 2, 3\}$.

Let $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/\mu\mathbb{Z}$ be the canonical quotient map, and let $A := \pi(P)$, $A_i := \pi(P_i)$. Given two subsets $X, Y \subseteq \mathbb{Z}/\mu\mathbb{Z}$, the *sumset* of X and Y is

$$X + Y := \{x + y \mid x \in X, y \in Y\}.$$

Proposition 2.1. *Let $S = \langle a_1, a_2, \dots, a_\nu\rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then, $\mathbb{Z}/\mu\mathbb{Z} = A \cup (A_1 + A_1) \cup (A_1 + A_2)$.*

Proof. Let $x \in \mathrm{Ap}(S)$, $x \neq 0$. Then, $x < c + \mu$ and x is a sum of elements of $P \setminus \{\mu\}$ (since $x - n\mu \notin S$ for $n > 0$). The sum of three elements of $P \setminus \{\mu\}$ is bigger than $c + \mu$, and thus cannot be equal to x; likewise, x cannot be the sum of two elements of $P_2 \cup P_3$, and it also cannot be the sum of an element of P_1 and an element of P_3. Hence, the unique possibilities are $x \in P$, $x \in P_1 + P_1$, or $x \in P_1 + P_2$. The claim follows by projecting onto $\mathbb{Z}/\mu\mathbb{Z}$. □

The following is a modification of an idea introduced by S. Eliahou in [2].

Definition 2.2. *Let* $(a,b) \in P_1 \times P_2$*. We say that* (a,b) *is an* Apéry pair *if* $a+b \in \mathrm{Ap}(S)$*, and we denote by* Σ *the set of all Apéry pairs.*

A subset $\{(a_i,b_i)\}_{i=1}^n \subseteq \Sigma$ *is* independent *if* $a_i \neq a_j$ *and* $b_i \neq b_j$ *for every* $i \neq j$*; we denote by* σ *the maximal cardinality of an independent set of Apéry pairs.*

We can relate Σ and σ through a graph-theoretic argument; see e.g. [6] for the terminology used in the proof.

Proposition 2.3. *Let* Σ *and* σ *as above. Then,*

$$|\Sigma| \leq \sigma \cdot \max\{q_1, q_2\}$$

Proof. Define a graph G by taking the disjoint union $P_1 \sqcup P_2$ as the set of vertices and Σ as the set of edges. Then, an independent subset of Σ is exactly an independent subset of edges of G, that is, a matching, and σ is exactly the matching number of G.

Since G is a bipartite graph, by König's theorem (see e.g. [6, Theorem 1.1.1]) the matching number of G is equal to the its point covering number, i.e., to the cardinality of the smallest set $S \subseteq V(G)$ such that every edge of G has a vertex in S.

For every $v \in V(G)$, the number of edges incident to v is at most q_1 if $v \in P_2$ and at most q_2 if $v \in P_1$; hence, the point covering number of G is at least $|E(G)|/\max\{q_1,q_2\}$. The claim follows. □

Using this terminology, we now relate quantitatively μ, ν, q_1 and q_2.

Proposition 2.4. *Let* $S = \langle a_1, a_2, \dots, a_\nu \rangle$ *be a numerical semigroup with* $a_2 > \frac{c(S)+\mu(S)}{3}$*. Then:*

(a) $\mu \leq \nu + \dfrac{q_1(q_1+1)}{2} + \sigma \cdot \max\{q_1,q_2\}$*;*
(b) $\mu \leq \nu + \dfrac{q_1(q_1+1)}{2} + q_1q_2$*;*
(c) $\mu \leq \dfrac{1}{2}\nu(\nu+1)$*;*
(d) $q_1 \geq \dfrac{2\nu - 1 - \sqrt{(2\nu+1)^2 - 8\mu}}{2}$.

Proof. (a) By Proposition 2.1, we have

$$\mu \leq |A| + |A_1 + A_1| + |A_1 + A_2|.$$

By definition, $|A| = \nu$, while $|A_1 + A_1| \leq q_1(q_1+1)/2$ by symmetry. If $x \in \mathrm{Ap}(S) \cap (P_1+P_2)$, then $x = a_1 + b_1$ for some Apéry pair $(a_1,b_1) \in \Sigma$; hence,

$$|\mathrm{Ap}(S) \cap (P_1 + P_2)| \leq |\Sigma| \leq \sigma \cdot \max\{q_1,q_2\},$$

with the last inequality coming from Proposition 2.3. Since $|A_1 + A_2| \leq |\mathrm{Ap}(S) \cap (P_1 + P_2)|$ the claim follows by summing the three bounds.

(b) is immediate from (a) and the fact that $\sigma \leq \min\{q_1,q_2\}$.

(c) Since $q_1 + q_2 \leq \nu - 1$, using (b) we have

$$\mu \leq \nu + \frac{q_1(q_1+1)}{2} + q_1 q_2 \leq$$
$$\leq \nu + \frac{q_1(q_1+1)}{2} + q_1(\nu - 1 - q_1) = \nu - \frac{1}{2}q_1^2 + \left(\nu - \frac{1}{2}\right) q_1,$$

and thus

$$q_1^2 - (2\nu - 1)q_1 + 2(\mu - \nu) \leq 0. \tag{1}$$

Therefore, the discriminant of the equation is nonnegative, that is,

$$0 \leq (2\nu - 1)^2 - 8(\mu - \nu) = (2\nu + 1)^2 - 8\mu,$$

or equivalently

$$\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu + \frac{1}{8}.$$

Moreover, since μ and ν are integers, so is $\frac{1}{2}\nu^2 + \frac{1}{2}\nu = \frac{\nu(\nu+1)}{2}$, and thus we can discard the $\frac{1}{8}$.

(d) The inequality (1) holds for

$$\frac{2\nu - 1 - \sqrt{(2\nu+1)^2 - 8\mu}}{2} \leq q_1 \leq \frac{2\nu - 1 + \sqrt{(2\nu+1)^2 - 8\mu}}{2}.$$

The claim follows. □

Remark 2.5. The bound in Proposition 2.4(d) may actually be negative: however, if $q_1 = 0$ then part (a) shows that $\mu \leq \nu$, and thus $\mu = \nu$. In this case, S is of maximal embedding dimension and Wilf's conjecture holds by [3, Theorem 20 and Corollary 2].

3. Estimates on $|L(S)|$

The goal of this section is to estimate the cardinality of $L := L(S)$.

Lemma 3.1. *Let x, y, b, p be real numbers, with $p > 0$ and $x < y$, and let $A := b + p\mathbb{Z} := \{b + pn \mid n \in \mathbb{Z}\}$. Then:*

(a) $|A \cap [x, y)| \geq \left\lfloor \frac{y-x}{p} \right\rfloor$;

(b) if $x \in A$ and $y \notin A$, then $|A \cap [x, y)| = \left\lfloor \frac{y-x}{p} \right\rfloor + 1$.

