When the Zariski space is a Noetherian space

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Abstract We characterize when the Zariski space $\mathrm{Zar}(K|D)$ (where D is an integral domain, K is a field containing D, and D is integrally closed in K) and the set $\mathrm{Zar}_{\min}(L|D)$ of its minimal elements are Noetherian spaces.

1. Introduction

The *Zariski space* $\mathrm{Zar}(K|D)$ of the valuation rings of a field K containing a subring D was introduced by O. Zariski (under the name *abstract Riemann surface*) during his study of resolution of singularities [24, 25]. In particular, he introduced a topology on $\mathrm{Zar}(K|D)$ (which was later called *Zariski topology*) and proved that it makes $\mathrm{Zar}(K|D)$ into a compact space [26, Chapter VI, Theorem 40]. Later, the Zariski topology on $\mathrm{Zar}(K|D)$ was studied more carefully, showing that it is a *spectral space* in the sense of Hochster [14]; i.e., that there is a ring R such that the spectrum of R (endowed with the Zariski topology) is homeomorphic to $\mathrm{Zar}(K|D)$ [4–6]. This topology has also been used to study representations of an integral domain by intersection of valuation rings [16–18] and, for example, in real and rigid algebraic geometry [15, 21].

In [22], it was shown that in many cases $\mathrm{Zar}(D)$ is not a Noetherian space; i.e., there are subspaces of $\mathrm{Zar}(D)$ that are not compact. In particular, it was shown that $\mathrm{Zar}(D) \setminus \{V\}$ (where V is a minimal valuation overring of D) is often non-compact; for example, this happens when $\dim(V) > 2\dim(D)$ [22, Proposition 4.3] or when D is Noetherian and $\dim(V) \geq 2$ [22, Corollary 5.2].

In this paper, we study integral domains such that $\mathrm{Zar}(D)$ is a Noetherian space, and, more generally, we study when the Zariski space $\mathrm{Zar}(K|D)$ is Noetherian. We show that, if $D = F$ is a field, then $\mathrm{Zar}(K|F)$ can be Noetherian only if the transcendence degree of K over F is at most 1 and, when $\mathrm{trdeg}_F K = 1$, we characterize when this happens in terms of the extensions of the valuation domains of $F[X]$, where X is an element of K transcendental over F (Proposition 4.2). In Section 5, we study the case where K is the quotient field of D. We first consider the local case, showing that if $\mathrm{Zar}(D)$ is Noetherian, then D must be a pseudo-valuation domain (Theorem 5.8) and, subsequently, we globalize this result to the non-local case, showing that $\mathrm{Zar}(D)$ is Noetherian if and only if $\mathrm{Spec}(D)$ and $\mathrm{Zar}(D_M)$ are as well, for every maximal ideal M of D (Theorem 5.11 and Corollary 5.12). We also prove the analogous results for the set $\mathrm{Zar}_{\min}(K|D)$ of the minimal elements of $\mathrm{Zar}(K|D)$.

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2. Background

Throughout the paper, when X_1 and X_2 are topological spaces, we shall use the notation $X_1 \simeq X_2$ to denote that X_1 and X_2 are homeomorphic.

2.1. Overrings and the Zariski space

Let D be an integral domain and let K be a ring containing D. We define $\mathrm{Over}(K|D)$ as the set of rings contained between D and K. The *Zariski topology* on $\mathrm{Over}(K|D)$ is the topology having, as a subbasis of open sets, the sets in the form

$$\mathcal{B}(x_1, \dots, x_n) := \{V \in \mathrm{Over}(K|D) \mid x_1, \dots, x_n \in V\},$$

as $x_1, \dots, x_n$ range in K. If K is the quotient field of D, an element of $\mathrm{Over}(K|D)$ is called an *overring* of D.

If K is the quotient field of D, a subset $X \subseteq \mathrm{Over}(K|D)$ is a *locally finite family* if every $x \in D$ (or, equivalently, every $x \in K$) is a non-unit in only finitely many $T \in \mathrm{Over}(K|D)$.

If K is a field containing D, the *Zariski space* of D in K is the set of all valuation domains containing D and whose quotient field is K; we denote it by $\mathrm{Zar}(K|D)$. The Zariski topology on $\mathrm{Zar}(K|D)$ is simply the Zariski topology inherited from $\mathrm{Over}(K|D)$. If K is the quotient field of D, then $\mathrm{Zar}(K|D)$ will simply be denoted by $\mathrm{Zar}(D)$, and its elements are called the *valuation overrings* of D.

Under the Zariski topology, $\mathrm{Zar}(K|D)$ is compact [26, Chapter VI, Theorem 40].

We denote by $\mathrm{Zar}_{\min}(K|D)$ the set of minimal elements of $\mathrm{Zar}(K|D)$, with respect to containment. If V is a valuation domain, we denote by $\mathfrak{m}_V$ its maximal ideal. Given $X \subseteq \mathrm{Zar}(D)$, we define

$$X^{\uparrow} := \{V \in \mathrm{Zar}(D) \mid V \supseteq W \text{ for some } W \in X\}.$$

Since a family of open sets is a cover of X if and only if it is a cover of $X^{\uparrow}$, we have that X is compact if and only if $X^{\uparrow}$ is compact.

If X is a subset of $\mathrm{Zar}(D)$, we denote by $A(X)$ the intersection $\bigcap\{V \mid V \in X\}$, called the *holomorphy ring* of X [20]. Clearly, $A(X) = A(X^{\uparrow})$.

The *center map* is the application

$$\begin{aligned} \gamma\colon \mathrm{Zar}(K|D) &\longrightarrow \mathrm{Spec}(D) \\ V &\longmapsto \mathfrak{m}_V \cap D. \end{aligned}$$

If $\mathrm{Zar}(K|D)$ and $\mathrm{Spec}(D)$ are endowed with the respective Zariski topologies, the map γ is continuous ([26, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [2, Theorem 5.21] or [11, Theorem 19.6]) and closed [4, Theorem 2.5].

In studying $\mathrm{Zar}(K|D)$, it is usually enough to consider the case where D is integrally closed in K; indeed, if $\overline{D}$ is the integral closure of D in K, then $\mathrm{Zar}(K|D) = \mathrm{Zar}(K|\overline{D})$.

2.2. Noetherian spaces

A topological space X is *Noetherian* if its open sets satisfy the ascending chain condition, or equivalently if all its subsets are compact. If $X = \mathrm{Spec}(R)$ is the spectrum of a ring, then X is a Noetherian space if and only if R satisfies the ascending chain condition on radical ideals; in particular, the spectrum of a Noetherian ring is always a Noetherian space. If $\mathrm{Spec}(R)$ is Noetherian, then every ideal of R has only finitely many minimal primes (see, e.g., the proof of [3, Chapter 4, Corollary 3, p. 102] or [2, Chapter 6, Exercises 5 and 7]).

Every subspace and every continuous image of a Noetherian space is again Noetherian; in particular, if $\mathrm{Zar}(D)$ is Noetherian, then so are $\mathrm{Zar}_{\min}(D)$ and $\mathrm{Spec}(D)$ [22, Proposition 4.1].

