ALGEBRA, LOGIC AND NUMBER THEORY
BANACH CENTER PUBLICATIONS, VOLUME 121
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2020

# TOPOLOGICAL PROPERTIES OF SUBSETS OF THE ZARISKI SPACE

DARIO SPIRITO

*Dipartimento di Matematica e Fisica, Università degli Studi "Roma Tre"*
*Roma, Italy*
*ORCID: 0000-0002-7318-7860 E-mail: spirito@mat.uniroma3.it*

**Abstract.** We study the properties of some distinguished subspaces of the Zariski space $\mathrm{Zar}(K|D)$ of a field $F$ over a domain $D$, in particular the topological properties of subspaces defined through algebraic means. We are mainly interested in two classes of problems: understanding when spaces of the form $\mathrm{Zar}(K|D) \setminus \{V\}$ are compact (which is strongly linked to the problem of determining when $\mathrm{Zar}(K|D)$ is a Noetherian space), and studying spaces of rings defined through pseudo-convergent sequences on an (arbitrary, but fixed) rank one valuation domain.

**1. Introduction and notation.** Let $D$ be an integral domain and $K$ be a field containing $D$ (not necessarily the quotient field of $D$). In the Thirties, studying the problem of resolution of singularities, Zariski introduced the *Zariski space* of $K$ over $D$ (under the name *generalized Riemann surface*) as the set $\mathrm{Zar}(K|D)$ of all valuation domains of $K$ containing $D$ [23, 24]. He introduced on this set a topology (later called the *Zariski topology*) which is generated by the open sets

$$\mathcal{B}(x_1,\dots,x_n) := \{V \in \mathrm{Zar}(K|D) \mid x_1,\dots,x_n \in V\},$$

as $x_1,\dots,x_n$ range in $K$, and showed that, under this topology, $\mathrm{Zar}(K|D)$ is a compact space [25, Chapter VI, Theorem 40].

Later, it was shown that $\mathrm{Zar}(K|D)$ is actually a *spectral space* (in the sense of Hochster [9]), that is, for every $K$ and $D$ there is a ring $R$ such that $\mathrm{Zar}(K|D) \simeq \mathrm{Spec}(R)$;

2010 *Mathematics Subject Classification*: Primary 13F30; Secondary 12J20, 13A15, 13A18, 54D30, 54E35.
*Key words and phrases*: Zariski space, Zariski topology, constructible topology, non-compact subspaces, Noetherian space, pseudo-convergent sequence, pseudo-limit.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc121-15

such an $R$ can also be constructed explicitly as a Bézout domain having quotient field $K(X)$ (called the *Kronecker function ring* of $K$ over $D$) [4, 5, 6]. As a spectral space, $\mathrm{Zar}(K|D)$ can also be endowed with the *inverse topology* (the topology generated by the complements of the open and compact subspaces of the original topology) and the *constructible* (or *patch*) *topology* (the topology generated by both the open and compact subspaces and their complements). These two topologies are both spectral (so, in particular, compact) and, more importantly, $\mathrm{Zar}(K|D)^{\mathrm{cons}}$ (i.e., $\mathrm{Zar}(K|D)$ under the constructible topology) is an Hausdorff space, something that does not happen for the Zariski or the inverse topology unless $D$ is a field and $K$ is an algebraic extension, i.e., unless $\mathrm{Zar}(K|D)$ is just $\{K\}$. Of particular importance are the closed sets of $\mathrm{Zar}(K|D)^{\mathrm{cons}}$: they are called *proconstructible* subsets, and they are again spectral spaces (in the Zariski topology).