Proof. Let $k := \left\lfloor \frac{y-x}{p} \right\rfloor$. Then,

$$x + kp \leq x + \frac{y-x}{p} \cdot p = y;$$

hence, the k sets $[x, x+p), [x+p, x+2p), \dots, [x+(k-1)p, x+kp)$ are disjoint subintervals of $[x, y)$. In each $[x+ip, x+(i+1)p)$ there is exactly one element of A; hence, $|A \cap [x, y)| \geq k$.

Moreover, if $x \in A$ then $x + kp \in A$; since $y \notin A$, then $x + kp \neq y$, and thus the interval $[x + kp, y)$ is nonempty and contains exactly one element of A (namely, $x + kp$). Hence, $|A \cap [x, y)| = k + 1$. ☐

Proposition 3.2. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then,*

(2)
$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor (1+\sigma) + \left(\left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 \right) (q_1 - \sigma) + \left(\left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 \right) (q_2 - \sigma).$$

Proof. If x is an integer, let

$$L_x := \{a \in L(S) \mid a \equiv x \bmod \mu\}.$$

Clearly, L_x and L_y are disjoint if $x \not\equiv y \bmod \mu$. Hence,

$$|L(S)| = \sum_{x \in \mathrm{Ap}(S)} |L_x| \geq |L_0| + \sum_{x \in P_1 \cup P_2} |L_x|.$$

If $x \in \mathrm{Ap}(S)$, then, $L_x = (x + \mu\mathbb{Z}) \cap [x, c)$; by Lemma 3.1(a) we have $|L_x| \geq \left\lfloor \frac{c - x}{\mu} \right\rfloor$. In particular, $|L_0| \geq \left\lfloor \frac{c}{\mu} \right\rfloor$.

Take an independent set $\{(a_t, b_t)\}_{i=1}^{\sigma}$ of Apéry pairs of maximal cardinality, and write $P_1 = \{a_1, \dots, a_\sigma, c_1, \dots, c_r\}$, $P_2 = \{b_1, \dots, b_\sigma, d_1, \dots, d_s\}$. Then,

$$\sum_{x \in P_1 \cup P_2} |L_x| \geq \sum_{t=1}^{\sigma} (|L_{a_i}| + |L_{b_i}|) + \sum_{j=1}^{r} |L_{c_j}| + \sum_{k=1}^{s} |L_{d_k}|.$$

Suppose (a, b) is an Apéry pair. By Lemma 3.1(a),

(3)
$$\begin{aligned} |L_a| + |L_b| &= \left\lfloor \frac{c - a}{\mu} \right\rfloor + \left\lfloor \frac{c - b}{\mu} \right\rfloor \geq \\ &\geq \frac{2c - (a + b)}{\mu} - 2 > \frac{c - \mu}{\mu} - 2 = \frac{c}{\mu} - 3. \end{aligned}$$

Moreover, $|L_a|$ and $|L_b|$ are both integers, and thus

$$|L_a| + |L_b| \geq \left\lfloor \frac{c}{\mu} \right\rfloor - 2.$$

On the other hand, if $x = c_j$ for some j then $x \in P_1$ and so

$$|L_x| \geq \left\lfloor \frac{c - \frac{1}{2}(c + \mu)}{\mu} \right\rfloor = \left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor,$$

while if $x = d_k$ for some k then $x \in P_2$ and thus

$$|L_x| \geq \left\lfloor \frac{c - \frac{2}{3}(c + \mu)}{\mu} \right\rfloor \geq \left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor.$$

Summing everything, we have

$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor + \sigma \left(\left\lfloor \frac{c}{\mu} \right\rfloor - 2 \right) + \left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor (q_1 - \sigma) + \left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor (q_2 - \sigma).$$

Applying Lemma 3.1(b), for every $x \in \{0\} \cup P_1 \cup P_2$, except possibly one (namely, the x such that $c \equiv x \bmod \mu$), there is a further element in $L_x \cap [x, c)$; hence, we can add $q_1 + q_2$ to the quantity on the right hand side. Distributing this quantity (adding 2σ to the second summand, $q_1 - \sigma$ to the third one and $q_2 - \sigma$ to the last one) and putting together the first two summand we have our claim. □

Due to the presence of the floor functions, we can get better estimates on the right hand side of (2) if we analyze it according to the integral part of c/μ. To do so, we introduce the following parameter.

Definition 3.3. *Let S be a numerical semigroup. Let k be the largest integer such that $\frac{c(S)}{\mu(S)} - (6k-1) < 6$: then, we set*

$$\theta(S) := \frac{c(S) - (6k-1)\mu(S)}{\mu(S)}.$$

It is immediate from the definition that $\theta(S)$ is a rational number and belongs to $[0, 6)$.

Proposition 3.4. *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$, and suppose that $c(S) > 3\mu(S)$. Let*

$$l := \begin{cases} 5 & \text{if } \theta(S) \in [0,4) \\ -1 & \text{if } \theta(S) \in [4,6). \end{cases}$$

and define

$$\begin{cases} \alpha := 1 - \dfrac{1}{l + \lfloor \theta \rfloor + 1}(\theta - \lfloor \theta \rfloor), \\ \beta := \min\left\{ \dfrac{1}{2}, \dfrac{1}{2} - \dfrac{1}{l + \lfloor \theta \rfloor + 1}\left(\dfrac{\theta}{2} - \left\lfloor \dfrac{\theta}{2} \right\rfloor - \dfrac{1}{2} \right) \right\}, \\ \gamma := \min\left\{ \dfrac{1}{3}, \dfrac{1}{3} - \dfrac{1}{l + \lfloor \theta \rfloor + 1}\left(\dfrac{\theta}{3} - \left\lfloor \dfrac{\theta}{3} \right\rfloor - \dfrac{1}{3} \right) \right\}, \end{cases}$$

Then,

$$\frac{\nu(S)|L(S)|}{c(S)} \geq \frac{\nu(S)}{\mu(S)}[\alpha(1+\sigma) + \beta(q_1 - \sigma) + \gamma(q_2 - \sigma)].$$

Note that the hypothesis $c(S) > 3\mu(S)$ is not really restrictive in our context, since, by [1], Wilf's conjecture holds when $c \leq 3\mu$.