2.3. Kronecker function rings

Let K be the quotient field of D. For every $V \in \mathrm{Zar}(D)$, let $V^b := V[X]_{\mathfrak{m}_V[X]} \subseteq K(X)$. If $\Delta \subseteq \mathrm{Zar}(D)$, the *Kronecker function ring* of D with respect to Δ is

$$\mathrm{Kr}(D, \Delta) := \bigcap \{V^b \mid V \in \Delta\};$$

we denote $\mathrm{Kr}(D, \mathrm{Zar}(D))$ simply by $\mathrm{Kr}(D)$.

The ring $\mathrm{Kr}(D, \Delta)$ is always a Bézout domain whose quotient field is $K(X)$, and, if Δ is compact, the intersection map $W \mapsto W \cap K$ establishes a homeomorphism between $\mathrm{Zar}(\mathrm{Kr}(D, \Delta))$ and the set $\Delta^{\uparrow}$ [4–6]. Since $\mathrm{Kr}(D, \Delta)$ is a Prüfer domain, furthermore, $\mathrm{Zar}(\mathrm{Kr}(D, \Delta))$ is homeomorphic to $\mathrm{Spec}(\mathrm{Kr}(D, \Delta))$; hence, $\mathrm{Spec}(\mathrm{Kr}(D, \Delta))$ is homeomorphic to $\Delta^{\uparrow}$, and asking if $\mathrm{Zar}(D)$ is Noetherian is equivalent to asking if $\mathrm{Spec}(\mathrm{Kr}(D))$ is Noetherian or, equivalently, if $\mathrm{Kr}(D)$ satisfies the ascending chain condition on radical ideals.

See [11, Chapter 32] or [10] for general properties of Kronecker function rings.

2.4. Pseudo-valuation domains

Let D be an integral domain with quotient field K. Then D is called a *pseudo-valuation domain* (for short, *PVD*) if, for every prime ideal P of D, whenever $xy \in P$ for some $x, y \in K$, then at least one of x and y is in P. Equivalently, D is a pseudo-valuation domain if and only if it is local and its maximal ideal M is also the maximal ideal of some valuation overring V of D (called the valuation domain *associated* to D) [12, Corollary 1.3 and Theorem 2.7]. If D is a valuation domain, then it is also a PVD, and the associated valuation ring is D itself.

The prototypical example of a pseudo-valuation domain that is not a valuation domain is the ring $F + XL[[X]]$, where $F \subseteq L$ is a field extension; its associated valuation domain is $L[[X]]$.

3. Examples and reduction

The easiest case for the study of the topology of $\mathrm{Zar}(D)$ is when D is a Prüfer domain; i.e., when D_M is a valuation domain for every maximal ideal M of D.

PROPOSITION 3.1

Let D be a Prüfer domain. Then

(a) $\mathrm{Zar}(D)$ *is a Noetherian space if and only if* $\mathrm{Spec}(D)$ *is Noetherian;*
(b) $\mathrm{Zar}_{\min}(D)$ *is Noetherian if and only if* $\mathrm{Max}(D)$ *is Noetherian.*

Proof

Since D is Prüfer, the center map $\gamma : \mathrm{Zar}(D) \longrightarrow \mathrm{Spec}(D)$ is a homeomorphism [4, Proposition 2.2]. This proves the first claim; the second one follows from the fact that the minimal valuation overrings of D correspond to the maximal ideals. □

Another example of a domain that has a Noetherian Zariski space is the pseudo-valuation domain $D := \mathbb{Q} + Y\mathbb{Q}(X)[[Y]]$, where X, Y are indeterminates on $\mathbb{Q}$; in this case $\mathrm{Zar}(D)$ can be written as the union of the quotient field of D and two sets homeomorphic to $\mathrm{Zar}(\mathbb{Q}[X]) \simeq \mathrm{Spec}(\mathbb{Q}[X])$, which are Noetherian. From this, it is possible to build examples of non-Prüfer domains whose Zariski spectrum is Noetherian and having arbitrary finite dimension [22, Example 4.7].

More generally, we have the following routine observation.

LEMMA 3.2

Let D be an integral domain, and suppose that a prime ideal P of D is also the maximal ideal of a valuation overring V of D. Then the quotient map $\pi : V \longrightarrow V/P$ establishes a homeomorphism between $\{W \in \mathrm{Zar}(D) \mid W \subseteq V\}$ and $\mathrm{Zar}(V/P|D/P)$, and between $\mathrm{Zar}_{\min}(D)$ and $\mathrm{Zar}_{\min}(V/P|D/P)$.

Proof

Consider the sets $\mathrm{Over}(V|D)$ and $\mathrm{Over}(V/P|D/P)$. Then the map

$$\widetilde{\pi}\colon \mathrm{Over}(V|D) \longrightarrow \mathrm{Over}(V/P|D/P)$$
$$A \longmapsto \pi(A) = A/P$$

is a bijection, whose inverse is the map sending B to $\pi^{-1}(B)$. Furthermore, it is a homeomorphism; indeed, if $x \in V/P$ then $\widetilde{\pi}^{-1}(\mathcal{B}(x)) = \mathcal{B}(y)$, for any $y \in \pi^{-1}(x)$, while if $x \in V$ then $\widetilde{\pi}(\mathcal{B}(x)) = \mathcal{B}(\pi(x))$.

The condition on P implies that D is a pullback in the diagram

$$\begin{array}{ccc} D & \xrightarrow{\pi} & D/P \\ \downarrow & & \downarrow \\ V & \xrightarrow{\pi} & V/P; \end{array}$$

hence, every $A \in \mathrm{Over}(V|D)$ arises as a pullback. By [8, Theorem 2.4(1)], A is a valuation domain if and only if $\pi(A)$ is a valuation domain and V/P is the quotient field of $\pi(A)$; hence, $\widetilde{\pi}$ restricts to a bijection between $\mathrm{Zar}(D) \cap \mathrm{Over}(V|D) = \{W \in \mathrm{Zar}(D) \mid W \subseteq V\}$ and $\mathrm{Zar}(V/P|D/P)$. Furthermore, since $\widetilde{\pi}$ is a homeomorphism, so is its restriction. The claim about $\mathrm{Zar}(D)$ and $\mathrm{Zar}(V/P|D/P)$ is proved; the claim for the space of minimal elements follows immediately. □

PROPOSITION 3.3

Let D be an integral domain, and let L be a field containing D. Then there is a domain R such that

- $\mathrm{Zar}(L|D) \simeq \mathrm{Zar}(R) \setminus \{F\}$*, where F is the quotient field of R;*
- $\mathrm{Zar}_{\min}(L|D) \simeq \mathrm{Zar}_{\min}(R)$.