These three topologies are closely linked with the algebraic properties of the valuation domains, and in particular there is a connection between the topological properties of $X \subseteq \mathrm{Zar}(K|D)$ and the algebraic properties of the intersection of the elements of $X$ (called the *holomorphy ring* $A(X)$ of $X$) [11, 12, 13, 14]: for example, if $X$ is a compact subset of one-dimensional valuation domains such that $\bigcap_{V\in X} \mathfrak{m}_V \neq (0)$, then $A(X)$ is a one-dimensional Bézout domain [14, Theorem 5.3]. In particular, for Prüfer domains, the set $\mathrm{Zar}(D)$ (that is, $\mathrm{Zar}(K|D)$ with $K$ being the quotient field of $D$) is homeomorphic to the spectrum of $D$ (under the Zariski topology). More generally, there is always a map $\gamma : \mathrm{Zar}(K|D) \longrightarrow \mathrm{Spec}(D)$, $V \mapsto \mathfrak{m}_V \cap D$, called the *center map*, which is continuous ([25, Chapter VI, §17, Lemma 1] or [4, Lemma 2.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [8, Theorem 19.6]) and closed [4, Theorem 2.5].

The space $\mathrm{Zar}(K|D)$ can also be considered as a subspace of the set $\mathrm{Over}(K|D)$ of the rings comprised between $D$ and $K$, as a subspace of the set of $D$-submodules of $K$ or, even more generally, as a subspace of the power set of $K$; all these sets become spectral spaces under the natural extension of the Zariski topology [3, 1.9.5(vi-vii)]. It is to be noted that a closer look at Zariski's proof of the compactness of $\mathrm{Zar}(K|D)$ actually shows that $\mathrm{Zar}(K|D)$ is a proconstructible subset of the power set $\mathcal{P}(K)$ [13, discussion after Proposition 2.1].

## 2. Compactness.

In general, it is hard to find subsets of $\mathrm{Zar}(K|D)$ that are *not* compact. A general algebro-geometric criterion was given in [7, Lemma 5.8(2)] through the theory of *semistar operations*; to be useful, however, it has to be applied together with the theory of the $b$-operation/integral closure, which can be defined either as the semistar operation induced by the whole $\mathrm{Zar}(D)$ or through integral dependence of ideals [21]. The first consequence is the following.

Theorem 2.1 ([19, Proposition 7.1]). *Let $D$ be a Noetherian ring with quotient field $K$, and let $\Delta$ be the set of Noetherian valuation overrings of $D$. Then, $\Delta$ is compact if and only if* $\dim(D) \leq 1$.

(Note that, when $\dim(D) \leq 1$, the set $\Delta$ is actually just $\mathrm{Zar}(D)$.) If $\Delta$ is as in the theorem, then we can write $\Delta = X(D) \cap \mathrm{Zar}(D)$, where $X(D)$ is the set of Noetherian overrings of $D$; in particular, $X(D)$ cannot be proconstructible in the Zariski topology of $\mathrm{Over}(D)$, since this would imply that $\Delta$, as the intersection of two proconstructible subspaces, is itself proconstructible. The same happens for other subsets of Noetherian rings.

PROPOSITION 2.2 ([19, Proposition 7.3 and Corollary 7.7]). *Let $D$ be a Noetherian domain. Then, the following are equivalent*:

(i) $\dim(D) = 1$;
(ii) $X(D)$ *is compact*;
(iii) *the set* $\{T \in \mathrm{Over}(D) \mid T$ *is a Dedekind domain*$\}$ *is compact*;
(iv) *the set* $\{T \in \mathrm{Over}(D) \mid T$ *is Noetherian of dimension* $1\}$ *is compact.*

*The same holds if "compact" is substituted with "proconstructible".*

Another interesting case is the one in which we delete just one valuation domain.

THEOREM 2.3 ([19, Theorem 3.6]). *Let $D$ be an integral domain and $V$ be a minimal element of* $\mathrm{Zar}(D)$*. If* $\mathrm{Zar}(D) \setminus \{V\}$ *is compact, then $V$ is equal to the integral closure of* $D[x_1, \dots, x_n]_M$ *for some* $x_1, \dots, x_n \in K$ *and some* $M \in \mathrm{Max}(D[x_1, \dots, x_n])$.

This condition is very strong; for example, it cannot happen in any of the following cases:

- $D$ is Noetherian and $\dim(V) \geq 2$;
- $\dim(V) > 2\dim(D)$ [19, Proposition 4.3];
- $D$ is local and $\bigcap\{P \mid P \in \mathcal{Y}\} = (0)$ for some family $\mathcal{Y}$ of nonzero incomparable prime ideals [19, Theorem 5.1].