Proof. Let $\theta := \theta(S)$, $\nu := \nu(S)$, $\mu := \mu(S)$, $c := c(S)$, $L := L(S)$. By definition, $c = (6k-1)\mu + \theta\mu$. Then,

$$\left\lfloor \frac{c}{\mu} \right\rfloor = \left\lfloor \frac{(6k-1)\mu + \theta\mu}{\mu} \right\rfloor = 6k - 1 + \lfloor \theta \rfloor = \frac{c}{\mu} - (\theta - \lfloor \theta \rfloor);$$

analogously,

$$\left\lfloor \frac{1}{2}\frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 = \frac{c}{2\mu} + \frac{1}{2} - \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor \right)$$

and

$$\left\lfloor \frac{1}{3}\frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 = \frac{c}{3\mu} + \frac{1}{3} - \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor \right).$$

Multiplying (2) by $\frac{\nu}{c}$, we obtain

$$\begin{aligned} \frac{\nu|L|}{c} \geq \frac{\nu}{\mu} & \left[1 - \frac{\mu}{c}(\theta - \lfloor \theta \rfloor) \right] (1+\sigma) + \\ & + \frac{\nu}{\mu} \left[\frac{1}{2} - \frac{\mu}{c} \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \right] (q_1 - \sigma) + \\ & + \frac{\nu}{\mu} \left[\frac{1}{3} - \frac{\mu}{c} \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor - \frac{1}{3} \right) \right] (q_2 - \sigma). \end{aligned} \tag{4}$$

Since $c > 3\mu$, we have $c \geq (l+\theta)\mu$; equivalently, $\dfrac{\mu}{c} \leq \dfrac{1}{l+\theta} \leq \dfrac{1}{l+\lfloor\theta\rfloor+1}$. Since $\theta \geq \lfloor \theta \rfloor$, it follows that

$$1 - \frac{\mu}{c}(\theta - \lfloor\theta\rfloor) \geq 1 - \frac{1}{l+\lfloor\theta\rfloor+1}(\theta - \lfloor\theta\rfloor),$$

and the right hand side is exactly α.

If $\frac{\theta}{2} - \left\lfloor\frac{\theta}{2}\right\rfloor - \frac{1}{2} \leq 0$, then the coefficient of $q_1 - \sigma$ in (4) is at least $\frac{1}{2}$; otherwise, as above,

$$\frac{1}{2} - \frac{\mu}{c}\left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \geq \frac{1}{2} - \frac{1}{l+\lfloor\theta\rfloor+1}\left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right).$$

Thus the coefficient of $q_1 - \sigma$ is larger than β. Analogously we obtain that the coefficient of $q_2 - \sigma$ in (4) is at least γ. Since, by definition, $1+\sigma$, $q_1 - \sigma$ and $q_2 - \sigma$ are always nonnegative, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}[\alpha(1+\sigma) + \beta(q_1-\sigma) + \gamma(q_2-\sigma)],$$

as claimed. □

Proposition 3.5. *Let α, β, γ, l be as in Proposition 3.4. Then:*

if $\lfloor\theta\rfloor = 0$: $\alpha \geq \frac{5}{6}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{3}$;

if $\lfloor\theta\rfloor = 1$: $\alpha \geq \frac{6}{7}$; $\beta \geq \frac{3}{7}$; $\gamma \geq \frac{2}{7}$;

if $\lfloor\theta\rfloor = 2$: $\alpha \geq \frac{7}{8}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{4}$;

if $\lfloor\theta\rfloor = 3$: $\alpha \geq \frac{8}{9}$; $\beta \geq \frac{4}{9}$; $\gamma \geq \frac{1}{3}$;

if $\lfloor\theta\rfloor = 4$: $\alpha \geq \frac{3}{4}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{4}$;

if $\lfloor\theta\rfloor = 5$**:** $\alpha \geq \frac{4}{5}$; $\quad \beta \geq \frac{2}{5}$; $\quad \gamma \geq \frac{1}{5}$;

Proof. The estimates for α follow immediately from Proposition 3.4 and the fact that $\theta - \lfloor\theta\rfloor \leq 1$.

If $\lfloor\theta\rfloor \equiv 0 \bmod 2$, then $\frac{\theta}{2} - \lfloor\frac{\theta}{2}\rfloor \leq \frac{1}{2}$; thus,

$$\frac{1}{2} - \left(\frac{\theta}{2} - \left\lfloor\frac{\theta}{2}\right\rfloor - \frac{1}{2}\right) \geq \frac{1}{2} - 0 = \frac{1}{2},$$

and so $\beta = \frac{1}{2}$. On the other hand, if $\lfloor\theta\rfloor \equiv 1 \bmod 2$ then $\frac{\theta}{2} - \lfloor\frac{\theta}{2}\rfloor \leq 1$ and so

$$\beta \geq \frac{1}{2} - \frac{1}{l + \lfloor\theta\rfloor + 1}\left(1 - \frac{1}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{l + \lfloor\theta\rfloor + 1}\right);$$

the claim is obtained substituting l and $\lfloor\theta\rfloor$ with their actual values.

Similarly, if $\lfloor\theta\rfloor \equiv 0 \bmod 3$ then $\frac{\theta}{3} - \lfloor\frac{\theta}{3}\rfloor \leq \frac{1}{3}$ and so $\gamma = \frac{1}{3}$. On the other hand, if $\lfloor\theta\rfloor \equiv 1 \bmod 3$ then $\frac{\theta}{3} - \lfloor\frac{\theta}{3}\rfloor \leq \frac{2}{3}$ and thus

$$\gamma \geq \frac{1}{3} - \frac{1}{l + \lfloor\theta\rfloor + 1}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3}\left(1 - \frac{1}{l + \lfloor\theta\rfloor + 1}\right).$$

If $\lfloor\theta\rfloor \equiv 2 \bmod 3$, the same reasoning gives a similar estimate, but with $\frac{2}{l+\lfloor\theta\rfloor+1}$ instead of $\frac{1}{l+\lfloor\theta\rfloor+1}$. Substituting the actual values of l and $\lfloor\theta\rfloor$ we get the values in the statement. □

Corollary 3.6. *Let $\alpha_i, \beta_i, \gamma_i$ be the values found in Proposition 3.5 when $\lfloor\theta\rfloor = i$, and let*

$$\ell_i(x, y, z) := \alpha_i(1 + z) + \beta_i(x - z) + \gamma_i(y - z).$$

If $i \leq 3$ and $j \geq 4$, then $\ell_i(x, y, z) \geq \ell_j(x, y, z)$ for all $x, y, z \in \mathbb{R}$ such that $x, y, z \geq 0$ and $z \leq \min\{x, y\}$.

Proof. With i, j as defined, we have $\alpha_i \geq \alpha_j$, $\beta_i \geq \beta_j$ and $\gamma_i \geq \gamma_j$; the fact that $1 + z$, $x - z$ and $y - z$ are nonnegative now guarantees that $\ell_i(x, y, z) \geq \ell_j(x, y, z)$. □

4. Some inequalities

We collect in this section some inequalities, obtained through analytic methods, that we will use in the proof of our main result.