Proof

Let X be an indeterminate over L, and define $R := D + XL[[X]]$. Then the prime ideal $P := XL[[X]]$ of R is also a prime ideal of the valuation domain $L[[X]]$; by Lemma 3.2, it follows that $\mathrm{Zar}(L|D) \simeq \Delta := \{W \in \mathrm{Zar}(R) \mid W \subseteq L[[X]]\}$. Furthermore, every valuation overring V of R contains $XL[[X]]$, and thus it is either in Δ or properly contains $L[[X]]$; however, since $L[[X]]$ has dimension 1, the latter case is possible only if $V = L((X))$ is the quotient field of R. The first claim is proved, and the second follows easily. □

Proposition 3.3 shows that, theoretically, it is enough to consider spaces of valuation rings between a domain and its quotient field. However, it is convenient to not be restricted to this case; the following Proposition 3.4 is an example, as will be the analysis of field extensions in Section 4.

PROPOSITION 3.4

Let D be an integral domain that is not a field, let K be its quotient field, and let L be a field extension of K. If $\mathrm{trdeg}_K L \geq 1$, then $\mathrm{Zar}(L|D)$ and $\mathrm{Zar}_{\min}(L|D)$ are not Noetherian.

Proof

If $\mathrm{trdeg}_K L \geq 1$, there is an element $X \in L \setminus K$ that is not algebraic over L. If $\mathrm{Zar}(L|D)$ is Noetherian, so is its subset $\mathrm{Zar}(L|D[X])$, and thus also $\mathrm{Zar}(K(X)|D[X]) = \mathrm{Zar}(D[X])$, which is the (continuous) image of $\mathrm{Zar}(L|D[X])$ under the intersection map $W \mapsto W \cap K(X)$. However, since D is not a field, $\mathrm{Zar}(D[X])$ is not Noetherian by [22, Proposition 5.4]; hence, $\mathrm{Zar}(L|D)$ cannot be Noetherian.

Consider now $\mathrm{Zar}_{\min}(L|D)$. It projects onto $\mathrm{Zar}_{\min}(K(X)|D)$, and thus we can suppose that $L = K(X)$. Let V be a minimal valuation overring of D. Then there is an extension W of V to L such that X is the generator of the maximal ideal of W; furthermore, W belongs to $\mathrm{Zar}_{\min}(K(X)|D)$. In particular, $\mathrm{Spec}(W) \setminus \mathrm{Max}(W)$ has a maximum, say P. Let $\Delta := \mathrm{Zar}(L|D) \setminus \{W\}$; then Δ can be written as the union of $\Lambda := (\mathrm{Zar}_{\min}(L|D) \setminus \{W\})^{\uparrow}$ and $\{W_P\}^{\uparrow}$. The latter is compact since $\{W_P\}$ is compact; if $\mathrm{Zar}_{\min}(L|D) \setminus \{W\}$ were compact, so would be Λ. In this case, Δ also would be compact, against the proof of [22, Proposition 5.4]. Hence, Δ is not compact, and so $\mathrm{Zar}_{\min}(L|D)$ is not Noetherian. □

4. Field extensions

In this section, we consider a field extension $F \subseteq L$ and analyze when the Zariski space $\mathrm{Zar}(L|F)$ and its subset $\mathrm{Zar}_{\min}(L|F)$ are Noetherian. By Proposition 3.3, this is equivalent to studying the Zariski space of the pseudo-valuation domain $F + XL[[X]]$.

This problem naturally splits into three cases, according to whether the transcendence degree of L over F is 0, 1 or at least 2. The first and the last cases have definite answers, and we collect them in the following proposition. Part (b) is a slight generalization of [22, Corollary 5.5(b)]. Recall that the *inverse topology* (with respect to the Zariski topology) on $\mathrm{Zar}(K|D)$ is the topology whose closed sets are the subsets $\Delta \subseteq \mathrm{Zar}(K|D)$ that are compact (in the Zariski topology) and such that $\Delta = \Delta^{\uparrow}$ (this is not the usual definition, but it is equivalent; see, for example, [6, Remark 2.2 and Proposition 2.6]). In particular, the intersection of two subsets with these properties is still compact in the Zariski topology.

PROPOSITION 4.1
Let $F \subseteq L$ be a field extension.

(a) *If* $\mathrm{trdeg}_F\, L = 0$*, then* $\mathrm{Zar}(L|F) = \{L\} = \mathrm{Zar}_{\min}(L|D)$*, and in particular both spaces are Noetherian.*
(b) *If* $\mathrm{trdeg}_F\, L \geq 2$*, then* $\mathrm{Zar}(L|F)$ *and* $\mathrm{Zar}_{\min}(L|F)$ *are not Noetherian.*

Proof
(a) is obvious. For (b), let X, Y be elements of L that are algebraically independent. Then the intersection map $\mathrm{Zar}_{\min}(L|F) \longrightarrow \mathrm{Zar}_{\min}(F(X,Y)|F)$ is surjective, and thus it is enough to prove that $\mathrm{Zar}_{\min}(F(X,Y)|F)$ is not Noetherian.

Let $V \in \mathrm{Zar}_{\min}(F(X,Y)|F)$ and, without loss of generality, suppose $X, Y \in V$. Let $\Delta := \mathrm{Zar}_{\min}(F(X,Y)|F) \setminus \{V\}$. Then $\Lambda := \mathrm{Zar}(F(X,Y)|F) \setminus \{V\}$ is the union of $\Delta^{\uparrow}$ and a finite set (the valuation domains properly containing V). If Δ were compact, so would be Λ, and thus Λ would be closed in the inverse topology. Since also $\mathrm{Zar}(F[X,Y])$ is closed in the inverse topology, it would follow that $\Lambda \cap \mathrm{Zar}(F[X,Y]) = \mathrm{Zar}(F[X,Y]) \setminus \{V\}$ is compact, against the proof of [22, Proposition 5.4]. Hence, Λ is not compact, and thus Δ cannot be compact. Therefore, $\mathrm{Zar}_{\min}(F(X,Y)|F)$ is not Noetherian. □

On the other hand, the case of transcendence degree 1 is more subtle. In [22, Corollary 5.5(a)], it was shown that $\mathrm{Zar}(L|F)$ is Noetherian if L is finitely generated over F; we now state a characterization.

PROPOSITION 4.2
Let $F \subseteq L$ be a field extension such that $\mathrm{trdeg}_F\, L = 1$*. Then the following are equivalent:*

(i) $\mathrm{Zar}(L|F)$ *is Noetherian;*
(ii) $\mathrm{Zar}_{\min}(L|F)$ *is Noetherian;*

(iii) *for every $X \in L$ transcendental over F, every valuation on $F[X]$ has only finitely many extensions to L;*
(iv) *there is an $X \in L$, transcendental over F, such that every valuation on $F[X]$ has only finitely many extensions to L;*
(v) *for every $X \in L$ transcendental over F, the integral closure of $F[X]$ in L has Noetherian spectrum;*
(vi) *there is an $X \in L$, transcendental over F, such that the integral closure of $F[X]$ in L has Noetherian spectrum.*

Proof
Every valuation domain of L containing F must contain the algebraic closure of F in L; hence, without loss of generality, we can suppose that F is algebraically closed in L.

(i) $\Longrightarrow$ (ii) is obvious; (ii) $\Longrightarrow$ (i) follows since $\mathrm{trdeg}_F L = 1$ and thus $\mathrm{Zar}(L|F) = \mathrm{Zar}_{\min}(L|F) \cup \{L\}$.