A topological space $X$ is *Noetherian* if all its subsets are compact; equivalently, if the open sets of $X$ satisfy the ascending chain condition. For example, the prime spectrum of any Noetherian ring is a Noetherian space [1, Chapter 6, Exercises 5–8]. On the other hand, by either of the previous two cases, $\mathrm{Zar}(D)$ is not a Noetherian space as soon as $D$ is a Noetherian domain of dimension 2 or more. Indeed, the Noetherianity of $\mathrm{Zar}(K|D)$ is an extremely rare phenomenon.

PROPOSITION 2.4. *Let $D$ be an integral domain and let $K$ be a field containing $D$; suppose that $D$ is integrally closed in $K$.*

(a) [20, Proposition 4.2] *If $D = F$ is a field, then* $\mathrm{Zar}(K|F)$ *is a Noetherian space if and only if* $\mathrm{trdeg}_F K \leq 1$ *and, for every $T \in K$ transcendental over $F$, every valuation on $F[T]$ extends to finitely many valuations of $K$.*
(b) [20, Theorem 5.11 and Corollary 5.12] *If $D$ is local and not a field, then* $\mathrm{Zar}(D)$ *is Noetherian if and only if $D$ is a pseudo-valuation domain,*[1] *$K$ is the quotient field of $D$ and* $\mathrm{Zar}(L|F)$ *is Noetherian, where $F$ is the residue field of $D$ and $L$ is the residue field of the associated valuation domain.*
(c) [20, Theorem 5.11 and Corollary 5.12] *If $D$ is not a field, then* $\mathrm{Zar}(K|D)$ *is Noetherian if and only if $K$ is the quotient field of $D$,* $\mathrm{Spec}(D)$ *is Noetherian and* $\mathrm{Zar}(D_M)$ *is Noetherian for every* $M \in \mathrm{Max}(D)$.

[1] A *pseudo-valuation domain* (PVD) is a local domain $(D, \mathfrak{m})$ having a valuation overring $V$ whose maximal ideal is $\mathfrak{m}$; such $V$ is called the valuation domain *associated* to $D$.

In particular, these domains have a fairly peculiar Zariski space: in the local case, the non-minimal valuations of $D$ are all comparable, and the valuative dimension of $D$ can be only $\dim(D)$ or $\dim(D)+1$ [20, Proposition 5.13].

**3. Pseudo-convergent sequences.** Let now $V$ be a one-dimensional valuation ring with valuation $v$, value group $\Gamma_v \subseteq \mathbb{R}$ and quotient field $K$. A *pseudo-convergent sequence* of $V$ is a sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ such that

$$v(s_n - s_{n-1}) < v(s_{n+1} - s_n)$$

for all $n \in \mathbb{N}$, $n \geq 1$. Pseudo-convergent sequences were introduced by Ostrowski to determine all the rank-one extensions of $V$ to $K(X)$ [15, 16], and subsequently used by Kaplansky to investigate maximal valued fields [10]. They can be generalized to *pseudo-monotone sequences* [2, Definition 4.6].

The *gauge* of $E$ is the sequence of the $\delta_n := v(s_{n+1} - s_n)$ [22, p. 327]; it is a strictly increasing sequence of real numbers, and its limit $\delta_E \in \mathbb{R} \cup \{\infty\}$ is called the *breadth* of $E$. In particular, $\delta_E$ is infinite if and only if $E$ is a Cauchy sequence (in the topology induced by the valuation). If $V$ is discrete, every pseudo-convergent sequence has infinite breadth. The ideal $\mathrm{Br}(E) := \{x \in V \mid v(x) \geq \delta_E\}$ is called the *breadth ideal* of $E$.