Definition 4.1. *Let $\rho \geq 1$ be a real number. Set*

$$\mathcal{I}(\rho) := \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 0, \frac{x(x+1)}{2} + xy = \rho\right\}$$

and

$$\mathcal{A}(\rho) := \left\{(x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 0, \frac{x(x+1)}{2} + xy \geq \rho\right\}.$$

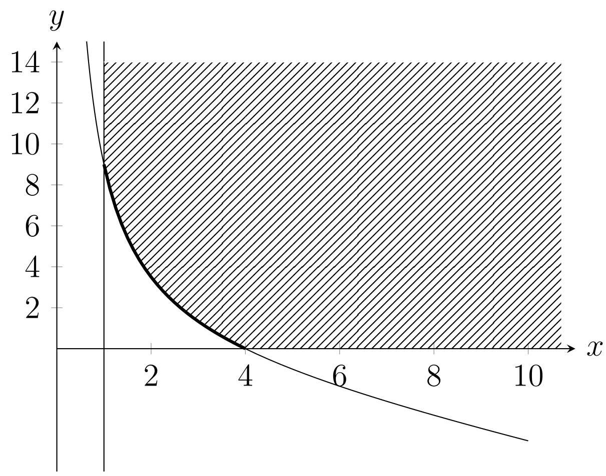

FIGURE 1. The line $\mathcal{I}(10)$ (the bolded subset of the hyperbola) and the region $\mathcal{A}(10)$ (shaded).

Graphically, the set $\left\{(x,y)\in\mathbb{R}^2 \mid x>0, \frac{x(x+1)}{2}+xy=\rho\right\}$ is a branch of a hyperbola having the y-axis as a vertical asymptote and crossing the x-axis at $x_0 := \frac{-1+\sqrt{1+8\rho}}{2} \geq 1$ (since $\rho \geq 1$). In particular, $\mathcal{I}(\rho)$ is the subset of this branch defined by x varying between 1 and x_0, while $\mathcal{A}(\rho)$ is the part of the first quadrant that is above $\mathcal{I}(\rho)$; see Figure 1.

Lemma 4.2. *Let $f(x,y) := \alpha+\beta x+\gamma y$, where α,β,γ are positive real numbers such that $2\beta\geq\gamma$. For every $(x,y)\in\mathcal{A}(\rho)$, we have*

$$f(x,y)\geq \alpha-\frac{\gamma}{2}+\sqrt{2\gamma(2\beta-\gamma)}\sqrt{\rho}.$$

Proof. For every $(x,y)\in\mathcal{A}(\rho)$ there is an $(x',y')\in\mathcal{I}(\rho)$ such that $x'\leq x$ and $y'\leq y$. Since f is a linear function with positive coefficients, the minimum of f on $\mathcal{A}(\rho)$ must belong to $\mathcal{I}(\rho)$ (and it exists since $\mathcal{I}(\rho)$ is compact).

Let (x_0,y_0) be the point of minimum of f on $\mathcal{I}(\rho)$. By Lagrange multipliers,

$$\begin{cases}\partial_x f(x_0,y_0) = x_0+\frac{1}{2}+y_0 = \beta\lambda\\ \partial_y f(x_0,y_0) = x_0 = \gamma\lambda\end{cases}$$

for some $\lambda\in\mathbb{R}$; imposing $(x_0,y_0)\in\mathcal{I}(\rho)$ we have

$$\begin{aligned}\rho &= x_0\left(\frac{1}{2}x_0+\frac{1}{2}+y_0\right)=\\ &= \partial_y f(x_0,y_0)\cdot\left(\partial_x f(x_0,y_0)-\frac{1}{2}\partial_y f(x_0,y_0)\right) = \frac{\gamma(2\beta-\gamma)}{2}\lambda^2\end{aligned}$$

and thus

$$\lambda=\sqrt{\frac{2}{\gamma(2\beta-\gamma)}}\sqrt{\rho}.$$

Substituting in f, we have

$$\begin{aligned} f(x_0, y_0) &= \alpha + \beta\gamma\lambda + \gamma\left[(\beta-\gamma)\lambda - \frac{1}{2}\right] = \\ &= \alpha - \frac{\gamma}{2} + \gamma(\beta+\beta-\gamma)\sqrt{\frac{2}{\gamma(2\beta-\gamma)}}\sqrt{\rho} = \\ &= \alpha - \frac{\gamma}{2} + \sqrt{2\gamma(2\beta-\gamma)}\sqrt{\rho}, \end{aligned}$$

as claimed. □

Lemma 4.3. *Let ζ, ξ be real numbers such that $\zeta \le 1$, and let $\lambda := \frac{4(1-\zeta)}{\xi^2}$. If x, y are positive real numbers such that*

$$\begin{cases} 2y \le x < \xi^2 y^2 - (2\zeta - 1)y - \dfrac{(1-\zeta)^2}{\xi^2} + \dfrac{\lambda^3\xi^2}{32}\cdot\dfrac{1}{y}\left(\dfrac{y}{y-\lambda}\right)^{5/2}, \\ y > \max\left\{\lambda, \dfrac{(\zeta-2)^2}{\xi^2}\right\} \end{cases} \tag{5}$$

then

$$\zeta - \xi\sqrt{x-y} \ge \frac{x}{y}. \tag{6}$$

Proof. Since $y > 0$, the inequality (6) is equivalent to

$$\xi y\sqrt{x-y} \ge x - \zeta y,$$

and squaring both sides (which makes since since $x - \zeta y \ge x - y > 0$) this is equivalent to

$$\xi^2 y^2(x-y) \ge x^2 - 2\zeta xy + \zeta^2 y^2,$$

that is,

$$x^2 - (2\zeta y + \xi^2 y^2)x + \zeta^2 y^2 + \xi^2 y^3 \le 0. \tag{7}$$

The claim will follow once we prove that the roots of (7) exist and lie outside the interval defined by the first inequality in (5).

For $x = 2y$, the left hand side of (7) becomes

$$\begin{aligned} &4y^2 - (2\zeta y + \xi^2 y^2)(2y) + \zeta^2 y^2 + \xi^2 y^3 = \\ &= (4 - 4\zeta + \zeta^2)y^2 + (\xi^2 - 2\xi^2)y^3 = \\ &= y^2[(\zeta-2)^2 - \xi^2 y] \end{aligned}$$

which is negative by the second hypothesis of (5).