(i) $\Longrightarrow$ (iii). Take $X \in L \setminus F$, and suppose there is a valuation w on $F[X]$ with infinitely many extensions to L; let W be the valuation domain corresponding to w. Then the integral closure $\overline{W}$ of W in L would have infinitely many maximal ideals. Since every maximal ideal of $\overline{W}$ contains the maximal ideal of W, the Jacobson radical J of $\overline{W}$ contains the maximal ideal of W, and in particular it is non-zero. It follows that J has infinitely many minimal primes; hence, $\mathrm{Max}(\overline{W})$ is not a Noetherian space. However, $\mathrm{Max}(\overline{W})$ is homeomorphic to a subspace of $\mathrm{Zar}(L|F)$, which is Noetherian by hypothesis; this is a contradiction, and so every valuation has only finitely many extensions.

(iii) $\Longrightarrow$ (v). Let T be the integral closure of $F[X]$, and suppose that $\mathrm{Spec}(T)$ is not Noetherian. We first claim that T is not locally finite; i.e., that there is an $\alpha \in T$ such that there are infinitely many maximal ideals of T containing α. Indeed, if T is locally finite and $\{I_\alpha\}_{\alpha \in A}$ is an ascending chain of radical ideals, then once $I_{\overline{\alpha}} \neq (0)$, the ideal $I_{\overline{\alpha}}$ is contained in only finitely many prime ideals (since T has dimension 1), and thus in only finitely many radical ideals; it follows that the chain stabilizes and $\mathrm{Spec}(R)$ is Noetherian, a contradiction.

Consider the norm $N(\alpha)$ of α over $F[X]$; i.e., the product of the algebraic conjugates of α over $F[X]$. Then $N(\alpha) \neq 0$, and it is both an element of $F[X]$ (being equal to the constant term of the minimal polynomial of α over $F[X]$) and an element of every maximal ideal containing α (since all the conjugates are in T). Since every maximal ideal of $F[X]$ is contained in only finitely many maximal ideals of T (since a maximal ideal of $F[X]$ corresponds to a valuation v and the maximal ideals of T containing it to the extensions of v), it follows that $N(\alpha)$ is contained in infinitely many maximal ideals of $F[X]$. However, this contradicts the Noetherianity of $\mathrm{Spec}(F[X])$; hence, $\mathrm{Spec}(T)$ is Noetherian.

Now (iii) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (vi) are obvious, while the proof of (iv) $\Longrightarrow$ (vi) is exactly the same as in the previous paragraph; hence, we need only to show (vi) $\Longrightarrow$ (i); the proof is similar to the one of [22, Corollary 5.5(a)].

Let $X \in L$, X transcendental over F, be such that the spectrum of the integral closure T of $F[X]$ is Noetherian. Since X is transcendental over F, there is an F-isomorphism ϕ of $F(X)$ sending X to X^{-1}; moreover, we can extend ϕ to an F-isomorphism $\overline{\phi}$ of L. Since $\phi(F[X]) = F[X^{-1}]$, the integral closure T of $F[X]$ is sent by $\overline{\phi}$ to the integral closure T' of $F[X^{-1}]$; in particular, $T \simeq T'$, and $\mathrm{Spec}(T) \simeq \mathrm{Spec}(T')$. Thus, also $\mathrm{Spec}(T')$ is Noetherian and so is $\mathrm{Spec}(T) \cup \mathrm{Spec}(T')$. Furthermore, $\mathrm{Zar}(T) \simeq \mathrm{Spec}(T) \simeq \mathrm{Spec}(L|F[X])$, and analogously for T'; hence, $\mathrm{Zar}(T) \cup \mathrm{Zar}(T')$ is Noetherian. But every $W \in \mathrm{Zar}(L|F)$ contains at least one between X and X^{-1}, and thus W contains $F[X]$ or $F[X^{-1}]$; i.e., $W \in \mathrm{Zar}(T)$ or $W \in \mathrm{Zar}(T')$. Hence, $\mathrm{Zar}(L|F) = \mathrm{Zar}(T) \cup \mathrm{Zar}(T')$ is Noetherian. □

We remark that there are field extensions that satisfy the conditions of Proposition 4.2 without being finitely generated. For example, if L is purely inseparable over some $F(X)$, then every valuation on $F[X]$ extends uniquely to L, and thus condition (iii) of the previous proposition is fulfilled; more generally, each valuation on $F(X)$ extends in only finitely many ways when the separable degree $[L : F(X)]_s$ is finite [11, Corollary 20.3]. There are also examples in characteristic 0; for example, [19, Section 12.2] gives examples of non-finitely generated algebraic extensions F of the rational numbers such that every valuation on $\mathbb{Q}$ has only finitely many extensions to F. The same construction works also on $\mathbb{Q}(X)$, and if L is such an example, then $\mathbb{Q} \subseteq L$ will satisfy the conditions of Proposition 4.2.

5. The domain case

We now want to study when the space $\mathrm{Zar}(D)$ is Noetherian, where D is an integral domain; without loss of generality, we can suppose that D is integrally closed since $\mathrm{Zar}(D) = \mathrm{Zar}(\overline{D})$. We start by studying intersections of Noetherian families of valuation rings.

Recall that a *treed domain* is an integral domain whose spectrum is a tree (i.e., such that, if P and Q are non-comparable prime ideals, then they are coprime). In particular, every Prüfer domain is treed.

LEMMA 5.1
Let R be a treed domain. If $\mathrm{Max}(R)$ *is Noetherian, then every ideal of R has only finitely many minimal primes.*

Note that we cannot improve this result to $\mathrm{Spec}(R)$ being Noetherian; for example, the spectrum of a valuation domain with unbranched maximal ideal is not Noetherian, while its maximal spectrum—a singleton—is Noetherian.

Proof
Let I be an ideal of R, and let $\{P_\alpha \mid \alpha \in A\}$ be the set of its minimal prime ideals. For every α, choose a maximal ideal M_α containing P_α; note that $M_\alpha \neq M_\beta$ if $\alpha \neq \beta$ since R is treed. Let Λ be the set of the M_α.

Let $X \subseteq \Lambda$, and define $J(X) := \bigcap\{IR_M \mid M \in X\} \cap R$. We claim that, if $M \in \Lambda$, then $J(X) \subseteq M$ if and only if $M \in X$. Indeed, clearly $J(X)$ is contained in every element of X. On the other hand, suppose $N \in \Lambda \setminus X$. Since $\mathrm{Max}(R)$ is Noetherian, X is compact, and thus also $\{R_M \mid M \in X\}$ is compact; by [7, Corollary 5]:

$$J(X)R_N = \Big(\bigcap_{M\in X} IR_M\Big)R_N \cap R_N = \bigcap_{M\in X} IR_M R_N \cap R_N.$$

Since $M, N \in \Lambda$, no prime contained in both M and N contains I; hence, $IR_M R_N$ contains 1 for each $M \in X$. Therefore, $1 \in J(X)R_N$, i.e., $J(X) \nsubseteq N$.