Pseudo-convergent sequences can be divided into two classes: $E$ is of *algebraic type* if $v(f(s_n))$ is definitively increasing for some polynomial $f \in K[X]$, while it is of *transcendental type* otherwise [10, Definitions, p. 306]. If $v(\alpha - s_n) < v(\alpha - s_{n+1})$ for all $n \in \mathbb{N}$ (or, equivalently, if $v(\alpha - s_n) = \delta_n$), then $\alpha$ is said to be a *pseudo-limit* of $E$; if $\alpha \in \overline{K}$ (the algebraic closure of $K$), then we can use the same definition once we fix an extension $u$ of $v$ to $\overline{K}$. In particular, $E$ is of algebraic type if and only if it has a pseudo-limit in $\overline{K}$ [10, Theorems 2 and 3]. Pseudo-limits are not unique, but if $\alpha$ is one of them, then the set $\mathcal{L}(E)$ of the pseudo-limits of $E$ is the coset $\alpha + \mathrm{Br}(E)$ [10, Lemma 3]. The name "algebraic" and "transcendental" derive from the fact that, if $E$ is a Cauchy sequence, the limit of $E$ in $\widehat{K}$ is algebraic (resp., transcendental) over $K$ if and only if $E$ is of algebraic (resp., transcendental) type.

To each pseudo-convergent sequence $E$ we associate the map $w_E : K(X) \longrightarrow \mathbb{R} \cup \{\infty\}$ such that [17, Propositions 4.3 and 4.4]

$$w_E(\phi) := \lim_{n\to\infty} v(\phi(s_n)).$$

Then, $w_E$ is a valuation on $K(X)$ if $E$ is of transcendental type or if $E$ is of algebraic type and $\delta_E < \infty$; if $E$ is of algebraic type and $\delta_E = \infty$, then $w_E$ is only a pseudo-valuation[2]. If $w_E$ is a valuation, the corresponding valuation ring $W_E$ is a one-dimensional extension of $V$ to $K(X)$; if $K$ is algebraically closed, then every rank-one extension of $V$ to $K(X)$ is in this form [15, 16]. We denote the set of all rings in the form $W_E$ as $\mathcal{W}$: then, the Zariski and the constructible topologies agree on $\mathcal{W}$, and under them $\mathcal{W}$ is a regular zero-dimensional space that is not compact [17, Propositions 6.3 and 6.4].

[2]A *pseudo-valuation* on $K$ is a map $v : K \longrightarrow \Gamma_v \cup \{\infty\}$ (where $(\Gamma_v, +)$ is a totally ordered abelian group) such that $v(a+b) \geq \min\{v(a), v(b)\}$ and $v(ab) = v(a) + v(b)$ for all $a, b \in K$; that is, it is a valuation without the hypothesis that only 0 goes to $\infty$. It is *not* linked with the notion of pseudo-valuation domain used in Section 2.

To every pseudo-convergent sequence $E$ can be associated another valuation domain, defined as

$$V_E := \{\phi \in K(X) \mid \phi(s_n) \in V \text{ for all large } n\}.$$

The ring $V_E$ is always an extension of $V$ to $K(X)$, and it is contained in $W_E$ (if $W_E$ is defined). If $E$ is of transcendental type, then $V_E = W_E$ is an immediate extension of $E$ [17, Theorem 4.9(a)]. On the other hand, if $E$ is of algebraic type, then the value group of $V_E$ is always isomorphic to $\Gamma_v \oplus \mathbb{Z}$, and the rank of $V_E$ depends on the breadth [17, Theorem 4.9(b,c)]:

- if $k\delta \in \Gamma_v$ for some positive $k \in \mathbb{N}$, then $V_E$ has rank 2 and $W_E$ has rank 1;
- if $\delta < \infty$ and $k\delta \notin \Gamma_v$ for all positive $k \in \mathbb{N}$, then $V_E = W_E$ has rank 1;
- if $\delta = \infty$, then $V_E$ has rank 2 and its one-dimensional overring is $K[X]_{(q)}$, where $q$ is the minimal polynomial of the limit of $E$.

The valuation $v_E$ can also be described explicitly as a map into $\mathbb{R}^2$ (see [17, Theorem 4.10]).