The larger root of the equation (7) is equal to

$$\begin{aligned} x_+ &:= \frac{(2\zeta y + \xi^2 y^2) + \sqrt{y^2\xi^2[4(\zeta-1)y + \xi^2 y^2]}}{2} = \\ &= \frac{2\zeta y + \xi^2 y^2 + \xi^2 y^2\sqrt{1 - \frac{\lambda}{y}}}{2}. \end{aligned}$$

By hypothesis, $\lambda < y$, and thus we can expand $\sqrt{1-\frac{\lambda}{y}}$ as a Taylor series in $\frac{\lambda}{y}$ around 0. Then,

$$\xi^2 y^2 \sqrt{1-\frac{\lambda}{y}} = \xi^2 y^2 \left(1 - \frac{1}{2}\cdot\frac{\lambda}{y} - \frac{1}{8}\frac{\lambda^2}{y^2} + R_2(\lambda/y)\right) =$$
$$= \xi^2 y^2 - 2(1-\zeta)y - \frac{2(1-\zeta)^2}{\xi^2} + \xi^2 y^2 R_2(\lambda/y),$$

where R_2 is the remainder, and thus

$$x_+ = \frac{2\zeta y + \xi^2 y^2 + \xi^2 y^2 - 2(1-\zeta)y - \frac{2(1-\zeta)^2}{\xi^2} + \xi^2 y^2 R_2(\lambda/y)}{2} =$$
$$= \xi^2 y^2 + (2\zeta - 1)y - \frac{(1-\zeta)^2}{\xi^2} + \frac{\xi^2 y^2 R_2(\lambda/y)}{2}.$$

Since $\frac{\mathrm{d}^3}{\mathrm{d}z^3}\sqrt{1-z} = -\frac{3}{8}\frac{1}{(1-z)^{5/2}}$, by Taylor's theorem there is an $\eta \in [0, \lambda/y]$ such that

$$|\xi^2 y^2 R_2(\lambda/y)| = \xi^2 y^2 \frac{1}{6}\left(\frac{\lambda}{y}\right)^3 \frac{3}{8}\frac{1}{(1-\eta)^{5/2}}.$$

The function $z \mapsto \frac{3}{8}\frac{1}{(1-z)^{5/2}}$ is increasing in $[0,1)$; hence, to bound above the remainder we can take $\eta = \lambda/y$. We obtain

$$|\xi^2 y^2 R_2(\lambda/y)| \leq \frac{\lambda^3 \xi^2}{16}\frac{1}{(1-(\lambda/y))^{5/2}},$$

from which the claim follows. ☐

5. Wilf's conjecture for large second generator

The estimates of the previous section are connected to Wilf's conjecture in the following way.

Proposition 5.1. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a semigroup of multiplicity μ, embedding dimension ν and such that $a_2 > \frac{c+\mu}{3}$. Let α, β, γ be defined as in Proposition 3.4, and let*

$$\ell(x,y,z) := \alpha(1+z) + \beta(x-z) + \gamma(y-z).$$

If, for every $(x,y) \in \mathcal{A}(\mu-\nu)$, we have

$$\begin{cases} \ell(x,y,x) \geq \mu/\nu, \\ \ell(x,y,y) \geq \mu/\nu, \end{cases}$$

then Wilf's conjecture holds for S.

Proof. By Proposition 3.4, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\ell(q_1, q_2, \sigma).$$

By Proposition 2.4(a), the triple (q_1, q_2, σ) satisfies the inequality

$$\frac{q_1(q_1+1)}{2} + \sigma \cdot \max\{q_1, q_2\} \geq \mu - \nu.$$

We distinguish two cases.

If $q_1 \geq q_2$, then the previous inequality becomes $\frac{q_1(q_1+1)}{2} + \sigma q_1 \geq \mu - \nu$, that is, $(q_1, \sigma) \in \mathcal{A}(\mu - \nu)$. Since the coefficients α, β, γ are positive and $q_2 \geq \sigma$, we have

$$\ell(q_1, q_2, \sigma) \geq \ell(q_1, \sigma, \sigma) \geq \frac{\mu}{\nu}$$

by hypothesis, and thus

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\frac{\mu}{\nu} = 1.$$

Thus, $\nu|L| \geq c$, i.e., Wilf's conjecture holds for S.

Suppose that $q_1 \leq q_2$, and consider the subset of the three-dimensional space

$$\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, z \leq x \leq y, \frac{x(x+1)}{2} + yz \geq \mu - \nu\right\}.$$

Then, $(q_1, q_2, \sigma) \in \Omega$. By definition, ℓ is a linear function with positive coefficients. Thus, as in the proof of Lemma 4.2, the point of minimum of ℓ on Ω belongs to the boundary

$$\Delta := \left\{(x, y, z) \mid x, y, z \geq 0, z \leq x \leq y, \frac{x(x+1)}{2} + yz = \mu - \nu\right\}.$$

Let also

$$\Delta' := \left\{(x, y, z) \mid x, y, z \geq 0, \frac{x(x+1)}{2} + yz = \mu - \nu\right\}.$$

The point of minimum (x_0, y_0, z_0) of ℓ on Δ', by Lagrange multipliers, must satisfy

$$\begin{cases} \partial_x \ell(x_0, y_0, z_0) = x_0 + \frac{1}{2} = \beta\lambda \\ \partial_y \ell(x_0, y_0, z_0) = z_0 = \gamma\lambda \\ \partial_z \ell(x_0, y_0, z_0) = y_0 = (\alpha - \beta - \gamma)\lambda. \end{cases}$$

for some λ. For every choice of $\lfloor\theta\rfloor$, we have $\gamma > \alpha - \beta - \gamma$; hence, we have $z_0 > y_0$. Since $z \leq y$ for all points of Ω, the point of minimum of ℓ on Δ must satisfy another boundary condition, that is, either $x_0 = z_0$ or $x_0 = y_0$. In the former case, $(x_0, y_0) \in \mathcal{I}(\mu - \nu)$, and thus as in the previous case $\ell(x_0, y_0, x_0) \geq \mu/\nu$ by hypothesis.

On the other hand, if $x_0 = y_0$ then $(x_0, z_0) \in \mathcal{I}(\mu - \nu)$, and thus $\ell(x_0, y_0, z_0) = \ell(x_0, x_0, z_0)$. However,

$$\ell(x_0, x_0, z_0) = \alpha(1+z_0) + (\beta+\gamma)(x_0 - z_0) \geq \alpha(1+z_0) + \beta(x_0 - z_0) = \ell(x_0, z_0, z_0)$$

since $x_0 \geq z_0$ and $\gamma > 0$. By hypothesis, $\ell(x_0, z_0, z_0) \geq \mu/\nu$, and thus also $\ell(x_0, x_0, z_0) \geq \mu/\nu$. In both cases, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\ell(q_1, q_2, \sigma) \geq \frac{\nu}{\mu}\frac{\mu}{\nu} = 1,$$

and thus Wilf's conjecture holds for S. □

Theorem 5.2. *For every $\epsilon > 0$ there is a $\nu_0(\epsilon)$ such that, if $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ is a numerical semigroup such that:*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu(S) = \nu \geq \nu_0(\epsilon)$, *and*
- $\mu(S) \leq \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \epsilon$,

then S satisfies Wilf's conjecture.

Proof. If $\mu \leq 3\nu$, then Wilf's conjecture holds by [10]. Thus, we can suppose $\mu > 3\nu$.