Hence, every subset X of Λ is closed in Λ since it is equal to the intersection between Λ and the closed set of $\mathrm{Spec}(R)$ determined by $J(X)$. Since Λ is Noetherian, it follows that Λ must be finite; therefore, also the set of minimal primes of I is finite. The claim is proved. □

As consequence of Lemma 5.1, we can generalize [16, Theorem 3.4(2)]. We premit an easy lemma.

LEMMA 5.2

Let D be an integral domain with quotient field K, and let $V, W \in \mathrm{Zar}(D)$. If $VW = K$, then $V^b W^b = K(X)$.

Proof

Let $Z := V^b W^b$. Then, since $\mathrm{Zar}(D)$ and $\mathrm{Zar}(\mathrm{Kr}(D))$ are homeomorphic, $Z = (Z \cap K)^b$; however, $K \subseteq VW \subseteq V^b W^b$, and thus $Z \cap K = K$. It follows that $Z = K^b = K(X)$, as claimed. □

THEOREM 5.3

Let $\Delta \subseteq \mathrm{Zar}(D)$ be a Noetherian space, and suppose that $VW = K$ for every $V \neq W$ in Δ. Then Δ is a locally finite space.

Proof

Let $\Delta^b := \{V^b \mid V \in \Delta\}$, and let $R := \mathrm{Kr}(D, \Delta)$; then (since, in particular, Δ is compact), $\mathrm{Zar}(R)$ is equal to $(\Delta^b)^\uparrow$.

Since R is a Bézout domain, it follows that $\mathrm{Spec}(R) \simeq (\Delta^b)^\uparrow$, while $\mathrm{Max}(R) \simeq \Delta^b$; in particular, $\mathrm{Max}(R)$ is Noetherian, and thus by Lemma 5.1 every ideal of R has only finitely many minimal primes. However, since $V^b W^b = K(X)$ for every $V \neq W$ in Δ (by Lemma 5.2), it follows that every non-zero prime of R is contained in only one maximal ideal; therefore, every non-zero ideal of R is contained in only finitely many maximal ideals, and thus the family $\{R_M \mid M \in \mathrm{Max}(R)\}$ is locally finite. This family coincides with Δ^b; since Δ^b is locally finite, also Δ must be locally finite, as claimed. □

We say that two valuation domains $V, W \in \mathrm{Zar}(D) \setminus \{K\}$ are *dependent* if $VW \neq K$. Since $\mathrm{Zar}(D)$ is a tree, being dependent is an equivalence relation on $\mathrm{Zar}(D) \setminus \{K\}$;

we call an equivalence class a *dependency class*. If $\mathrm{Zar}(D)$ is finite-dimensional (i.e., if every valuation overring of D has finite dimension), then the dependency classes of $\mathrm{Zar}(D)$ are exactly the sets in the form $\{W \in \mathrm{Zar}(D) \mid W \subseteq V\}$, as V ranges among the one-dimensional valuation overrings of D.

Under this terminology, the previous theorem implies that, if D is local and $\mathrm{Zar}(D)$ is Noetherian, then $\mathrm{Zar}(D)$ can only have finitely many dependency classes; indeed, otherwise, we could form a Noetherian but not locally finite subset of $\mathrm{Zar}(D)$ by taking one minimal overring in each dependency class, against the theorem. We actually can say (and will need) something more.

Given a set $X \subseteq \mathrm{Zar}(D)$, we define $\mathrm{comp}(X)$ as the set of all valuation overrings of D that are comparable with some elements of X; i.e.,

$$\mathrm{comp}(X) := \left\{W \in \mathrm{Zar}(D) \,\middle|\, \exists\, V \in X \text{ such that } W \subseteq V \text{ or } V \subseteq W\right\}.$$

If $X = \{V\}$ is a singleton, we write $\mathrm{comp}(V)$ for $\mathrm{comp}(X)$. Note that, for every subset X, $\mathrm{comp}(\mathrm{comp}(X)) = \mathrm{Zar}(D)$ since $\mathrm{comp}(X)$ contains the quotient field of D.

The purpose of the following propositions is to show that, if D is local and $\mathrm{Zar}(D)$ is Noetherian, then $\mathrm{Zar}(D)$ can be written as $\mathrm{comp}(W)$ for some valuation overring $W \neq K$. The first step is showing that $\mathrm{Zar}(D)$ is equal to $\mathrm{comp}(X)$ for some finite X.

PROPOSITION 5.4

Let D be a local integral domain. If $\mathrm{Zar}_{\min}(D)$ is Noetherian, then there are valuation overrings $W_1, \dots, W_n$ of D, $W_i \neq K$, such that $\mathrm{Zar}(D) = \mathrm{comp}(W_1) \cup \cdots \cup \mathrm{comp}(W_n)$.

Proof

Let $R := \mathrm{Kr}(D)$ be the Kronecker function ring of D. Then the extension $N := MR$ of the maximal ideal M of D is a proper ideal of R, and the prime ideals containing N correspond to the valuation overrings of R where N survives; i.e., to the valuation overrings of D centered on M.

Since $\mathrm{Zar}_{\min}(D)$ is Noetherian, so is $\mathrm{Max}(R)$; since R is treed (being a Bézout domain), by Lemma 5.1 N has only finitely many minimal primes. Thus, there are finitely many valuation overrings of D, say $W_1, \dots, W_n$, such that every $V \in \mathrm{Zar}_{\min}(D)$ is contained in one W_i. We claim that $\mathrm{Zar}(D) = \mathrm{comp}(W_1) \cup \cdots \cup \mathrm{comp}(W_n)$. Indeed, let V be a valuation overring of D. Since $\mathrm{Zar}(D)$ is compact, V contains some minimal valuation overring V', and by construction $V' \in \mathrm{comp}(W_i)$ for some i; in particular, $W_i \supseteq V'$. The valuation overrings containing V' (i.e., the valuation overrings of V') are linearly ordered; thus, V must be comparable with W_i, i.e., $V \in \mathrm{comp}(W_i)$. The claim is proved. □

The following result can be seen as a generalization of the classical fact that, if $X = \{V_1, \dots, V_n\}$ is finite, then $\mathrm{Zar}(A(X))$ is the union of the various $\mathrm{Zar}(V_i)$ (since $A(X)$ will be a Prüfer domain, and its localization at the maximal ideals will be a subset of X).

PROPOSITION 5.5

Let D be an integral domain and let $X \subseteq \mathrm{Zar}(D)$ be a finite set. Then $\mathrm{Zar}(A(\mathrm{comp}(X))) = \mathrm{comp}(X)$.

Proof

Since $\mathrm{comp}(V) \subseteq \mathrm{comp}(W)$ if $V \subseteq W$, we can suppose without loss of generality that the elements of X are pairwise incomparable. Let $X = \{V_1, \dots, V_n\}$, $A_i := A(\mathrm{comp}(V_i))$ and let $A := A(\mathrm{comp}(X)) = A_1 \cap \cdots \cap A_n$. Note that $D \subseteq A$, and thus the quotient field of A coincides with the quotient field of D and of the V_i.

If $V \in \mathrm{comp}(X)$, then clearly $A \subseteq V$; thus, $\mathrm{comp}(X) \subseteq \mathrm{Zar}(A)$.