We denote the set of all the $V_E$ as $\mathcal{V}$: then, $\mathcal{V}$ is a regular space in both the Zariski and the constructible topologies [17, Theorem 6.15], but the two topologies agree on $\mathcal{V}$ if and only if the residue field of $V$ is finite [17, Proposition 6.11]. There is also a map

$$\begin{aligned} \mathcal{W} &\longrightarrow \mathcal{V} \\ W_E &\longmapsto V_E \end{aligned}$$

that, under the Zariski topology, is continuous and injective, but *not* a topological embedding [17, Proposition 6.13].

There are two natural ways to partition $\mathcal{V}$, either by fixing the breadth of the sequences or by fixing a pseudo-limit.

Let $\delta \in \mathbb{R} \cup \{\infty\}$, and define $\mathcal{V}(\bullet, \delta) := \{V_E \in \mathcal{V} \mid \delta_E = \delta\}$. Then, the Zariski and the constructible topologies agree on $\mathcal{V}(\bullet, \delta)$ [18, Theorem 3.5]; furthermore, this topology is also generated by the ultrametric distance

$$d_\delta(V_E, V_F) := \lim_{n\to\infty} \max\{d(s_n, t_n) - e^{-\delta}, 0\},$$

where $E := \{s_n\}_{n\in\mathbb{N}}$ and $F := \{t_n\}_{n\in\mathbb{N}}$. Under this metric, $\mathcal{V}(\bullet, \delta)$ is complete, and is the completion of the subspace [18, Proposition 3.4]

$$\mathcal{V}_K(\bullet, \delta) := \{V_E \in \mathcal{V}(\bullet, \delta) \mid E \text{ has a pseudo-limit in } K\}.$$

When $\delta = \infty$, the space $\mathcal{V}(\bullet, \infty)$ is canonically isomorphic to the completion $\widehat{K}$, and $d_\infty$ reduces to the distance induced by $\widehat{v}$; furthermore, $\mathcal{V}_K(\bullet, \infty)$ corresponds to $K$. Hence, $\mathcal{V}(\bullet, \delta)$ can be seen as a generalization of the completion of $V$, with the elements of $\mathcal{V}(\bullet, \delta)$ corresponding to the closed balls of $V$ of radius $e^{-\delta}$. Note that the various $d_\delta$ *cannot* be unified to a metric on the whole $\mathcal{V}$ (since otherwise they would define closed subspaces of $\mathcal{V}$, but the $\mathcal{V}(\bullet, \delta)$ are not closed) [18, Proposition 3.8].

Let $\beta \in \overline{K}$, fix an extension $u$ of $v$ to $\overline{K}$ and let

$$\mathcal{V}^u(\beta, \bullet) := \{V_E \in \mathcal{V} \mid \beta \text{ is a pseudo-limit of } E \text{ w.r.t. } u\}.$$

Then, each $\mathcal{V}^u(\beta,\bullet)$ is a closed subspace of $\mathcal{V}$ [18, Proposition 4.2], and the Zariski and the constructible topologies agree on $\mathcal{V}^u(\beta,\bullet)$ [18, Proposition 4.6]; furthermore, the elements of $\mathcal{V}^u(\beta,\bullet)$ are parametrized by the breadth, and so there is a bijection between $\mathcal{V}^u(\beta,\bullet)$ and $(-\infty,\delta(\beta,K)]$ (given by $E\mapsto\delta_E$), where $\delta(\beta,K):=\sup\{u(\beta-x)\mid x\in K\}$ represent (the valuation relative to) the distance between $\beta$ and $K$. The topology induced by $\mathcal{V}^u(\beta,\bullet)$ on $(-\infty,\delta(\beta,K)]$ is generated by the sets $(a,b]$, with $b\in\mathbb{Q}\Gamma_v$ [18, Theorem 4.4]. This topology is metrizable if and only if $\Gamma_v$ is countable; in particular, we have the following.

Proposition 3.1 ([18, Corollary 4.8]). *If $\Gamma_v$ is not countable, then* $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ *is not metrizable.*

To conclude, we list some open problems on the topological properties of $\mathcal{V}$, $\mathcal{W}$ and their subsets.