Let $\ell(x, y, z) := \alpha(1 + z) + \beta(x - z) + \gamma(y - z)$, where α, β, γ are defined as in Proposition 3.4. By Proposition 5.1, we need to show that $\ell(x, y, x) \geq \mu/\nu$ and $\ell(x, y, y) \geq \mu/\nu$ for all $(x, y) \in \mathcal{A}(\mu - \nu)$.

Let $f(x, y) := A+Bx+Cy$ be either $\ell(x, y, x)$ or $\ell(x, y, y)$. By Lemma 4.2, we have

$$f(x, y) \geq \left(A - \frac{C}{2}\right) + \sqrt{2C(2B - C)}\sqrt{\mu - \nu}.$$

Set $\zeta := A - \frac{C}{2}$ and $\xi := \sqrt{2C(2B - C)}$. By Lemma 4.3, for every $\epsilon > 0$ and for large enough ν the inequality $f(x, y) \geq \mu/\nu$ holds as long as

$$\mu < \xi^2\nu^2 + (2\zeta - 1)\nu - \frac{(1 - \zeta)^2}{\xi^2} - \epsilon \tag{8}$$

since the last term of the first equation of (5) goes to 0 as ν goes to infinity.

Since we are interested in the asymptotic worst case for the right hand side of (8), we need to take the smallest ξ^2 form the possible ones. If $f(x, y) = \ell(x, y, x)$, then we have $f(x, y) = \alpha + (\alpha - \gamma)x + \gamma y$ and thus

$$\xi^2 = 2\gamma(2\alpha - 2\gamma - \gamma) = 2\gamma(2\alpha - 3\gamma);$$

on the other hand, if $f(x, y) = \ell(x, y, y)$ then $f(x, y) = \alpha+\beta x+(\alpha-\beta)y$ and thus

$$\xi^2 = 2(\alpha - \beta)(2\beta - (\alpha - \beta)) = 2(\alpha - \beta)(3\beta - \alpha).$$

Substituting the α, β and γ with the estimates found in Proposition 3.5 (and note that, by Corollary 3.6, it would be enough to consider the cases $\lfloor\theta\rfloor = 4$ and $\lfloor\theta\rfloor = 5$), the minimum among these values is $8/25$, which happens for $\lfloor\theta\rfloor = 5$ and for the case $\ell(x, y, y)$ (see Table 1). Therefore, for large ν, if μ is smaller than the right hand side of (8) when ξ and ζ are calculated in this case then they are smaller in every case. Making the substitution we have our claim. □

	$\ell(x,y,x)$	$\ell(x,y,y)$
$\lfloor\theta\rfloor = 0$	$4/9$	$4/9$
$\lfloor\theta\rfloor = 1$	$24/49$	$18/49$
$\lfloor\theta\rfloor = 2$	$1/2$	$15/32$
$\lfloor\theta\rfloor = 3$	$14/27$	$32/81$
$\lfloor\theta\rfloor = 4$	$3/8$	$3/8$
$\lfloor\theta\rfloor = 5$	$2/5$	$8/25$

TABLE 1. Values of ξ^2 for $f(x,y) = \ell(x,y,x)$ and $f(x,y) = \ell(x,y,y)$.

A more explicit version is the following.

Proposition 5.3. *Let $S = \langle a_1, a_2, \ldots, a_\nu\rangle$ be a numerical semigroup with $\nu(S) = \nu \geq 10$. If $a_2 > \frac{c(S)+\mu(S)}{3}$ and*

$$\mu(S) \leq \frac{8}{25}\nu(S)^2 + \frac{1}{5}\nu(S) - \frac{5}{4},$$

then S satisfies Wilf's conjecture.

Proof. As in the proof of Theorem 5.2, we can suppose $\mu > 3c$.

Following the proof of Theorem 5.2, let $\ell(x,y,z) := \alpha(1+z)+\beta(x-z)+\gamma(y-z)$, where α,β,γ are as in Proposition 3.4. By Corollary 3.6, it is enough to consider the cases $(\alpha,\beta,\gamma) = \left(\frac{3}{4},\frac{1}{2},\frac{1}{4}\right)$ and $(\alpha,\beta,\gamma) = \left(\frac{4}{5},\frac{2}{5},\frac{1}{5}\right)$.

We need to consider four different $f(x,y) := A + Bx + Cy$:

- $\lfloor\theta\rfloor = 4$ and $f(x,y) = \ell(x,y,x)$: then, $A = \frac{3}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{4}$;
- $\lfloor\theta\rfloor = 4$ and $f(x,y) = \ell(x,y,y)$: then, $A = \frac{3}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{4}$;
- $\lfloor\theta\rfloor = 5$ and $f(x,y) = \ell(x,y,x)$: then, $A = \frac{4}{5}$, $B = \frac{3}{5}$, $C = \frac{1}{5}$;
- $\lfloor\theta\rfloor = 5$ and $f(x,y) = \ell(x,y,y)$: then, $A = \frac{4}{5}$, $B = \frac{2}{5}$, $C = \frac{2}{5}$.

Set $\xi := \sqrt{2C(2B-C)}$, $\zeta := A - \frac{C}{2}$ and $\lambda := \frac{4(1-\zeta)}{\xi^2}$. By Lemmas 4.2 and 4.3, we have $f(x,y) \geq \mu/\nu$ on $\mathcal{A}(\mu-\nu)$ if

$$\mu < \xi^2\nu^2 + (2\zeta-1)\nu - \frac{(1-\zeta)^2}{\xi^2} - \frac{\lambda^3\xi^2}{32}\cdot\frac{1}{\nu}\left(\frac{\nu}{\nu-\lambda}\right)^{5/2}.$$

Substituting the values of ξ,ζ,λ of each case and using $\nu \geq 10$ we have that $f(x,y) \geq \mu/\nu$ under the following conditions:

- if $\lfloor\theta\rfloor = 4$, when

$$\mu < \frac{3}{8}\nu^2 + \frac{1}{4}\nu - \frac{3}{8} - \frac{3}{4}\cdot\frac{1}{10}\left(\frac{10}{6}\right)^{5/2};$$

- if $\lfloor\theta\rfloor = 5$ and $f(x,y) = \ell(x,y,x)$, when
$$\mu < \frac{2}{5}\nu^2 + \frac{2}{5}\nu - \frac{9}{40} - \frac{27}{80}\cdot\frac{1}{10}\left(\frac{10}{7}\right)^{5/2};$$
- if $\lfloor\theta\rfloor = 5$ and $f(x,y) = \ell(x,y,y)$, when
$$\mu < \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \frac{5}{4}\cdot\frac{1}{10}\left(\frac{10}{5}\right)^{5/2}.$$

Among these, the worst case (for all coefficients) is the last one, and thus it is enough to consider that one. We have
$$\frac{5}{4}\cdot\frac{1}{10}\left(\frac{10}{5}\right)^{5/2} = \frac{5\cdot 4\sqrt{2}}{40} = \frac{\sqrt{2}}{2} < \frac{3}{4},$$
and thus $f(x,y) \geq \mu/\nu$ if
$$\mu < \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \frac{3}{4}.$$
By Proposition 5.1, Wilf's conjecture holds in this case, as claimed. □

Remark 5.4. Let $\rho := \lceil\frac{\mu}{\nu}\rceil$. In [7], it is shown that Wilf's conjecture holds when $\mu \geq \frac{f(\rho)}{g(\rho)}$, where f is a polynomial of degree 5 and g a polynomial of degree 1, and if all prime factors of μ are at least equal to ρ. Asymptotically, the first condition can be rephrased as $\mu \geq c\rho^4$, where c is a constant. Since $\rho \in [\mu/\nu, \mu/\nu + 1)$, this is equivalent to
$$\mu \geq c'\left(\frac{\mu}{\nu}\right)^4 \Longrightarrow \mu \leq c'\nu^{4/3}$$
for some constant c'.