Conversely, let $V \in \mathrm{Zar}(A)$, and let $\mathfrak{m}_i$ be the maximal ideal of V_i. Then $\mathfrak{m}_i \subseteq W$ for every $W \in \mathrm{comp}(V_i)$; in particular, $\mathfrak{m}_i \subseteq A_i$. Therefore, $P := \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \subseteq A$; since $A \subseteq V$, this implies that $PV \subseteq V$.

Suppose $V \notin \mathrm{comp}(X)$, and let $T := V \cap V_1 \cap \cdots \cap V_n$. Since the rings $V, V_1, \dots, V_n$ are pairwise incomparable, T is a Bézout domain whose localizations at the maximal ideals are $V, V_1, \dots, V_n$. In particular, V is flat over T, and each $\mathfrak{m}_i$ is a T-module; hence,

$$PV = \left(\bigcap_{i=1}^{n} \mathfrak{m}_i\right) V = \bigcap_{i=1}^{n} \mathfrak{m}_i V.$$

Since V is not comparable with V_i, for each i, the set $\mathfrak{m}_i$ is not contained in V; in particular, the family $\{\mathfrak{m}_i V \mid i = 1, \dots, n\}$ is a family of V-modules not contained in V. Since the V-submodules of the quotient field K are linearly ordered, the family has a minimum, and thus $\bigcap_{i=1}^{n} \mathfrak{m}_i V$ is not contained in V. However, this contradicts $PV \subseteq V$; hence, V must be in $\mathrm{comp}(X)$, and $\mathrm{Zar}(A) = \mathrm{comp}(X)$. □

The proof of part (a) of the following proposition closely follows the proof of [13, Proposition 1.19].

PROPOSITION 5.6

Let $X := \{V_1, \dots, V_n\}$ be a finite family of valuation overrings of the domain D, and suppose that $V_i V_j = K$ for every $i \neq j$, where K is the quotient field of D. Let $A_i := A(\mathrm{comp}(V_i))$, and let $A := A(\mathrm{comp}(X))$. Then

(a) *each A_i is a localization of A;*
(b) *for each ideal I of A, there is an i such that $IA_i \neq A_i$;*
(c) *if $i \neq j$, then $A_i A_j = K$.*

Proof

(a) By induction and symmetry, it is enough to prove that $B := A_2 \cap \cdots \cap A_n$ is a localization of A. Let J be the Jacobson radical of B; then $J \neq (0)$ since it contains the intersection $\mathfrak{m}_{V_2} \cap \cdots \cap \mathfrak{m}_{V_n}$. Furthermore, if $W \neq K$ is a valuation overring of V_1, then $J \not\subseteq W$ since otherwise (as in the proof of Proposition 5.5) $\mathfrak{m}_{V_2} \cap \cdots \cap \mathfrak{m}_{V_n}$ would be contained in $\mathfrak{m}_W \cap (W \cap V_2 \cap \cdots \cap V_n)$, against the fact that $\{W, V_2, \dots, V_n\}$ are independent valuation overrings.

Hence, for every such W we can apply [13, Proposition 1.13] to $D := B \cap W$, obtaining that B is a localization of D, say $B = S^{-1}D$, where S is a multiplicatively closed subset of D; in particular, there is a $s_W \in S \cap \mathfrak{m}_W$. Each s_W is in $B \cap A_1 = A$ (since $\mathfrak{m}_W$ is contained in every member of comp(V_1)); let T be the set of all s_W. Then

$$T^{-1}A = T^{-1}(B \cap A_2) = T^{-1}B \cap T^{-1}A_1.$$

Each s_W is a unit of B, and thus $T^{-1}B = B$. On the other hand, no valuation overring $W \neq K$ of V_1 can be an overring of $T^{-1}A_1$ since T contains s_W, which is inside the maximal ideal of W. Since $\mathrm{Zar}(A_1) = \mathrm{comp}(V_1)$, it follows that $T^{-1}A_1 = K$, and thus $T^{-1}A = B$; in particular, B is a localization of A.

(b) Without loss of generality, we can suppose $I = P$ to be prime. There is a valuation overring W of A whose center on A is P; since $\mathrm{Zar}(A) = \mathrm{comp}(X)$ by Proposition 5.5, there is a V_i such that $W \in \mathrm{comp}(V_i)$. Hence, $PA_i \neq A_i$.

(c) By Proposition 5.5, $\mathrm{Zar}(A_i) \cap \mathrm{Zar}(A_j) = \{K\}$. It follows that K is the only common valuation overring of A_iA_j; in particular, A_iA_j must be K. □

By [23, Proposition 4.3], Proposition 5.6 can also be rephrased by saying that the set $\{A_1, \dots, A_n\}$ is a *Jaffard family* of A, in the sense of [9, Section 6.3].

PROPOSITION 5.7

Let D be an integrally closed domain; suppose that $\mathrm{Zar}(D) = \mathrm{comp}(V_1) \cup \cdots \cup \mathrm{comp}(V_n)$*, where $X := \{V_1, \dots, V_n\}$ is a family of incomparable valuation overrings of D such that $V_iV_j = K$ if $i \neq j$. Then*

(a) *the restriction of the center map γ to X is injective;*
(b) $|\mathrm{Max}(D)| \geq |X|$.

Proof

(a) If P is the image of both V_i and V_j, then P survives in both A_i and A_j; however, since A_i and A_j are localizations of A (Proposition 5.6(a)), A_P would be a common overring of A_i and A_j, against the fact that $A_iA_j = K$ (Proposition 5.6(c)). Therefore, the center map is injective on X.

(b) Let M be a maximal ideal; then there is a unique i such that $MA_i \neq A_i$. In particular, M can contain only one element of $\gamma(X)$, namely $\gamma(V_i)$; thus, $|\mathrm{Max}(D)| \geq |\gamma(X)| = |X|$, as claimed. □

We are ready to prove the pivotal result of the paper.

THEOREM 5.8

Let D be an integrally closed local domain. If $\mathrm{Zar}_{\min}(D)$ *is a Noetherian space, then D is a pseudo-valuation domain.*

Proof

Since D is local, by Proposition 5.4 there are $W_1, \dots, W_n$, not equal to K, such that

$\mathrm{Zar}(D) = \mathrm{comp}(W_1) \cup \cdots \cup \mathrm{comp}(W_n)$. By eventually passing to bigger valuation domains, we can suppose without loss of generality that $W_i W_j = K$ if $i \neq j$; since D is local, by Proposition 5.7(b) we have $1 \geq n$, and so $\mathrm{Zar}(D) = \mathrm{comp}(V)$ for some $V \neq K$.

Let Δ be the set of $W \in \mathrm{Zar}(D)$ such that $\mathrm{comp}(W) = \mathrm{Zar}(D)$; then Δ is a chain, and thus it has a minimum in $\mathrm{Zar}(D)$, say V_0 (explicitly, V_0 is the intersection of the elements of Δ); furthermore, clearly $V_0 \in \Delta$. Since $V \in \Delta$, we have $V_0 \subseteq V$, and in particular $V_0 \neq K$. Let M be the maximal ideal of V_0; then M is contained in every $W \in \mathrm{comp}(V_0) = \mathrm{Zar}(D)$, and thus $M \subseteq D$.