- Is $\mathcal{V}$ zero-dimensional?
- Is $\mathcal{V}$ a normal space?
- Are $\mathcal{V}(\delta_1,\bullet)$ and $\mathcal{V}(\delta_2,\bullet)$ homeomorphic for $\delta_1\neq\delta_2$? (This is true if $\delta_1-\delta_2\in\Gamma_v$ [18, Proposition 3.9].)
- If $\Gamma_v$ is countable, are $\mathcal{V}$ and $\mathcal{W}$ metrizable?
- If $\Gamma_v$ is countable, is $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ metrizable?
- More generally, when is $\mathrm{Zar}(K|D)^{\mathrm{cons}}$ metrizable?
- If any of them is metrizable, can we find an *ultrametric* distance?
- What happens to $\mathcal{V}$ when the rank of $V$ is not 1?

## References


[1] M. F. Atiyah, I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass., 1969.

[2] J.-L. Chabert, *On the polynomial closure in a valued field*, J. Number Theory 130 (2010), 458–468.

[3] J. Dieudonné, A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, Inst. Hautes Études Sci. Publ. Math. 20 (1964), 5–259.

[4] D. E. Dobbs, R. Fedder, M. Fontana, *Abstract Riemann surfaces of integral domains and spectral spaces*, Ann. Mat. Pura Appl. (4) 148 (1987), 101–115.

[5] D. E. Dobbs, M. Fontana, *Kronecker function rings and abstract Riemann surfaces*, J. Algebra 99 (1986), 263–274.

[6] C. A. Finocchiaro, M. Fontana, K. A. Loper, *The constructible topology on spaces of valuation domains*, Trans. Amer. Math. Soc. 365 (2013), 6199–6216.

[7] C. A. Finocchiaro, M. Fontana, D. Spirito, *Spectral spaces of semistar operations*, J. Pure Appl. Algebra 220 (2016), 2897–2913.

[8] R. Gilmer, *Multiplicative Ideal Theory*, Pure and Appl. Math. 12, Marcel Dekker, New York, 1972.

[9] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969), 43–60.

[10] I. Kaplansky, *Maximal fields with valuations I*, Duke Math. J. 9 (1942), 303–321.

[11] B. Olberding, *Noetherian spaces of integrally closed rings with an application to intersections of valuation rings*, Comm. Algebra 38 (2010), 3318–3332.

[12] B. Olberding, *Affine schemes and topological closures in the Zariski–Riemann space of valuation rings*, J. Pure Appl. Algebra 219 (2015), 1720–1741.

[13] B. Olberding, *Topological aspects of irredundant intersections of ideals and valuation rings*, in: Multiplicative Ideal Theory and Factorization Theory, Springer Proc. Math. Stat. 170, Springer, Cham, 2016, 277–307.

[14] B. Olberding, *A principal ideal theorem for compact sets of rank one valuation rings*, J. Algebra 489 (2017), 399–426.

[15] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper. Teil I*, Math. Z. 39 (1935), 269–320.

[16] A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper. Teil II und III*, Math. Z. 39 (1935), 321–404.

[17] G. Peruginelli, D. Spirito, *The Zariski–Riemann space of valuation domains associated to pseudo-convergent sequences*, Trans. Amer. Math. Soc. 373 (2020), 7959–7990.

[18] G. Peruginelli, D. Spirito, *Metrizability of spaces of valuation domains associated to pseudo-convergent sequences*, submitted.

[19] D. Spirito, *Non-compact subsets of the Zariski space of an integral domain*, Illinois J. Math. 60 (2016), 791–809.

[20] D. Spirito, *When the Zariski space is a Noetherian space*, Illinois J. Math. 63 (2019), 299–316.

[21] I. Swanson, C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, London Math. Soc. Lecture Note Ser. 336, Cambridge Univ. Press, Cambridge, 2006.

[22] S. Warner, *Topological Fields*, North-Holland Math. Stud. 157, Notas de Matemática 126, North-Holland, Amsterdam, 1989.

[23] O. Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) 40 (1939), 639–689.

[24] O. Zariski, *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. 50 (1944), 683–691.

[25] O. Zariski, P. Samuel, *Commutative Algebra*, Vol. II, Grad. Texts in Math. 29, Springer, New York, 1975.