Written in this way, this result can be compared with Theorem 5.2: in the latter, we are able to increase the exponent of ν from 4/3 to 2, at the price of adding the hypothesis $a_2 > \frac{c+\mu}{3}$.

To conclude the paper, we give three variants of Theorem 5.2 that can be proved with a similar argument. The first one looks at case $c \equiv 0 \bmod \mu$, the second one strengthens the coefficient $\frac{8}{25}$ and the third one weakens Wilf's conjecture.

Proposition 5.5. *If $S = \langle a_1, a_2, \dots, a_\nu\rangle$ is a numerical semigroup such that*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu(S) \geq 10$ *and*
- $c \equiv 0 \bmod \mu$,

then S satisfies Wilf's conjecture.

Proof. Let α, β, γ, l be as in Proposition 3.4. Since $c \equiv 0 \bmod \mu$, the number θ is an integer. Thus, α is always equal to 1; likewise, β is always equal to $\frac{1}{2}$, since $\frac{\theta}{2} - \left\lfloor\frac{\theta}{2}\right\rfloor$ is either 0 or $\frac{1}{2}$. If θ is equivalent to 0 or 1 modulo 3, then in the same way $\gamma = \frac{1}{3}$; on the other hand, if $\theta = 2$ we have

$$\gamma = \frac{1}{3} - \frac{1}{8}\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{7}{24},$$

while if $\theta = 5$ then

$$\gamma = \frac{1}{3} - \frac{1}{5}\left(\frac{2}{3} - 1 - \frac{1}{3}\right) = \frac{4}{15}.$$

Since the minimal γ is $\frac{4}{15}$, we have always $\ell(x,y,z) \geq (1+z) + \frac{1}{2}(x - z) + \frac{4}{15}(y-z)$.

Let $f(x,y) := \ell(x,y,y)$. Then, $f(x,y) = 1 + \frac{1}{2}x + \frac{1}{2}y$. Let $A := 1$, $B := \frac{1}{2} =: C$. As in the proof of Theorem 5.2, define:

$$\begin{cases} \xi := \sqrt{2C(2B-C)} = \sqrt{\dfrac{1}{2}}, \\ \zeta := A - \dfrac{C}{2} = \dfrac{3}{4}, \\ \lambda := \dfrac{4(1-\zeta)}{\xi^2} = 2. \end{cases}$$

By Lemma 4.3, the inequality $f(x,y) \geq \mu/\nu$ holds as long as

$$\mu < \xi^2\nu^2 + (2\zeta - 1)\nu - \frac{(1-\zeta)^2}{\xi^2} - \frac{\lambda^3\xi^2}{32} \cdot \frac{1}{\nu}\left(\frac{\nu}{\nu - \lambda}\right)^{5/2} =$$

$$= \frac{1}{2}\nu^2 + \frac{1}{2}\nu - \frac{1}{4} - \frac{1}{2\nu}\left(\frac{\nu}{\nu-\lambda}\right)^{5/2}.$$

Since $\nu \geq 10$, we have

$$\frac{1}{2\nu}\left(\frac{\nu}{\nu-\lambda}\right)^{5/2} \leq \frac{1}{20}\left(\frac{5}{4}\right)^{5/2} < \frac{1}{10}$$

and thus the claim holds for

$$\mu < \frac{1}{2}\nu^2 + \frac{1}{2}\nu - \frac{1}{4} - \frac{1}{10}.$$

Since $\frac{1}{2}\nu^2 + \frac{1}{2}\nu$ is always an integer and $\frac{1}{4} + \frac{1}{10} < \frac{1}{2}$, we have $f(x,y) \geq \mu/\nu$ (and thus that Wilf's conjecture holds) whenver $\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu - 1$.

By Proposition 2.4(c), we always have $\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu$, so the only case left to consider is $\mu = \frac{1}{2}\nu^2 + \frac{1}{2}\nu = \frac{\nu(\nu+1)}{2}$. Under this condition, we have, by Proposition 2.4(d),

$$q_1 \geq \frac{2\nu - 1 - 1}{2} = \nu - 1.$$

Since also $q_1 \leq \nu - 1$, we must have $q_1 = \nu - 1$ and $q_2 = 0$. In this case,

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left[1 + \frac{1}{2}(\nu - 1)\right] = \frac{\nu}{\mu} \cdot \frac{\nu+1}{2} = \frac{\nu(\nu+1)}{2\mu} = 1$$

and thus S satisfies Wilf's conjecture.

Suppose now $f(x,y) = \ell(x,y,x) = \alpha + (\alpha-\gamma)x + \gamma y$. Since $\gamma \geq \frac{4}{15}$, it is enough to consider the case $\alpha = 1$, $\gamma = \frac{4}{15}$. Define A, B, C, ξ, ζ, λ as above; then, we have $A = 1$, $B = \frac{11}{15}$, $C = \frac{4}{15}$ and so

$$\begin{cases} \xi := \sqrt{2C(2B-C)} = \sqrt{\dfrac{48}{75}}, \\ \zeta := A - \dfrac{C}{2} = \dfrac{13}{15}, \\ \lambda := \dfrac{4(1-\zeta)}{\xi^2} = \dfrac{55}{12}. \end{cases}$$

Then,

$$\frac{\lambda^3\xi^2}{32} \cdot \frac{1}{\nu}\left(\frac{\nu}{\nu-\lambda}\right)^{5/2} \leq \frac{\left(\frac{55}{12}\right)^3 \cdot \frac{48}{75}}{32} \cdot \frac{1}{10}\left(\frac{10}{10-\frac{55}{12}}\right)^{5/2} \leq \frac{9}{10}$$

and so Wilf's conjecture holds for

$$\mu < \frac{48}{75}\nu^2 + \frac{11}{15}\nu - \frac{1}{18} - \frac{9}{10}.$$

Since both $\frac{48}{75}$ and $\frac{11}{15}$ are larger than $1/2$ and $\frac{1}{18}+\frac{9}{10} < 1$, the quantity on the right hand side is strictly larger than $\frac{1}{2}\nu^2 + \frac{1}{2}\nu$. Hence, the previous inequality holds whenever $a_2 > \frac{c+\mu}{3}$, and thus Wilf's conjecture holds also in this case. ☐