Consider now the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ \pi\ } & D/M \\ \big\downarrow & & \big\downarrow \\ V_0 & \xrightarrow{\ \pi\ } & V_0/M. \end{array}$$

Clearly, $D = \pi^{-1}(D/M)$; let F_1 be the quotient field of D/M. By Lemma 3.2, the set of minimal valuation overrings of D is homeomorphic to $\mathrm{Zar}_{\min}(V_0/M|D/M)$, which thus is Noetherian; by Proposition 3.4, it follows that either D/M is a field and $\mathrm{trdeg}_{D/M}(V_0/M) = 1$ (in which case D is a pseudo-valuation domain with associated valuation domain V_0) or $\mathrm{trdeg}_{F_1}(V_0/M) = 0$.

In the latter case, we note that D/M is integrally closed in V_0/M since D/M is the intersection of all the elements of $\mathrm{Zar}(V_0/M|D/M)$; hence, V_0/M is the quotient field of D/M. If D/M is not a field, by the same argument of the first part of the proof, it follows that $\mathrm{Zar}(D/M) = \mathrm{comp}(W_0)$ for some valuation overring $W_0 \neq F_1$; however, this contradicts the choice of V_0 because $\pi^{-1}(W_0)$ would be comparable with every element of $\mathrm{Zar}(D)$. Hence, it must be $V_0/M = D/M$ (i.e., $V_0 = D$); that is, D is a valuation domain and, in particular, a pseudo-valuation domain. □

With this result, we can find the possible structures of $\mathrm{Zar}(D)$ and $\mathrm{Zar}_{\min}(D)$ when D is local and $\mathrm{Zar}_{\min}(D)$ is Noetherian. Indeed, D is a pseudo-valuation domain; let V be its associated valuation overring. Then we have two cases: either $D = V$ (i.e., D itself is a valuation domain) or $D \neq V$.

In the first case, $\mathrm{Zar}_{\min}(D)$ is a singleton, while $\mathrm{Zar}(D)$ is homeomorphic to $\mathrm{Spec}(D)$; in particular, $\mathrm{Zar}(D)$ is linearly ordered, and it is a Noetherian space if and only if $\mathrm{Spec}(D)$ is Noetherian.

In the second case, we can separate $\mathrm{Zar}(D)$ into two parts: $\mathrm{Zar}_{\min}(D)$ and $\Delta := \mathrm{Zar}(D) \setminus \mathrm{Zar}_{\min}(D)$. The former must be homeomorphic to $\mathrm{Zar}_{\min}(L|F) = \mathrm{Zar}(L|F) \setminus \{L\}$ (where F and L are the residue fields of D and V, respectively); on the other hand, the latter is linearly ordered, and is composed of the valuation overrings of V, so in particular it is homeomorphic to $\mathrm{Spec}(V)$, which is (set-theoretically) equal to $\mathrm{Spec}(D)$. In other words, $\mathrm{Zar}(D)$ is composed of a long "stalk" (Δ), under which there is an infinite family of minimal valuation overrings. In particular, we get the following.

PROPOSITION 5.9
Let D, V, F, L as above. Then

(a) $\mathrm{Zar}_{\min}(D)$ *is Noetherian if and only if* $\mathrm{Zar}(L|F)$ *is Noetherian;*
(b) $\mathrm{Zar}(D)$ *is Noetherian if and only if* $\mathrm{Zar}(L|F)$ *and* $\mathrm{Spec}(V)$ *are Noetherian.*

Proof
If $\mathrm{Zar}_{\min}(D)$ is Noetherian, then $\mathrm{Zar}_{\min}(L|F)$ is Noetherian as well. By Propositions 4.1 and 4.2, $\mathrm{Zar}(L|F)$ is Noetherian.

If $\mathrm{Zar}(D)$ is Noetherian, so are $\mathrm{Spec}(D) = \mathrm{Spec}(V)$ and $\Delta \simeq \mathrm{Zar}(L|F)$ (in the notation above). Conversely, if $\mathrm{Zar}(L|F)$ and $\mathrm{Spec}(V)$ are Noetherian, then so are $\mathrm{Zar}_{\min}(D)$ and Δ, and thus also $\mathrm{Zar}_{\min}(D) \cup \Delta = \mathrm{Zar}(D)$ is Noetherian. □

Furthermore, we can now apply Propositions 4.1 and 4.2 to characterize when $\mathrm{Zar}(L|F)$ is Noetherian (see Corollary 5.12).

We now study the non-local case.

LEMMA 5.10
Let D be an integral domain such that D_M is a PVD for every $M \in$ $\mathrm{Max}(D)$ *and, for every M, let $V(M)$ be the valuation overring associated to D_M. Then the space* $\{V(M) \mid M \in \mathrm{Max}(D)\}$ *is homeomorphic to* $\mathrm{Max}(D)$.

Proof
Let $\Delta := \{V(M) \mid M \in \mathrm{Max}(D)\}$. If γ is the center map, then $\gamma(V(M)) = M$ for every M; thus, γ restricts to a bijection between Δ and $\mathrm{Max}(D)$. Since γ is continuous and closed, it follows that it is a homeomorphism. □

THEOREM 5.11
Let D be an integrally closed domain. Then

(a) $\mathrm{Zar}_{\min}(D)$ *is Noetherian if and only if* $\mathrm{Max}(D)$ *is Noetherian and* $\mathrm{Zar}_{\min}(D_M)$ *is Noetherian for every* $M \in \mathrm{Max}(D)$*;*
(b) $\mathrm{Zar}(D)$ *is Noetherian if and only if* $\mathrm{Spec}(D)$ *is Noetherian and* $\mathrm{Zar}(D_M)$ *is Noetherian for every* $M \in \mathrm{Max}(D)$.

Proof
(a) If $\mathrm{Zar}_{\min}(D)$ is Noetherian, then $\mathrm{Max}(D)$ is Noetherian since it is the image of $\mathrm{Zar}_{\min}(D)$ under the center map, while each $\mathrm{Zar}_{\min}(D_M)$ is Noetherian since they are subspaces of $\mathrm{Zar}_{\min}(D)$.

Conversely, suppose that $\mathrm{Max}(D)$ is Noetherian and that $\mathrm{Zar}(D_M)$ is Noetherian for every $M \in \mathrm{Max}(D)$. By the latter property and Theorem 5.8, every D_M is a PVD; by Lemma 5.10, the space $\Delta := \{V(M) \mid M \in \mathrm{Max}(D)\}$ (in the notation of the lemma) is homeomorphic to $\mathrm{Max}(D)$, and thus Noetherian. Let β be the map sending a $W \in \mathrm{Zar}_{\min}(D)$ to $V(\mathfrak{m}_W \cap D)$.