Proposition 5.6. *There is an integer N such that, for every $\nu \geq N$, there are only finitely many numerical semigroups $S = \langle a_1, a_2, \ldots, a_\nu\rangle$ with*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu = \nu(S)$, *and*
- $\mu(S) \leq \frac{4}{9}\nu^2$,

and that do not *satisfy Wilf's conjecture.*

Proof. Fix any $\chi \in (0, 1/3)$, and consider the function

$$f(q_1, q_2) := 1 - \chi + \frac{1}{2}q_1 + \frac{1}{3}q_2.$$

By Lemmas 4.2 and 4.3, for every $\epsilon > 0$ there is an $N_1(\chi,\epsilon)$ such that, for every point $(q_1,q_2) \in \mathcal{A}(\mu,\nu)$, with $\nu \geq N_1(\chi,\epsilon)$, we have $f(q_1,q_2) \geq \mu/\nu$ whenever

$$\mu \leq \frac{4}{9}\nu^2 + \left(\frac{2}{3} - 2\chi\right)\nu - \frac{1}{16} - \epsilon.$$

Let $N_2(\chi,\epsilon) := \left(\epsilon + \frac{1}{16}\right)\left(\frac{2}{3} - 2\chi\right)^{-1}$: then, for $\nu \geq N_2(\chi,\epsilon)$, we have

$$\left(\frac{2}{3} - 2\chi\right)\nu - \frac{1}{16} - \epsilon \geq 0.$$

Therefore, for every $\nu \geq N := N(\chi,\epsilon) := \max\{N_1(\chi,\epsilon), N_2(\chi,\epsilon)\}$ we have $f(q_1,q_2) \geq \mu/\nu$ whenever $\mu \leq \frac{4}{9}\nu^2$. Equivalently, we have

$$1 + \frac{1}{2}q_1 + \frac{1}{3}q_2 \geq \frac{\mu}{\nu} + \chi.$$

Using the inequality $\lfloor x \rfloor > x - 1$ on Proposition 3.2, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu}\left(1 + \frac{1}{2}q_1 + \frac{1}{3}q_2\right) - \frac{\nu}{c}\left(1 + \frac{1}{2}q_1 + \frac{2}{3}q_2\right)$$

which for $\nu \geq N$ is bigger than

$$\frac{\nu}{\mu}\left(\frac{\mu}{\nu} + \chi\right) - \frac{\nu}{c}\left(\frac{\mu}{\nu} + \chi + \frac{1}{3}q_2\right) \geq 1 + \frac{\nu}{\mu}\chi - \frac{1}{c}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right),$$

using also the fact that $q_2 \leq \nu - 1$. The quantity on the right hand side is bigger than 1 when

$$\frac{\nu}{\mu}\chi - \frac{1}{c}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right) \geq 0;$$

since c, ν, μ and χ are positive, this is equivalent to

$$c \geq \frac{\mu}{\chi\nu}\left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right), \tag{9}$$

and all semigroups satisfying this inequality satisfy Wilf's conjecture.

In particular, for any value of ν, μ and χ, there are only a finite number of semigroups that do not satisfy this condition. For any ν, there are also a finite number of multiplicities μ satisfying $\mu \leq \frac{4}{9}\nu^2$; hence, for any fixed $\nu \geq N$ there are only finitely many numerical semigroups that verify the hypothesis of the theorem and that do not satisfy Wilf's conjecture. □

We note that the right hand side of (9) is very large: for example, if $\nu = 10$, $\mu = 50$ and $\chi = \frac{1}{6}$, then it is equal to 26050. The strategy used in the proof of Theorem 5.2 (i.e., writing $c = (6k-1)\mu + \theta\mu$ and using different estimates for different $\lfloor\theta\rfloor$) can be employed to obtain numerically better bounds (but still with the hypothesis $\mu \leq \frac{4}{9}\nu^2$).

Proposition 5.7. *For every $\lambda < \frac{4}{5}$ there is a $\nu_0(\lambda)$ such that, if $S = \langle a_1, a_2, \ldots, a_\nu\rangle$ is a numerical semigroup such that $a_2 > \frac{c(S)+\mu(S)}{3}$ and $\nu \geq \nu_0(\lambda)$, then*

$$\nu(S)|L(S)| \geq \lambda \cdot c(S). \tag{10}$$

Proof. Fix a $\lambda < \frac{4}{5}$. Let $\ell(x,y,z) := \alpha(1+z) + \beta(x-z) + \gamma(y-z)$, where α, β, γ are defined as in Proposition 3.4. Let $f(x,y) := A + Bx + Cy$ be either $\ell(x,y,x)$ or $\ell(x,y,y)$; as in the proof of Theorem 5.2, we need to prove that $f(x,y) \geq \lambda(\mu/\nu)$ for both choices of f and all $(x,y) \in \mathcal{I}(\mu-\nu)$, that is, we have to show that $\lambda^{-1}f(x,y) \geq \mu/\nu$.

By Lemmas 4.2 and 4.3, for every $\epsilon > 0$ there is a $\nu_1(\epsilon)$ such that, for every $\nu \geq \nu_1(\epsilon)$ this inequality holds for

$$\mu \leq \left(2(C\lambda^{-1})(2B\lambda^{-1} - C\lambda^{-1}) - \epsilon\right)\nu^2 = \left(\frac{2C(2B-C)}{\lambda^2} - \epsilon\right)\nu^2.$$

By the proof of Theorem 5.2, $2C(2B-C)$ is at least $\frac{8}{25}$; if $\lambda < \frac{4}{5}$, then

$$\frac{2C(2B-C)}{\lambda^2} > \frac{8}{25} \cdot \frac{25}{16} = \frac{1}{2}.$$

Therefore, we can choose an ϵ satisfying

$$0 < \epsilon < \frac{2C(2B-C)}{\lambda^2} - \frac{1}{2},$$

and for such an ϵ there is a $\nu_2(\epsilon, \lambda)$ such that

$$\left(\frac{2C(2B-C)}{\lambda^2} - \epsilon\right)\nu^2 > \frac{1}{2}\nu^2 + \frac{1}{2}\nu$$

for all $\nu \geq \nu_1(\epsilon, \lambda)$. Setting $\nu_0(\lambda) := \max\{\nu_1(\epsilon), \nu_2(\epsilon, \lambda)\}$, we have that the inequality (10) holds for $\nu \geq \nu_0(\lambda)$ and $\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu$. Since every semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$ satisfies the latter condition (by Proposition 2.4(c)), the claim holds. □

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