Let X be any subset of $\mathrm{Zar}_{\min}(D)$, and let Ω be an open cover of X; without loss of generality, we can suppose $\Omega = \{\mathcal{B}(f_\alpha) \mid \alpha \in A\}$, where the f_α are elements of K. Then Ω is also a cover of $X' := \{\beta(V) \mid V \in X\}$; since X' is compact (being a subset of the Noetherian space Δ), there is a finite subfamily of Ω, say $\Omega' := \{\mathcal{B}(f_1), \ldots, \mathcal{B}(f_n)\}$, that covers X'. For each i, let $X_i := \{V \in X \mid f_i \in \beta(V)\}$; then $X = X_1 \cup \cdots \cup X_n$. We want to find, for each i, a finite subset $\Omega_i \subset \Omega$ that is a cover of X_i.

Fix thus an i, let $f := f_i$, and let $I := (D :_D f)$ be the conductor ideal. For every $M \in \mathrm{Max}(D)$, let $Z(M) := \gamma^{-1}(M) \cap X_i = \{V \in X_i \mid \mathfrak{m}_V \cap D = M\}$, where γ is the center map. The union of the $Z(M)$ is X_i; we separate the cases $I \nsubseteq M$ and $I \subseteq M$.

If $I \nsubseteq M$, then $1 \in ID_M = (D_M :_{D_M} f)$, and thus $f \in D_M$; hence, in this case, $\mathcal{B}(f)$ contains $Z(M)$.

Suppose $I \subseteq M$; clearly, we can suppose $Z(M) \neq \emptyset$. We claim that in this case M is minimal over I. Indeed, if there is a $V \in Z(M)$, then $f \in V$, and thus $f \in \beta(V)$; therefore, $f \in D_P$ for every prime ideal $P \subsetneq M$ (since $D_P \supsetneq \beta(V)$ for every such P), and thus $I \nsubseteq P$. Therefore, M is minimal over I. By Lemma 5.1, I has only finitely many minimal primes; hence, there are only finitely many maximal ideals M such that $I \subseteq M$ and $Z(M) \neq \emptyset$. For each of these M, the set of valuation domains in X centered on M is a subset of $\mathrm{Zar}_{\min}(D_M)$, and thus it is compact; hence, for each of them, Ω admits a finite subcover $\Omega(M)$. It follows that $\Omega_i := \{\mathcal{B}(f)\} \cup \bigcup \Omega(M)$ is a finite subset of Ω that is a cover of X_i.

Hence, $\bigcup_i \Omega_i$ is a finite subset of Ω that covers X; thus, X is compact. Since X was arbitrary, $\mathrm{Zar}_{\min}(D)$ is Noetherian.

(b) If $\mathrm{Zar}(D)$ is Noetherian, then $\mathrm{Spec}(D)$ and every $\mathrm{Zar}(D_M)$ are Noetherian.

Conversely, suppose that $\mathrm{Spec}(D)$ is Noetherian and that $\mathrm{Zar}(D_M)$ is Noetherian for every $M \in \mathrm{Max}(D)$. By the previous point, $\mathrm{Zar}_{\min}(D)$ is Noetherian. Furthermore, if $P \in \mathrm{Spec}(D) \setminus \mathrm{Max}(D)$, then D_P is a valuation domain; hence, $\mathrm{Zar}(D) \setminus \mathrm{Zar}_{\min}(D)$ is homeomorphic to $\mathrm{Spec}(D) \setminus \mathrm{Max}(D)$, which is Noetherian by hypothesis. Being the union of two Noetherian subspaces, $\mathrm{Zar}(D)$ itself is Noetherian. □

COROLLARY 5.12

Let D be an integral domain that is not a field, and let L be a field containing D; suppose that D is integrally closed in L. Then $\mathrm{Zar}(L|D)$ *(resp.,* $\mathrm{Zar}_{\min}(L|D)$*) is Noetherian if and only if the following hold:*

- *L is the quotient field of D;*
- $\mathrm{Spec}(D)$ *is Noetherian (resp.,* $\mathrm{Max}(D)$ *is Noetherian);*
- *for every $M \in \mathrm{Max}(D)$, the ring D_M is a pseudo-valuation domain such that* $\mathrm{Zar}(L|F)$ *is Noetherian, where F is the residue field of D_M and L is the residue field of the associated valuation overring of D_M.*

Proof

Join Proposition 3.4, Theorem 5.11, and Proposition 5.9. □

For our last result, we recall that the *valuative dimension* $\dim_v(D)$ of an integral domain D is the supremum of the dimensions of the valuation overrings of D; a domain D is called a *Jaffard domain* if $\dim(D) = \dim_v(D) < \infty$, while it is a *locally Jaffard domain* if D_P is a Jaffard domain for every $P \in \mathrm{Spec}(D)$ [1]. Any locally Jaffard domain is Jaffard, but the converse does not hold [1, Example 3.2]. The class of Jaffard domains includes, for example, finite-dimensional Noetherian domains, Prüfer domains, and universally catenarian domains.

PROPOSITION 5.13
Let D be an integrally closed integral domain of finite Krull dimension, and suppose that $\mathrm{Zar}_{\min}(D)$ is a Noetherian space. Then

(a) $\dim_v(D) \in \{\dim(D), \dim(D) + 1\}$;
(b) *D is locally Jaffard if and only if D is a Prüfer domain.*

Proof
(a) Let M be a maximal ideal of D. Then $\mathrm{Zar}_{\min}(D_M)$ is Noetherian, and thus D_M is a pseudo-valuation domain; by [1, Proposition 2.9], $\dim_v(D_M) = \dim(D_M) + \mathrm{trdeg}_F L$, where F is the residue field of D_M, and L is the residue field of the associated valuation ring of D_M. By Propositions 5.9 and 4.1, $\mathrm{trdeg}_F L \leq 1$, and thus $\dim_v(D_M) \leq \dim(D_M) + 1$. Hence, $\dim_v(D) \leq \dim(D) + 1$; since $\dim_v(D) \geq \dim(D)$ always, we have the claim.

(b) If D is a Prüfer domain, then it is locally Jaffard. Conversely, if D is locally Jaffard, then $\dim_v(D_P) = \dim(D_P)$ for every prime ideal P of D. Take any maximal ideal M, and let F, L as above; using $\dim_v(D_M) = \dim(D_M) + \mathrm{trdeg}_F L$, it follows that $\mathrm{trdeg}_F L = 0$. Since D (and so D_M) is integrally closed, it must be $F = L$; i.e., D_M itself is a valuation domain. Therefore, D is a Prüfer domain. □

Note that there are domains D that are Jaffard domains and have $\mathrm{Zar}(D)$ Noetherian but are not Prüfer domains. Indeed, the construction presented in [1, Example 3.2] gives a ring R with two maximal ideals, M and N, such that R_M is a two-dimensional valuation ring while R_N is a one-dimensional pseudo-valuation domain with $\dim_v(R_N) = 2$; in particular, it is a Jaffard domain that is not Prüfer. Choosing $k = K(Z_1)$ in the construction (or, more generally, choosing k such that $K(Z_1, Z_2)$ is finite over k), the Zariski space of R_N is Noetherian (being homeomorphic to $\mathrm{Zar}(K(Z_1, Z_2)|k)$, which is Noetherian by Proposition 4.2), and thus $\mathrm{Zar}(R)$ is Noetherian.

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